数理解析研究所講究録
1021巻 1997年 48-62

UNIVERSE OF QUANTUM SET THEORY

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ABSTRACT. Quantum logic is a logic whose model is the complete orthomodular lattice of closed subspaces of a Hilbert space. G. Takeuti developed in [1] a quantum set theory based on the quantum logic.

In the present paper, we introduce a strong implication into the Takeuti's quantum set theory, and consider it as a lattice valued set theory in [4].

INTRODUCTION

Let $\mathcal{Q}$ be a complete lattice consisting of all closed linear subspaces of a Hilbert space $\mathcal{H}$. $\mathcal{Q}$ is an orthomodular lattice, and a model of *quantum logic.* We can construct the $\mathcal{Q}$-valued universe $V^{\mathcal{Q}}$ in our universe V of ZFC. The set theory on $V^{\mathcal{Q}}$ is based on quantum logic, and is called the *quantum set theory.*

On $\mathcal{Q}$ we have the unary operation $^{\perp}$ of orthogonal complementation, besides the lattice operations $\bigwedge$ and $\bigvee$. The operation $\rightarrow_{\mathcal{Q}}$ defined by

$$a \rightarrow_{\mathcal{Q}} b \overset{\text{def}}{\Longleftrightarrow} a^{\perp} \vee (a \wedge b)$$

is an implication, in the sense that the following I 1, I 2 are satisfied.

I 1 : $(a \rightarrow_{\mathcal{Q}} b) = 1$ iff $a \leqslant b$

I 2 : $a \wedge (a \rightarrow_{\mathcal{Q}} b) \leqslant b$

We call $\rightarrow_{\mathcal{Q}}$ the *quantum implication.*

G.Takeuti developed a quantum set theory with the quantum implication $\rightarrow_{\mathcal{Q}}$ in [1]. Then the corresponding equality and the membership relation on $V^{\mathcal{Q}}$ are defined by:

$$\begin{aligned}
[\![u =_Q v]\!] &= \bigwedge_{x\in\mathcal{D}u} (u(x) \to_Q [\![x \in_Q v]\!]) \wedge \bigwedge_{x\in\mathcal{D}v} (v(x) \to_Q [\![x \in_Q u]\!]) \\
[\![u \in_Q v]\!] &= \bigvee_{x\in\mathcal{D}v} [\![u =_Q x]\!] \wedge v(x).
\end{aligned}$$

Each implication defines equality and membership relation. We call the above equality $=_Q$ and the membership relation $\in_Q$ the *quantum equality* and the *quantum membership relation*, respectively.

Unfortunately, the equality axioms are not valid for the quantum equality on $V^{\mathcal{Q}}$. This means that quantum implication is not strong enough to define the equality of the set theory. We need a stronger equality to develop a set theory with equality axioms on $V^{\mathcal{Q}}$.

On the other hand, any complete lattice has an implication $\to$ defined by

$$(a \to b) = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{otherwise.} \end{cases}$$

The above defined $\to$ is the strongest implication which represents the order relation of the lattice, and we call it the *basic implication.* For a complete lattice $\mathcal{L}$, we can construct the lattice valued universe $V^{\mathcal{L}}$. We showed in [4] that the lattice valued set theory LZFZ on $V^{\mathcal{L}}$ can be formulated by introducing the basic implication $\to$. That is, if we interprete the equality and membership relation by using the basic implication, i.e.

$$\begin{aligned}
[\![u = v]\!] &= \bigwedge_{x\in\mathcal{D}u} (u(x) \to [\![x \in v]\!]) \wedge \bigwedge_{x\in\mathcal{D}v} (v(x) \to [\![x \in u]\!]) \\
[\![u \in v]\!] &= \bigvee_{x\in\mathcal{D}v} [\![u = x]\!] \wedge v(x).
\end{aligned}$$

then the equality axioms are valid on $V^{\mathcal{Q}}$.

In this paper, we introduce the basic implication $\to$, and the corresponding basic equality and membership relation to the Takeuti's quantum set theory, as primitive symbols. The quantum equality and quantum membership relation are defined in the set theory.

Now, we have two equalities $=$, $=_Q$, and two membership relations $\in$, $\in_Q$ on V^Q such that

$$u = v \leftrightarrow \forall x(x \in u \leftrightarrow x \in v) \quad \text{and} \quad u \in v \leftrightarrow \exists x(u = x \wedge x \in v),$$

$$u =_Q v \leftrightarrow \forall x(x \in_Q u \leftrightarrow_Q x \in_Q v) \quad \text{and} \quad u \in_Q v \leftrightarrow \exists x(u =_Q x \wedge x \in_Q v)$$

We denote $(1 \rightarrow a)$ by $\Box a$, that is,

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1. \end{cases}$$

An element a of Q is said to be $\Box$-closed if $\Box a = a$.

