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Crispness and Representation Theorem in Dedekind Categories

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1 Introduction

Since Zadeh's invention the concept of fuzzy sets has been extensively investigated in mathematics, science and engineering. The notion of fuzzy relations is also a basic one in processing fuzzy information in relational structures, see e.g. Pedrycz [15]. Goguen [5] generalized the concepts of fuzzy sets and relations taking values from partially ordered sets. Fuzzy relational equations were initiated and applied to medical models of diagnosis by Sanchez [17].

On the other hand, the theory of relations, namely relational calculus, has a long history, see [13, 18, 19] for more details. Almost all modern formalizations of relation algebras are affected by the work of Tarski [20]. Mac Lane [12] and Puppe [16] exposed a categorical basis for the calculus of additive relations. Freyd and Scedrov [2] developed and summarized categorical relational calculus, which they called allegories. Concerning applications to the relational theory of graphs and programs, Schmidt and Ströhlein [18] gave a simple proof of a representation theorem for Boolean relation algebras satisfying the Tarski rule and the point axiom. They also wrote an excellent text book [19] on relations and graphs with many useful examples from computer science. In relational calculus one calculates with relations in an element-free style, which makes relational calculus a very useful framework for the study of mathematics [8] and theoretical computer science [1, 7, 11] and also a useful tool for applications. Some element-free formalizations of fuzzy relations and proofs of representation theorems were provided in [3, 9, 10].

In this paper we consider Dedekind categories named by Olivier and Serrato [14]. One of the aim of this paper is to study notions of crispness and scalar relations in Dedekind categories. A notion of crispness was introduced in [10] under the assumption that Dedekind categories have unit objects which are an abstraction of singleton (or one-point) sets. To capture the notion of crispness without such assumption, we use a notion of scalar relations. The notion of scalar relations in homogeneous relation algebras was introduced in [4]. The other aim of this paper is to prove a representation theorem for Dedekind categories. Such a theorem for Dedekind categories with a unit object satisfying strict point axiom was also proved in [10].
This paper is organized as follows:
In section 2 we first state the definition of complete Dedekind categories [14] as a categorical structure formed by L-relations [5] with sup-inf composition. Also we define a preoder among objects of Dedekind categories which compares the lattice structures on objects in a sense. Section 3 studies notions of scalars and crispness for Dedekind categories. The scalars on an object form a distributive lattice, which would be seen as the underlying lattice structure. In section 4 we recall the definition of L-relations, due to Goguen [5], and illustrate a few

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relationships between crispness and lattice structures of scalars. In section 5 we show a representation theorem for connected Dedekind categories satisfying the strict point axiom without the assumption of existence of unit objects, and it is proved that the representation function is a bijection preserving all operations of Dedekind categories.

2 Dedekind Categories

In this section we recall the fundamentals on relation categories, which we will call Dedekind categories following Olivier and Serrato [14].

Throughout this paper, a morphism α from an object X into an object Y in a Dedekind category (which will be defined below) will be denoted by a half arrow $\alpha : X \rightharpoondown Y$, and the composite of a morphism $\alpha : X \rightharpoondown Y$ followed by a morphism $\beta : Y \rightharpoondown Z$ will be written as $\alpha\beta : X \rightharpoondown Z$.

Definition 2.1 A Dedekind category $\mathcal{D}$ is a category satisfying the following:
D1. [Complete Distributive Lattice] For all pairs of objects X and Y the hom-set $\mathcal{D}(X,Y)$ consisting of all morphisms of X into Y is a complete distributive lattice with the least morphism 0_{XY} and the greatest morphism ∇_{XY}.
D2. [Involution] An involution $^\sharp : \mathcal{D} \to \mathcal{D}$ is a monotone contravariant functor. That is, for all morphisms $\alpha, \alpha' : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$,
(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$.
D3. [Dedekind Formula] For all morphisms $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [Residues] For all morphphsms $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the residue (or division, weakest precondition) $\gamma \div \beta : X \rightharpoondown Y$ is a morphism such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightharpoondown Y$. $\square$

Note that complete distributive lattices are equivalent to complete Brouwerian lattices or complete Heyting algebras.

Throughout this section, all discussions will assume a fixed complete Dedekind category $\mathcal{D}$. We denote the identity morphism on an object X of $\mathcal{D}$ by id_X. The greatest morphism ∇_{XY} is called the universal morphism and the least morphism 0_{XY} the zero morphism. A morphism is nonzero if it is not equal to the zero morphism. An object X is called empty if $\nabla_{XX} = 0_{XX}$, and nonempty if $\nabla_{XX} \neq 0_{XX}$.