For each set u in our external universe V, We define $\check{u} \in V^Q$ by

$$\begin{cases} \mathcal{D}\check{u} = \{\check{x} \mid x \in u\} \\ \check{u}(\check{x}) = 1. \end{cases}$$

Then for sets $u, v \in V$, $[\![\check{u} = \check{v}]\!]$ and $[\![\check{u} \in \check{v}]\!]$ are $\Box$-closed and

$$u = v \Longleftrightarrow [\![\check{u} = \check{v}]\!] = 1\ ; \quad u \in v \Longleftrightarrow [\![\check{u} \in \check{v}]\!] = 1.$$

A set in V^Q which is equal to a set of the form $\check{u}$ is called a *check set*. As far as check sets concern, the quantum equality $=_Q$ and the quantum membership relation $\in_Q$ are identical with the basic ones $=$ and $\in$, respectively, i.e.

$$\check{u} =_Q \check{v} \Longleftrightarrow \check{u} = \check{v}\ ; \quad \check{u} \in_Q \check{v} \Longleftrightarrow \check{u} \in \check{v}.$$

If we take the implication $\rightarrow$ in the definition of numbers, then the set of real numbers is a check set (cf. [4]). On the other hand, it is known that the set R^Q of real numbers in $\langle V^Q, \wedge, \vee, \rightarrow_Q \rangle$ is represented by the set of self-adjoint operators on the Hilbert space $\mathcal{H}$.

1. Quantum Logic

Let $\mathcal{H}$ be a Hilbert space. We consider the complete lattice

$$\mathcal{Q} = \langle \mathcal{Q}, \leqslant, \bigwedge \bigvee, {}^{\perp}, 0, 1 \rangle$$

consisting of closed linear subspaces of $\mathcal{H}$, as a model of quantum logic, where the least upper bound of $\{p_\alpha\}_\alpha$ of $\mathcal{Q}$ is denoted by $\bigvee_\alpha p_\alpha$, and the greatest lower bound of $\{p_\alpha\}_\alpha$ is denoted by $\bigwedge_\alpha p_\alpha$. The smallest element and the largest element of $\mathcal{Q}$ are denoted by 0 and 1, respectively.

1.1. Properties of $\mathcal{Q}$.

The complete lattice $\mathcal{Q} = \langle \mathcal{Q}, \leqslant, \wedge, \vee, {}^{\perp}, 0, 1 \rangle$ of all closed linear subspaces of $\mathcal{H}$ is an orthomodular lattice. That is, for $p, q \in \mathcal{Q}$,

(1) $p^{\perp\perp} = p$
(2) $p \wedge p^{\perp} = 0$; $p \vee p^{\perp} = 1$
(3) $(p \wedge q)^{\perp} = p^{\perp} \vee q^{\perp}$; $(p \vee q)^{\perp} = p^{\perp} \wedge q^{\perp}$
(4) $p \wedge (p^{\perp} \vee (p \wedge q)) = p \wedge q$

1.2. Implications.

In general, an operator $\rightarrow_*$ on a complete lattice which satisfies the following properties is called an *implication.*
I 1 : $(a \rightarrow_* b) = 1$ iff $a \leqslant b$
I 2 : $a \wedge (a \rightarrow_* b) \leqslant b$.

The operator $\rightarrow$ on a lattice defined by

$$(a \rightarrow b) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

represents the order relation of the lattice, and is the strongest implication on the lattice in the sense that

$$(p \rightarrow q) \leqslant (p \rightarrow_* q) \text{ for every implication } \rightarrow_* .$$

We call $\rightarrow$ the ***basic implication.*** The corresponding complement $\neg a$ of a is defined by

$$\neg a \stackrel{\text{def}}{=} (a \rightarrow 0)$$

Here we denote the formula $(1 \to a)$ by $\Box a$, that is,

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1 \end{cases}$$

Proposition 1.1. *For all elements* a, b, a_k, b_k, c_k $(k \in K)$ *of* $\mathcal{Q}$,

G1 : $\Box a \leqslant a$
G2 : $\neg \Box a = \Box \neg \Box a$
G3 : $\bigwedge_k \Box a_k \leqslant \Box \bigwedge_k a_k$
G4 : *If* $\Box a \leqslant b$, *then* $\Box a \leqslant \Box b$
G5 : $\Box a \wedge \bigvee_k b_k = \bigvee_k (\Box a \wedge b_k)$; $\quad a \wedge \bigvee_k \Box b_k = \bigvee_k (a \wedge \Box b_k)$;
$\Box a \vee \bigwedge_k b_k = \bigwedge_k (\Box a \vee b_k)$; $\quad a \vee \bigwedge_k \Box b_k = \bigwedge_k (a \vee \Box b_k)$
G6 : $\Box a \vee \neg \Box a = 1$
G7 : *If* $a \wedge \Box c \leqslant b$, *then* $\neg b \wedge \Box c \leqslant \neg a$.
G8 : $(a \to b) = \bigvee \{c \in \mathcal{L} \mid c = \Box c,\ a \wedge c \leqslant b\}$

Definition 1.1. On the orthomoduler lattice $\mathcal{Q}$, the binary operator $\to_Q$ defined by

$$(p \to_Q q) \stackrel{\text{def}}{=} p^\perp \vee (p \wedge q)$$

is also an implication on $\mathcal{Q}$. We call $\to_Q$ the *quantum implication.*

1.3. Commutability.

Definition 1.2. p is said to be *commutable with* q, in symbols $p \,\rfloor_{\circ}\, q$, if $p \leqslant (p \wedge q) \vee (p \wedge q^\perp)$.

$$p \,\rfloor_{\circ}\, q \stackrel{\text{def}}{\Longleftrightarrow} p \leqslant (p \wedge q) \vee (p \wedge q^\perp)$$

Note that $p \,\rfloor_{\circ}\, q$, $q \,\rfloor_{\circ}\, p$, $p^\perp \,\rfloor_{\circ}\, q$, $p \,\rfloor_{\circ}\, q^\perp$ are all equivalent, and $(\Box p) \,\rfloor_{\circ}\, q$ for all $p, q \in \mathcal{Q}$.

Definition 1.3. $A \subset \mathcal{Q}$ is *commutable*, in symbols $\rfloor_{\circ}(A)$, if elements of A are mutually commutable.

$$\rfloor_{\circ}(A) \stackrel{\text{def}}{=} \bigwedge_{p,q \in A} (p \,\rfloor_{\circ}\, q).$$

Proposition 1.2. *If $p \,\rfloor\!\!\circ\, q_\alpha$ for each α, then*

$$p \wedge \bigvee_\alpha q_\alpha = \bigvee_\alpha (p \wedge q_\alpha).$$

Theorem 1.1. *Let M be a linearly ordered subset of $\mathcal{Q}$. Then M is commutable, and there exists a complete Boolean subalgebra B of $\mathcal{Q}$ including M.*

Proof. It is obvious that M is commutable. Let B be a maximal commutable subset of $\mathcal{Q}$ containing M. B is a complete Boolean subalgebra. $\square$

2. $\mathcal{Q}$-VALUED UNIVERSE $V^{\mathcal{Q}}$

$\mathcal{Q}$-valued universe $V^{\mathcal{Q}}$ is constructed by induction, in the same way as Boolean valued universe V^B.

$$\begin{aligned} V_\alpha^{\mathcal{Q}} &= \{u \mid \exists\beta<\alpha\, \exists \mathcal{D}u \subset V_\beta^{\mathcal{Q}} (u : \mathcal{D}u \to \mathcal{Q})\} \\ V^{\mathcal{Q}} &= \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{\mathcal{Q}} \end{aligned}$$

The primitive relations $u = v$, $u \in v$ are interpreted on $V^{\mathcal{Q}}$ as:

$$\begin{aligned} [\![u=v]\!] &= \bigwedge_{x \in \mathcal{D}u} (u(x) \to [\![x \in v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \to [\![x \in u]\!]) \\ [\![u \in v]\!] &= \bigvee_{x \in \mathcal{D}v} [\![u = x]\!] \wedge v(x). \end{aligned}$$

We say an element p of $\mathcal{Q}$ is $\square$-*closed* if $p = \square p$. As an immediate consequence of the definition of $[\![u = v]\!]$, we have:

Lemma 2.1. *For every $u, v \in V^{\mathcal{Q}}$, $[\![u=v]\!]$ is $\square$-closed.*

Lemma 2.2. *For $u, v \in V^{\mathcal{Q}}$ and $\{b_k\}_k \subset \mathcal{Q}$,*

$$[\![u=v]\!] \wedge \bigvee_k b_k = \bigvee_k [\![u=v]\!] \wedge b_k$$

Proof. By Proposition 1.1, G5. $\square$

2.1. Nonlogical axioms.

The following nonlogical axioms, GA1–GA11, of lattice valued set theory **LZFZ** are valid on $\mathcal{Q}$-valued universes $V^{\mathcal{Q}}$ (cf. [4]) :