Proposition 2.2 *Let $\alpha, \alpha' : X \rightharpoondown Y$ and $\beta, \beta' : Y \rightharpoondown Z$ be morphisms in $\mathcal{D}$.*

(a) $\nabla_{XX}\nabla_{XY} = \nabla_{XY}\nabla_{YY} = \nabla_{XY}$.

(b) *If $\alpha \sqcup \alpha' = \nabla_{XY}$, $\alpha \sqcap \alpha' = 0_{XY}$ and $\nabla_{XX}\alpha = \alpha$, then $\nabla_{XX}\alpha' = \alpha'$.*

(c) *If $u \sqsubseteq \mathrm{id}_X$ and $v \sqsubseteq \mathrm{id}_X$, then $u^\sharp = uu = u$ and $uv = u \sqcap v$.*

(d) *If $u \sqsubseteq \mathrm{id}_X$ and $v \sqsubseteq \mathrm{id}_Y$, then $u\alpha = \alpha \sqcap u\nabla_{XY}$ and $\alpha v = \alpha \sqcap \nabla_{XY}v$.*

The statement (a) in the last proposition indicates that if $\nabla_{XY} \neq 0_{XY}$, then both of X and Y are nonempty.

Proposition 2.3 *Let $\alpha : X \rightharpoondown Y$ be a morphism such that $\nabla_{XX}\alpha = \alpha$. Then the following three conditions are equivalent: (a) $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$, (b) $\nabla_{XX} = \alpha\alpha^\sharp$, (c) $\nabla_{XX} = \alpha\nabla_{YX}$.*

A binary relation $\prec$ among objects of $\mathcal{D}$ is defined as follows: For two objects X and Y a relation $X \prec Y$ holds if and only if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$. Then $\prec$ is a preorder, that is, reflexive and transitive. For $\nabla_{XX} = \nabla_{XX}\nabla_{XX}$, and if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ and $\nabla_{YY} = \nabla_{YZ}\nabla_{ZY}$, then $\nabla_{XX} = \nabla_{XY}\nabla_{YY}\nabla_{YX} = \nabla_{XY}\nabla_{YZ}\nabla_{ZY}\nabla_{YX} \sqsubseteq \nabla_{XZ}\nabla_{ZX}$. Hence its symmetric closure $X \sim Y$, which means $X \prec Y$ and $Y \prec X$, is an equivalence relation.

Proposition 2.4 *Assume that $X \prec Y$. If $u\nabla_{XY} \sqsubseteq v\nabla_{XY}$ for $u, v : X \rightharpoonup X$ such that $u \sqsubseteq \mathrm{id}_X$ and $u \sqsubseteq \mathrm{id}_X$, then $u \sqsubseteq v$.*

Definition 2.5 A Dedekind category $\mathcal{D}$ is connected if all pairs of objects of $\mathcal{D}$ are equivalent, that is, if $X \sim Y$ for all objects X and Y of $\mathcal{D}$.

3 Scalars and Crispness

We now introduce the two notions of scalars and s-crisp relations to define a concept of points with a separation property that two different points does not meet.

Definition 3.1 A scalar k on X is a morphism $k : X \rightharpoonup X$ of $\mathcal{D}$ such that $k \sqsubseteq \mathrm{id}_X$ and $k\nabla_{XX} = \nabla_{XX}k$.

A scalar k on X commutes with all morphisms $\alpha : X \rightharpoonup X$, that is, $k\alpha = \alpha k$, because

$$k\alpha = \alpha \sqcap k\nabla_{XX} = \alpha \sqcap \nabla_{XX}k = \alpha k.$$

It is trivial that the zero morphism $0_{XX} : X \rightharpoonup X$ and the identity morphism $\mathrm{id}_X : X \rightharpoonup X$ are scalars on X. The set of all scalars on X is denoted by $\mathcal{F}(X)$. It is clear that $\mathcal{F}(X)$ is a complete distributive lattice for all objects X.

Lemma 3.2 *For a morphism $\xi : X \rightharpoonup Y$ and an object W define a morphism*

$$\phi_{XYW}(\xi) = \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W : W \rightharpoonup W.$$

Then

(a) *$\phi_{XYW}(\xi)\nabla_{WZ} = \nabla_{WX}\xi\nabla_{YZ}$ and $\nabla_{ZW}\phi_{XYW}(\xi) = \nabla_{ZX}\xi\nabla_{YW}$ for each object Z,*

(b) *$\phi_{XYW}(\xi)$ is a scalar on W,*

(c) *$\phi_{XXW}\phi_{XYX}(\xi) = \phi_{YYW}\phi_{XYY}(\xi) = \phi_{XYW}(\xi)$,*

(d) *If $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then $\xi \sqsubseteq \nabla_{XW}\phi_{XYW}(\xi)\nabla_{WY}$,*

(e) *If $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, an identity $\phi_{XYW}(\xi) = 0_{WW}$ is equivalent to $\xi = 0_{XY}$.*

From the above Lemma 3.2(b) one have a function $\phi_{XYW} : \mathcal{D}(X, Y) \to \mathcal{F}(W)$. Note that if $W = X$ or $W = Y$, then $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$.