GA1. Equality: $\forall u \forall v\,(u = v \wedge \varphi(u) \to \varphi(v))$.
GA2. Extensionality: $\forall u, v\,(\forall x(x \in u \leftrightarrow x \in v) \to u = v)$.
GA3. Pairing: $\forall u, v \exists z\,(\forall x(x \in z \leftrightarrow (x = u \vee x = v)))$.
The set z satisfying $\forall x(x \in z \leftrightarrow (x = u \vee x = v))$ is denoted by $\{u, v\}$.
GA4. Union: $\forall u \exists z\,(\forall x(x \in z \leftrightarrow \exists y \in u(x \in y)))$.
The set z satisfying $\forall x(x \in z \leftrightarrow \exists y \in u(x \in y))$ is denoted by $\bigcup u$.
GA5. Power set: $\forall u \exists z\,(\forall x(x \in z \leftrightarrow x \subset u))$, where

$$x \subset u \overset{\text{def}}{\iff} \forall y(y \in x \to y \in u).$$

The set z satisfying $\forall x(x \in z \leftrightarrow x \subset u)$ is denoted by $\mathcal{P}(u)$.
GA6. Infinity: $\exists u\,(\exists x(x \in u) \wedge \forall x(x \in u \to \exists y \in u(x \in y)))$.
GA7. Separation: $\forall u \exists v\,(\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x)))$.
The set v satisfying $\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x))$ is denoted by $\{x \in u \mid \varphi(x)\}$.
GA8. Collection:

$$\forall u \exists v \left(\forall x(x \in u \to \exists y \varphi(x, y)) \to \forall x(x \in u \to \exists y \overset{\square}{\in} v \varphi(x, y))\right).$$

GA9. $\in$-induction: $\forall x\,(\forall y(y \in x \to \varphi(y)) \to \varphi(x)) \to \forall x \varphi(x)$.
GA10. Zorn: $\mathrm{Gl}(u) \wedge \forall v\,(\mathrm{Chain}(v, u) \to \bigcup v \in u) \to \exists z\, \mathrm{Max}(z, u)$, where

$$\begin{aligned}
\mathrm{Gl}(u) &\overset{\text{def}}{\iff} \forall x(x \in u \to x \overset{\square}{\in} u), \\
\mathrm{Chain}(v, u) &\overset{\text{def}}{\iff} v \subset u \wedge \forall x, y(x, y \in v \to x \subset y \vee y \subset x), \\
\mathrm{Max}(z, u) &\overset{\text{def}}{\iff} z \in u \wedge \forall x(x \in u \wedge z \subset x \to z = x).
\end{aligned}$$

GA11. Axiom of $\Diamond$: $\forall u \exists z \forall t(t \in z \leftrightarrow \Diamond(t \in u))$.
The set z satisfying $\forall t(t \in z \leftrightarrow \Diamond(t \in u))$ is denoted by $\Diamond u$.

2.2. Subuniverse V^B of $V^{\mathcal{Q}}$.

Let $B = \langle B, \bigwedge, \bigvee, \rightarrow_{\mathrm{B}}, 0, 1\rangle$ be a Boolean subalgebra of $\mathcal{Q}$, and let

$$(p \rightarrow_B q) \stackrel{\text{def}}{=} p^{\perp} \vee q.$$

$\rightarrow_B$ is an implication on B, and

$$(p \rightarrow_B q) = (p \rightarrow_{\mathcal{Q}} q) \quad (= p^{\perp} \vee (p \wedge q)) \quad \text{for } p, q \in B.$$

The corresponding equality and membership relation to $\rightarrow_{\mathcal{Q}}$ and $\rightarrow_{\mathrm{B}}$, $=_{\mathcal{Q}}$, $\in_{\mathcal{Q}}$, $=_{\mathrm{B}}$, $\in_{\mathrm{B}}$, are defined by:

$$\begin{aligned} [\![u =_{\mathcal{Q}} v]\!] &= \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow_{\mathcal{Q}} [\![x \in_{\mathcal{Q}} v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \rightarrow_{\mathcal{Q}} [\![x \in_{\mathcal{Q}} u]\!]) \\ [\![u \in_{\mathcal{Q}} v]\!] &= \bigvee_{x \in \mathcal{D}v} [\![u =_{\mathcal{Q}} x]\!] \wedge v(x). \end{aligned}$$

$$\begin{aligned} [\![u =_{\mathrm{B}} v]\!] &= \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow_{\mathrm{B}} [\![x \in_{\mathrm{B}} v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \rightarrow_{\mathrm{B}} [\![x \in_{\mathrm{B}} u]\!]) \\ [\![u \in_{\mathrm{B}} v]\!] &= \bigvee_{x \in \mathcal{D}v} [\![u =_{\mathrm{B}} x]\!] \wedge v(x). \end{aligned}$$

It is obvious from the fact that $(p \rightarrow_B q) = (p \rightarrow_{\mathcal{Q}} q)$ for $p, q \in B$ that

$$[\![u =_{\mathrm{B}} v]\!] = [\![u =_{\mathcal{Q}} v]\!], \quad [\![u \in_{\mathrm{B}} v]\!] = [\![u \in_{\mathcal{Q}} v]\!] \quad \text{for } u, v \in V^B.$$

Hence $\langle B, \bigwedge, \bigvee, \rightarrow_{\mathrm{B}}, 0, 1\rangle$ is a subalgebra of $\langle \mathcal{Q}, \bigwedge, \bigvee, \rightarrow_{\mathcal{Q}}, 0, 1\rangle$, and V^B is a subuniverse of $V^{\mathcal{Q}}$.