Proposition 3.3 (a) *If $X \prec Y$, then $\phi_{YYX}(\phi_{XXY}(k)) = k$ for all scalars $k \in \mathcal{F}(X)$,*

(b) *If $X \sim Y$, then $\mathcal{F}(X)$ is isomorphic to $\mathcal{F}(Y)$ as lattices.*

(c) *$\phi_{ZZX}(k)\alpha = \alpha\phi_{ZZY}(k)$ for all scalars k on Z and all morphisms $\alpha : X \rightharpoonup Y$.*

(d) *For every nonzero morphism* $\xi : X \rightharpoonup Y$ *in* $\mathcal{D}$ *there is a nonzero scalar* $k \in \mathcal{F}(X)$ *such that* $\nabla_{XX}\xi\nabla_{YY} = k\nabla_{XY}$.

Definition 3.4 A morphism $\alpha : X \rightharpoonup Y$ is s-crisp if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all nonzero scalars $k : X \rightharpoonup X$ and all morphisms $\tau : X \rightharpoonup Y$. □

It is trivial from the above definition that all universal morphism ∇_{XY} is s-crisp.

Proposition 3.5 *If the identity morphism* id_Y *is s-crisp, then so are all total functions* $f : X \to Y$.

Proof. Let $f : X \to Y$ be a total function. Assume that $k\tau \sqsubseteq f$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$. First note that $k\tau = \tau\phi_{XXY}(k)$ by 3.3(c). Then we have

$$\phi_{XXY}(k)\tau^\sharp f = (\tau\phi_{XXY}(k))^\sharp f = (k\tau)^\sharp f \sqsubseteq f^\sharp f \sqsubseteq \mathrm{id}_Y$$

and so $\tau^\sharp f \sqsubseteq \mathrm{id}_Y$ from the assumtion. Therefore $\tau^\sharp \sqsubseteq \tau^\sharp f f^\sharp \sqsubseteq f^\sharp$, which completes the proof. □

Lemma 3.6 *A morphism* $\alpha : X \rightharpoonup Y$ *is s-crisp if and only if a relatively pseudo-complemet* $\alpha' \Rightarrow \alpha$ *is s-crisp for all morphisms* $\alpha' : X \rightharpoonup Y$.

Proof. First assume that $\alpha : X \rightharpoonup Y$ is s-crisp and $k\tau \sqsubseteq \alpha' \Rightarrow \alpha$ for a nonzero scalar k and morphisms $\tau, \alpha' : X \rightharpoonup Y$. Then we have

$$k(\tau \sqcap \alpha') = k\tau \sqcap \alpha' \sqsubseteq \alpha$$

and so $\tau \sqcap \alpha' \sqsubseteq \alpha$, since $\alpha : X \rightharpoonup Y$ is s-crisp. Therefore $\tau \sqsubseteq \alpha' \Rightarrow \alpha$. Conversely if $\alpha' \Rightarrow \alpha$ is s-scrisp for all morphisms $\alpha' : X \rightharpoonup Y$, then $\alpha = \nabla_{XY} \Rightarrow \alpha$ is s-crisp. This completes the proof. □

Theorem 3.7 *The following three statements are equivalent:*

(a) *If* $k \neq 0_{XX}$ *and* $k \sqcap k' = 0_{XX}$ *for scalars* $k, k' \in \mathcal{F}(X)$, *then* $k' = 0_{XX}$,

(b) *The zero morphism* 0_{XY} *is s-crisp for all objects* Y, *(that is, if* $k\tau = 0_{XY}$ *for a nonzero scalar* k *on* X *and a morphism* $\tau : X \rightharpoonup Y$, *then* $\tau = 0_{XY}$*),*

(c) *For every morphism* $\alpha : X \rightharpoonup Y$ *its pseudo-complement* $\neg\alpha : X \rightharpoonup Y$ *is s-crisp for all objects* Y,

(d) *Every complemented morphism* $\alpha : X \rightharpoonup Y$ *is s-crisp for all objects* Y.