3. Numbers in $V^{\mathcal{Q}}$

The set ω of all natural numbers is constructed from 0 by the successor function $x \mapsto x+1$, where 0 is the empty set and $x+1 = x \cup \{x\}$. The integers are constructed as equivalence classes of pairs of natural numbers, the rational numbers are constructed as equivalence classes of pairs of integers, and finally, the real numbers are constructed by Dedekind's cuts of rational numbers. We denote the set of all integers by $\mathbb{Z}$, the set of all rational numbers by $\mathbb{Q}$, the set of all real numbers by $\mathbb{R}$ and the set of all complex numbers by $\mathbb{C}$.

If we use the basic implication $\to$ in the definitions of these numbers in $V^{\mathcal{Q}}$, then they are all check sets (cf. [4]), that is, the sets of natural numbers, integers, rational numbers, real numbers, and complex numbers are equal to $\check{\omega}$, $\check{\mathbb{Z}}$, $\check{\mathbb{Q}}$, $\check{\mathbb{R}}$ and $\check{\mathbb{C}}$, respectively, where $\check{u}$ is the copy in $V^{\mathcal{Q}}$ of u, which is defined by

$$\mathcal{D}\check{u} = \{\check{x} \mid x \in u\}, \quad u(\check{x}) = 1 \text{ for } x \in u.$$

We call a set v of $V^{\mathcal{Q}}$ a *check set*, if $[\![v = \check{u}]\!] = 1$ for some set u.

If we use the quantum implication $\to_Q$ instead of the basic implication $\to$, then the sets of natural numbers, integers, rational numbers are still $\check{\omega}$, $\check{\mathbb{Z}}$, and $\check{\mathbb{Q}}$, respectively. A real number that is an upper segment of a Dedekind cut defined by using the quantum implication $\to_Q$, we call a *Q-real*. It is known that each Q-real is in some Boolean subuniverse V^B of $V^{\mathcal{Q}}$, and represented by a self-adjoint operator on the Hilbert space $\mathcal{H}$.

3.1. Q-reals.

Definition 3.1. We call $u \in \mathcal{P}(\check{\mathbb{Q}})$ a *Q-real*, if

$$(1) : \exists r \in \check{\mathbb{Q}}(r \in_Q u) \wedge \exists s \in \check{\mathbb{Q}}(s \in_Q u)^{\perp}$$
$$(2) : \forall r \in \check{\mathbb{Q}} \Big((r \in_Q u) \longleftrightarrow_Q \exists s \in \check{\mathbb{Q}}(s < r \wedge (s \in_Q u)\Big)$$

Proposition 3.1. *If $[\![u$ is a Q-real $]\!] = p$, then there exists $v \in V^{\mathcal{Q}}$ such that*

$$\mathcal{D}v = \mathcal{D}\check{\mathbb{Q}}, \quad [\![v \textit{ is a Q-real }]\!] = 1, \quad \textit{and} \quad [\![u =_Q v]\!] \geqslant p.$$

Proof. Such an element v of $V^{\mathcal{Q}}$ is defined by

$$\mathcal{D}v = \mathcal{D}\check{\mathbb{Q}}$$

$$v(\check{r}) = \begin{cases} ([\![\check{r} \in u]\!] \wedge p) \vee p^{\perp}, & \text{if } r > 0 \\ ([\![\check{r} \in u]\!] \wedge p), & \text{if } r \leq 0. \end{cases}$$

□

Let R^{Q} be the set of Q-reals in V^{Q}.

$$R^{\mathrm{Q}} = \{u \in V^{\mathrm{Q}} \mid [\![u \text{ is a Q-real }]\!] = 1\}$$

Definition 3.2. Q-reals $u, v \in R^{\mathrm{Q}}$ are said to be *commutable* if

$$\forall r, s \in \mathbb{Q}([\![\check{r} \in u]\!] \downarrow_{\circ} [\![\check{s} \in v]\!]).$$

Proposition 3.2. (1) *If $\{u, v, w\}$ is a set of mutually commutable Q-reals, then*

$$[\![(u =_Q v) \wedge (v =_Q w)]\!] \leqslant [\![u =_Q w]\!]$$

(2) *If u is a Q-real in V^{Q} which is a check set, then u is commutable with every Q-real in V^{Q}.*

Proof. (1)

$$\{[\![\check{r} \in u]\!] \mid r \in \mathbb{Q}\} \cup \{[\![\check{s} \in v]\!] \mid s \in \mathbb{Q}\} \cup \{[\![\check{t} \in w]\!] \mid t \in \mathbb{Q}\} \cup \{[\![u =_Q v]\!], [\![v =_Q w]\!]\}$$

is commutable. Hence (1) follows from

$$[\![\check{r} \in u]\!] \wedge [\![(u =_Q v) \wedge (v =_Q w)]\!] \leqslant [\![\check{r} \in w]\!], \quad \text{and}$$

$$[\![\check{r} \in w]\!] \wedge [\![(u =_Q v) \wedge (v =_Q w)]\!] \leqslant [\![\check{r} \in u]\!].$$

(2) follows from the fact that $0, 1 \in \mathcal{Q}$ are commutable with every element of $\mathcal{Q}$.

□

Definition 3.3. If u, v are commutable Q-reals in R^{Q}, then

$$u \leqslant_Q v \overset{\text{def}}{\iff} \forall r (r \in v \rightarrow_Q r \in u)$$

$$u <_Q v \overset{\text{def}}{\iff} (u \leqslant_Q v) \wedge (v \leqslant_Q u)^{\perp}$$

$$u + v \overset{\text{def}}{=} \{r \in \check{\mathbb{Q}} \mid \exists r_1, r_2 \in \check{\mathbb{Q}}((r = r_1 + r_2) \wedge (r_1 \in u) \wedge (r_2 \in v)\}$$

$$-u \overset{\text{def}}{=} \{r \in \check{\mathbb{Q}} \mid \exists s \in \check{\mathbb{Q}}((s < r) \wedge (-s \in u)^{\perp})\}$$

If A is a commutable set of Q-reals in R^{Q}, and $u, v \in A$, then $u+v$, $-u$ are Q-reals, and $A \cup \{u + v, -u\}$ is commutable. Hence we have the following propositions.

Proposition 3.3. *If u_1, u_2, v_1, v_2 are mutually commutable Q-reals in R^{Q}, then the following sentenses are valid in V^{Q}.*

(1) $u_1 =_Q u_2 \wedge v_1 =_Q v_2 \wedge u_1 \leqslant_Q v_1 \longrightarrow u_2 \leqslant_Q v_2$

(2) $u_1 =_Q u_2 \wedge v_1 =_Q v_2 \longrightarrow u_1 + v_1 =_Q u_2 + v_2$
(3) $u_1 =_Q u_2 \longrightarrow -u_1 =_Q -u_2$

Proposition 3.4. *If u, v, w are mutually commutable Q-reals in V^Q, then the following sentenses are valid in V^Q.*

(1) $u + v = v + u$
(2) $u + (v + w) = (u + v) + w$
(3) $u + \check{0} = u$
(4) $u + (-u) = \check{0}$
(5) $u \leqslant_Q \check{0} \leftrightarrow -u \geqslant_Q \check{0}$

3.2. Projections. For each element p of $\mathcal{Q}$, let $\hat{p}$ be the element of V^Q defined by

$$\mathcal{D}\hat{p} = \mathcal{D}\check{\mathbb{Q}}$$

$$\hat{p}(\check{r}) = \begin{cases} 0, & \text{if } r \leq 0 \\ p^{\perp}, & \text{if } 0 < r \leq 1 \\ 1, & \text{if } 1 < r \end{cases}$$

Then $\hat{p}$ is a Q-real in R^Q, and

$$[\![\hat{p} =_Q \check{1}]\!] = p, \quad [\![\hat{p} =_Q \check{0}]\!] = p^{\perp},$$

where $\check{1}$ and $\check{0}$ are identified with Q-reals defined by

$$\mathcal{D}\check{1} = \mathcal{D}\check{0} = \mathcal{D}\check{\mathbb{Q}}$$

$$\check{1}(\check{r}) = \begin{cases} 0, & \text{if } r \leq 1 \\ 1, & \text{if } 1 < r \end{cases}, \qquad \check{0}(\check{r}) = \begin{cases} 0, & \text{if } r \leq 0 \\ 1, & \text{if } 0 < r \end{cases}.$$

Proposition 3.5. *If a Q-real u satisfies*

$$[\![u =_Q \check{1}]\!] = [\![(u =_Q \check{0})^{\perp}]\!] = p,$$

then $[\![u = \hat{p}]\!] = 1$.

Proof. From the definition of $\hat{p}$. □

Definition 3.4. We say a Q-real u is a *projection on $p \in \mathcal{Q}$* if

$$[\![u =_Q \check{1}]\!] = [\![(u =_Q \check{0})^{\perp}]\!] = p.$$

Proposition 3.6. *Let* $p, q \in \mathcal{Q}$.