4 *L*-Relations

Let L be a complete distributive lattice (or, a complete Heyting algebra) with the least element 0 and the greatest element 1. The supremum (the least upper bound) and the infimum (the greatest lower bound) of a family $\{k_\lambda\}$ of elements in L will be denoted by $\vee_\lambda k_\lambda$ and $\wedge_\lambda k_\lambda$, respectively. For two elements $a, b \in L$ the relative pseudo-complement of a relative to b will be written as $a \Rightarrow b$. Now recall some fundamentals on L-relations [5].

Let X and Y be sets. An L-relation R from X into Y, written $R : X \rightharpoonup Y$, is a function $R : X \times Y \to L$. The set of all L-relations from X into Y will be denoted by $L - Rel(X)$. An L-relation R is contained in an L-relation S, written $R \subseteq S$, if $R(x,y) \leq S(x,y)$ for all $(x,y) \in X \times Y$. The zero relation O_{XY} and the universal relation ∇_{XY} are L-relations with $O_{XY}(x,y) = 0$ and $\nabla_{XY}(x,y) = 1$ for all $(x,y) \in X \times Y$, respectively. It is trivial that $\subseteq$ is a partial order, and $O_{XY} \subseteq R \subseteq \nabla_{XY}$ for all fuzzy relations R. For a family $\{R_\lambda\}_\lambda$ of fuzzy relations we define fuzzy relations $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$ as follows:

$$(\cup_\lambda R_\lambda)(x,y) = \vee_\lambda R_\lambda(x,y)$$

and

$$(\cap_\lambda R_\lambda)(x,y) = \wedge_\lambda R_\lambda(x,y)$$

for all $x, y \in X$. It is obvious that $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$ are the least upper bound and the greatest lower bound of a family $\{R_\lambda\}_\lambda$, respectively, with respect to the order $\subseteq$. The composite $RS(= R;S) : X \rightharpoonup Z$ of an L-relation $R : X \rightharpoonup Y$ followed by an L-relation $S : Y \rightharpoonup Z$ is defined by

$$(RS)(x,z) = \vee_{y \in Y}[R(x,y) \wedge S(y,z)]$$

for all $(x,z) \in X \times Z$. This composition of L-relations is called as sup-inf composition. The associativity $(RS)T = R(ST)$ holds for all L-relations R, S and T. The identity relation id_X of a set X is an L-relation such that $\mathrm{id}_X(x,x') = 1$ if $x = x'$ and $\mathrm{id}_X(x,x') = 0$ otherwise. The unitary law $\mathrm{id}_X R = R\,\mathrm{id}_Y = R$ holds for all $R : X \rightharpoonup Y$. The inverse (or transpose) $R^\sharp : Y \rightharpoonup X$ of an L-relation $R : X \rightharpoonup Y$ is defined by

$$R^\sharp(y,x) = R(x,y)$$

for all $(y,x) \in Y \times X$. For L-relations $S : Y \rightharpoonup Z$ and $T : X \rightharpoonup Z$ the residue $T \div S : X \rightharpoonup Y$ is defined by

$$(T \div S)(x,y) = \wedge_{z \in Z}[S(y,z) \Rightarrow T(x,z)]$$

for all $(x,y) \in X \times Y$. The readers can easily see that L-relations and their operations defined above satisfy almost all axioms of Dedekind categories, except for D3(Dedekind formula) and D4(Residues), which will be proved in the following:

Proposition 4.1 *Let $R : X \rightharpoonup Y, S : Y \rightharpoonup Z$ and $T : X \rightharpoonup Z$ be L-relations. Then*

(a) *$RS \cap T \subseteq R(S \cap R^\sharp T)$ (Dedekind formula),*

(b) *$RS \subseteq T$ if and only if $R \subseteq T \div S$.*

In relational calculus ([2, 8, 19]) a function R on X is a relation satisfying the univalency $R^\sharp R \subseteq I$ and the totality $I \subseteq RR^\sharp$.

An L-relation $k : X \rightharpoonup X$ is a scalar on X if and only if

$$\forall x, x' \in X : k(x,x) = k(x',x') \text{ and } x \neq x' \Rightarrow k(x,x') = 0.$$

An L-relation $R : X \rightharpoonup Y$ is 0-1 crisp ([5]) if $R(x,y) = 0$ or $R(x,y) = 1$ for all $(x,y) \in X \times Y$. Of course O_{XY}, ∇_{XY} and id_X are 0-1 crisp. For a 0-1 crisp L-relation $R : X \rightharpoonup Y$ define an L-relation $\bar{R} : X \rightharpoonup Y$ by $\bar{R}(x,y) = 0$ if $R(x,y) = 1$ and $\bar{R}(x,y) = 1$ otherwise. Then $R \cup \bar{R} = \nabla_{XY}$ and $R \cap \bar{R} = O_{XY}$. This fact means that all 0-1 crisp L-relations are complemented.