(1) $p \leq q \Longleftrightarrow [\![\hat{p} \leq_Q \hat{q}]\!] = 1$

(2) *If* p, q *are commutable and* $p \wedge q = 0$, *then* $[\![\hat{p} + \hat{q} =_Q (p \vee q)\hat{}\,]\!] = 1$

Proof. (1) is obvious.

(2) If $p \downarrow_\circ q$ and $p \wedge q = 0$, then $p \leqslant q^\perp$ and $q \leqslant p^\perp$. Hence,

$$\begin{aligned} p &= p \wedge q^\perp = [\![\hat{p} =_Q \check{1}]\!] \wedge [\![\hat{q} =_Q \check{0}]\!] \leqslant [\![\hat{p} + \hat{q} =_Q \check{1}]\!] \\ q &= q \wedge p^\perp = [\![\hat{q} =_Q \check{1}]\!] \wedge [\![\hat{p} =_Q \check{0}]\!] \leqslant [\![\hat{p} + \hat{q} =_Q \check{1}]\!] \end{aligned}$$

It follows that $p \vee q \leqslant [\![\hat{p} + \hat{q} =_Q \check{1}]\!]$.

$$(p \vee q)^\perp = p^\perp \wedge q^\perp \leqslant [\![\hat{p} =_Q \check{0}]\!] \wedge [\![\hat{q} =_Q \check{0}]\!] \leqslant [\![\hat{p} + \hat{q} =_Q \check{0}]\!].$$

□

3.3. Product on R^Q.

Definition 3.5. the product $u \cdot \hat{p}$ of a Q-real u and a projection $\hat{p}$ which is commutable with u is defined by

$$u \cdot \hat{p} \stackrel{\text{def}}{=} \{r \in \check{\mathbb{Q}} \mid (r \in u \wedge \hat{p} =_Q \check{1}) \vee (r > 0 \wedge \hat{p} =_Q \check{0})\}$$

We abbrevite the symbol $\cdot$ of product, if there is no possible confusion.

Proposition 3.7. $[\![\check{0} \cdot \hat{p} = \check{0}]\!] = [\![\check{1} \cdot \hat{p} = \hat{p}]\!] = 1$

Proof. Since $\check{r} \in \check{0} \cdot \hat{p} \rightarrow \check{r} \in \check{0}$ and $\check{r} \in \check{0} \rightarrow \check{r} \in \check{0} \cdot \hat{p}$ are valid, we have

$$[\![\check{0} \cdot \hat{p} = \check{0}]\!] = 1.$$

$[\![\check{1} \cdot \hat{p} = \hat{p}]\!] = 1$ is shown similarly. □

Proposition 3.8. *Let* u, v *be a Q-real in* R^Q, $p, q \in \mathcal{Q}$, *and let* $u, v, \hat{p}, \hat{q}$ *be mutually commutable. Then*

(1) $p \leqslant [\![u\hat{p} =_Q u]\!]$, $\quad p^\perp \leqslant [\![u\hat{p} =_Q \check{0}]\!]$

(2) $p \leqslant [\![u \leqslant_Q v]\!] \Longrightarrow [\![u\hat{p} \leqslant_Q v\hat{p}]\!] = 1$

(3) $[\![u(\hat{p}_1 + \hat{p}_2) =_Q u\hat{p}_1 + u\hat{p}_2]\!] = 1$

(4) $[\![\hat{p}\hat{q} =_Q \check{1}]\!] = p \wedge q$.

Proof. (1) follows from the fact that $\{\hat{p}, u, \check{0}, \check{1}\}$ is a commutable set of Q-reals, and

$$\begin{aligned} p \wedge [\![\check{r} \in u\hat{p}]\!] &= [\![\hat{p} =_Q \check{1} \wedge \check{r} \in u\hat{p}]\!] \leqslant [\![\check{r} \in u]\!], \\ p \wedge [\![\check{r} \in u]\!] &= [\![\hat{p} =_Q \check{1} \wedge \check{r} \in u]\!] \leqslant [\![\check{r} \in u\hat{p}]\!]. \end{aligned}$$

(2) By (1) and Proposition 3.3, we have

$$p \leqslant [\![u\hat{p} \leqslant_Q v\hat{p}]\!], \quad \textit{and} \quad p^{\perp} \leqslant [\![u\hat{p} =_Q v\hat{p} =_Q \check{0}]\!].$$

It follows that $p \leqslant [\![u \leqslant_Q v]\!] \Longrightarrow [\![u\hat{p} \leqslant_Q v\hat{p}]\!] = 1$.

(3)

$$\begin{aligned} [\![\check{r} \in u(\hat{p}_1 + \hat{p}_2)]\!] &= [\![\exists r_1, r_2 \in \check{\mathbb{Q}}((\check{r} = r_1 + r_2) \wedge (r_1 \in up_1) \wedge (r_2 \in up_2)]\!] \\ &= \bigvee_{\substack{r_1, r_2 \in \mathbb{Q} \\ r = r_1 + r_2}} [\![\Big((\check{r}_1 \in u) \wedge (\hat{p}_1 =_Q \check{1}) \wedge (\check{r}_2 > 0) \wedge (\hat{p}_2 =_Q \check{0})\Big) \vee \\ &\qquad \Big((\check{r}_2 \in u) \wedge (\hat{p}_2 =_Q \check{1}) \wedge (\check{r}_1 > 0) \wedge (\hat{p}_1 =_Q \check{0})\Big) \vee \\ &\qquad \Big((\check{r}_1 > 0) \wedge (\check{r}_2 > 0) \wedge (\hat{p}_1 =_Q \check{1}) \wedge (\hat{p}_2 =_Q \check{0})\Big)]\!] \\ &= [\![r \in u(\hat{p}_1 + \hat{p}_2)]\!] \end{aligned}$$

(4) follows from (1)

□

Definition 3.6. For commutable Q-reals u, v in $R^{\mathcal{Q}}$, let

$$[\![(u \geqslant_Q 0) \wedge (v \geqslant_Q 0)]\!] = p.$$

$(uv)\hat{p}$ is defined as

$$\{r \in \check{\mathbb{Q}} \mid \exists r_1, r_2 \Big((r = r_1 r_2) \wedge (r_1 \in u) \wedge (r_2 \in v) \wedge (\hat{p} =_Q \check{1})\Big) \vee \Big((r > 0) \wedge (\hat{p} =_Q \check{0})\Big)\}.$$

Proposition 3.9. *For commutable Q-reals u, v in $R^{\mathcal{Q}}$ such that*

$$[\![u \geqslant_Q 0 \wedge v \geqslant_Q 0]\!] = p,$$

we have

$$[\![uv\hat{p} =_Q vu\hat{p}]\!] = 1.$$

Definition 3.7. Let u, v be commutable Q-reals in $R^{\mathcal{Q}}$, and

$$p_1 = [\![0 \leqslant_Q u]\!], \quad p_2 = [\![u <_Q 0]\!], \quad p_3 = [\![0 \leqslant_Q v]\!], \quad p_4 = [\![v <_Q 0]\!].$$

Then p_1, p_2, p_3, p_4 are mutually orthogonal, and $p_1 \vee p_2 \vee p_3 \vee p_4 = 1$. Thus, the product uv is defined as follows.

$$uv \stackrel{\text{def}}{=} uv\hat{p}_1\hat{p}_3 + u(-v)\hat{p}_1\hat{p}_4 + (-u)v\hat{p}_2\hat{p}_3 + (-u)(-v)\hat{p}_2\hat{p}_4$$

Proposition 3.10. *Let u_1, u_2, v_1, v_2 be mutually commutable Q-reals in $R^{\mathcal{Q}}$. Then*

$$[\![(u_1 =_Q u_2) \wedge (v_1 =_Q v_2)]\!] \leqslant [\![u_1v_1 =_Q u_2v_2]\!]$$

Proof. Immediate from the definition. □

3.4. Representation of Q-reals.

Let u be a Q-real in $V^{\mathcal{Q}}$. Then

$$r, s \in \mathbb{Q}, \quad r \leqslant s \Longrightarrow [\![\check{r} \in u]\!] \leqslant [\![\check{s} \in u]\!].$$

Hence the set $M = \{[\![\check{r} \in u]\!] \mid r \in \mathbb{Q}\}$ is a linearly ordered subset of $\mathcal{Q}$, and commutable. Let B be a maximal commutable subset of $\mathcal{Q}$ including M. Then B is a complete Boolean subalgebra of $\mathcal{Q}$, and $u \in V^B \subset V^{\mathcal{Q}}$.

Proposition 3.11. *If $B = \langle B, \wedge, \vee, \rightarrow_Q, ^\perp, 1, 0\rangle$ is a complete Boolean subalgebra of $\mathcal{Q}$, then*

$$[\![v \textit{ is a Q-real }]\!]_B = [\![v \textit{ is a Q-real }]\!] \quad \textit{for each } v \in V^B,$$

where $[\![\quad]\!]_B$ means the truth value in V^B.

Each Q-real u is represented as a self-adjoint operator $\int \lambda dE(\lambda)$ on $\mathcal{H}$, where $E(\lambda) = \bigwedge_{r>\lambda} [\![\check{r} \in u]\!] \in B$ (cf. [2]).

Proposition 3.12. *Every Q-real u in $R^{\mathcal{Q}}$ is in some Boolean valued subuniverse V^B of $V^{\mathcal{Q}}$, and the set of Q-reals in V^B, $R^{\mathcal{Q}} \cap V^B$, form a commutative ring, and also $\check{\mathbb{R}}$-linear space.*

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