Proposition 4.2 *All s-crisp L-relations are 0-1 crisp.*

Proposition 4.3 *For L-relations the following statements are equivalent:*

C0. $\forall a, b \in L : a \wedge b = 0 \Rightarrow a = 0$ *or* $b = 0$.

K0. All 0-1 crisp L-relations are s-crisp.

Proposition 4.4 *For L-relations the following statements are equivalent:*

C1. $\forall a, b \in L : a \wedge b = 0$ *and* $a \vee b = 1 \Rightarrow a = 0$ *or* $b = 0$.

K1. All complemented L-relations are 0-1 crisp.

K2. All totally functional L-relations are 0-1 crisp.

5 Representation Theorem

Definition 5.1 Let $\mathcal{D}$ be a complete Dedekind category. A point x of X is an s-crisp morphism $x : X \rightharpoonup X$ such that $\nabla_{XX}x = x$, $x^\sharp x \sqsubseteq \mathrm{id}_X$ and $\mathrm{id}_X \sqsubseteq xx^\sharp$. □

Proposition 5.2 *Let x and x' be points of X. Then*

(a) *If $\nabla_{XX}\rho = \rho$ and $\rho \sqsubseteq x$ for a morphism $\rho : X \rightharpoonup X$, then $\rho = kx$ for a unique scalar k on X.*

(b) *If $x \neq x'$, then $x \sqcap x' = 0_{XX}$ and $xx'^\sharp = 0_{XX}$.*

Set $L = \mathcal{F}(W)$ for a fixed object W. Then L is a complete disributive lattice. A function $\chi(\alpha) : \chi(X) \times \chi(Y) \rightarrow L$ assigning $\chi(\alpha)(x, y) = \phi_{XYW}(x\alpha y^\sharp) \in L$ to a pair (x, y) of points x of X and y of Y, gives an L-relation of $\chi(X)$ into $\chi(Y)$. Thus we have a function $\chi : \mathcal{D}(X, Y) \rightarrow L\text{-}Rel(\chi(X), \chi(Y))$.

Proposition 5.3 *If $\mathcal{D}$ is a connected Dedekind category, then the function $\chi : \mathcal{D}(X,Y) \rightarrow L$-$Rel(\chi(X), \chi(Y))$ satisfies the following properties:*

(a) $\chi(O_{XY}) = O_{\chi(X)\chi(Y)}$, $\chi(\nabla_{XY}) = \nabla_{\chi(X)\chi(Y)}$ *and* $\chi(\mathrm{id}_X) = \mathrm{id}_{\chi(X)}$,

(b) $\chi(\alpha \sqcup \alpha') = \chi(\alpha) \cup \chi(\alpha')$ *and* $\chi(\alpha \sqcap \alpha') = \chi(\alpha) \cap \chi(\alpha')$,

(c) $\chi(\alpha^\sharp) = \chi(\alpha)^\sharp$,

(d) $\chi(\alpha)\chi(\beta) = \chi(\alpha(\sqcup_{y \in \chi(Y)} y^\sharp y)\beta)$.

(e) *The function $\chi : \mathcal{D}(X, Y) \rightarrow L - Rel(\chi(X), \chi(Y))$ is surjective.*

Definition 5.4 A complete Dedekind category $\mathcal{D}$ satisfies the strict point axiom if and only if

$$\sqcup_{x \in \chi(X)} x = \nabla_{XX}$$

for all objects X, where $\chi(X)$ denotes the set of all points of X. □

Proposition 5.5 *A complete Dedekind category $\mathcal{D}$ satisfies the strict point axiom if and only if the function $\chi : \mathcal{D}(X, X) \rightarrow L$-$Rel(\chi(X), \chi(X))$ is injective for all objects X.*

Proposition 5.6 *If a complete Dedekind category $\mathcal{D}$ satisfies the strict point axiom, then for all objects X the identity morphism id_X is complemented. Moreover, if the condition C1 is in addition valid in $\mathcal{D}$, then id_X is s-crisp for all objects X.*

Theorem 5.7 *(Representation Theorem) Assume that $\mathcal{D}$ satisfies the strict point axiom. Then every morphism $\alpha : X \rightharpoonup Y$ has a unique representation*

$$\alpha = \sqcup_{x \in \chi(X)} \sqcup_{y \in \chi(Y)} \chi_X(\alpha)(x, y)x^\sharp \nabla_{XY} y.$$

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