数理解析研究所講究録
1022巻 1997年 113-116

Remarks on algebraic convergence of discrete Möbius groups

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1 Theorems of algebraic convergence of discrete groups

In 1982, T. Jørgensen and P. Klein proved the following result on algebraic convergence of a sequence of non-elementary finitely generated Kleinian groups.

THEOREM 1. (Jørgensen-Klein [3]) *Let* $\{G_m\}$ *be a sequence of non-elementary* $r-$ *generator Kleinian groups converging algebraically to the group* G. *Then* G *is also a non-elementary Kleinian group and the correspondence from the generators of* G *to their approximants in* G_m *extends for all sufficiently large* $m \in \mathrm{N}$ *to a homomorphism of* G *onto* G_m.

Theorem 1 is an extension of the preceding theorem of the first author ([2]) in 1976. Main tool to establish these two theorems is the followig propositon which is known as Jørgensen's inequality.

PROPOSITION 2. (Jørgensen's inequality [2]) *Let* f *and* g *be two linear fractional transformations which generate a non-elementary discrete group. Then the following inequality holds*

$$|\, tr[f,g]-2 \,| + |\, tr^2(f)-4 \,| \;\geq\; 1.$$

Attempts to extend Jørgensen's inequality to all dimensions were made in several manners. (For example see [1] and [4].) In 1989, G.J. Martin showed a theorem on algebraic convergence of a sequence of non-elementary finitely generated discrete Möbius groups in several dimensions by use of his generalization of Jørgensen's inequality. In the case of several dimensions, the uniform bound of the order of elliptic cyclic groups in a sequence of Möbius groups plays an important role .

THEOREM 3. (Martin [3]) *Let G be the algebraic limit of a sequence $\{G_m\}$ of non-elementary $r-$ generator discrete subgroups of $M(B^n)$ of uniformly bounded torsion. Then G is a non-elementary discrete group.*

In this note, we clarify the difference between two convergence theorems (Theorem 1 and Theorem 2) by constructing some examples.

2 Examples

We need some notations and definitions. The unit ball $B^n (n = 2, 3, 4, \cdots)$ in R^n with the Poincaré metric is a model of the $n-$ dimensional hyperbolic space. Let $M(B^n)$ be a subgroup of the general Möbius group $M(\overline{R^n})$ which keeps B^n invariant. For $f, g \in M(B^n)$ we set

$$D(f,g) = sup\{\mid f(x) - g(x) \mid \mid \ x \in S^{n-1} = \partial B^n\}$$

and regard $M(B^n)$ as a metric space. We say that a subgroup G of $M(B^n)$ is a non-elementary group if G contains two elements of infinite order with distinct fixed points.

Let $\{G_m\}$ be a sequence of subgroups of $M(B^n)$ each with same finite number of generators $\{g_{m,1}, g_{m,2}, \cdots, g_{m,r}\}$ for $m = 1, 2, \cdots$. If we have $D(g_{m,i}, g_i) \to 0$ as $m \to \infty$ and $g_i \in M(B^n)$ for $i = 1, 2, \cdots$, then we say that the sequence of groups $\{G_m\}$ converges algebraically to the limit group $G =< g_1, g_2, \cdots, g_r >$. For any Möbius transformation g, we denote the order of g by $ord(g)$. Let $\{G_i\}_{i \in I}$ be a family of groups. We say that $\{G_i\}_{i \in I}$ has uniformly bounded torsion if there is an integer m_0 with the following properties : if $g \in G_i$ for some i, then $ord(g) = \infty$ or $ord(g) \leq m_0$. It is important to note the order of elliptic elements of a sequence of subgroups of $M(B^n)$.

EXAMPLE 1. For $n \geq 4$ we construct a sequence $\{G_m\}$ of non-elementary discrete subgroups of $M(B^n)$ which converges algebraically to a non-discrete subgroup. With no loss of generarities, we may assume $n = 4$. Let $G_0 =< g_1, g_2, \cdots, g_r > \subset M(B^2)$ be a purely hyperbolic non-elementary Fuchsian group and representing a compact Riemann surface, that is a surface group. The group G_0 acts on B^2 which is embeded in B^4 by the map $(x, y) \mapsto (x, y, 0, 0)$. The action of each $g \in G_0$ extends uniquely to B^4 by requiring that the extension is hyperbolic. In this way G_0 becomes a non-elementary finitely generated discrete subgroup of $M(B^4)$. Let

$$h_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_m & -\sin\theta_m \\ 0 & 0 & \sin\theta_m & \cos\theta_m \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix},$$

where $2\pi/\theta_m$ is rational $(m = 1, 2, \cdots), 2\pi/\theta$ is irrational and $\theta_m \to \theta$ as $m \to \infty$. We set

$G_m = < G_0, h_m >$ and $G = < G_0, h >$. Since every hyperbolic element has no rotation part and $h_m (m = 1, 2, \cdots)$ fixes every point of $B^2 \hookrightarrow B^4$, h_m commutes to each $g \in G_0$. We can easily see that $G_m (m = 1, 2, \cdots)$ and G are non-elementary groups. Since G contains an elliptic element h of infinite order, G is not discrete.

Now we show that $G_m (m = 1, 2, \cdots)$ is discrete. It is well known that the following three statesments are equivalent to each other : (i) G_m is a discrete group. (ii) G_m acts discontinuously on B^4. (iii) G_m is discontinuous at some point of B^4. So it suffices to show that G_m is discontinuous at the origin. Let B be an open ball centered at the origin whose radius is sufficiently small. Denote by $b = B \bigcap B^2$. Since $G_m \mid_{B^2} = G_0$ acts discontinuously on B^2 as a surface group, $\{g \in G_0 \mid g(b) \bigcap b \neq \emptyset\}$ is trivial. Recall that $h_m (m = 1, 2, \cdots)$ commutes to any $g \in G_0$. So any element $g \in G_m$ is wrtten in the form $g = \tilde{g} \circ (h_m)^k$(for some $\tilde{g} \in G_0$ and $k \in \mathbf{Z}$). Let g_0 be an element of G_m such that $g_0(B) \bigcap B \neq \emptyset$. Then $g_0(b) \bigcap b \neq \emptyset$ and we obtain $g_0 = (h_m)^j$ for some $j \in \mathbf{Z}$. Since h_m is elliptic of finite order, we conclude the subgroup $\{g \in G_m \mid g(B) \bigcap B \neq \emptyset\}$ of G_m is finite for $m = 1, 2, \cdots$. Therefore G_m is discontinuous at the origin. Here we obtain that $\{G_m\}$ is a sequence of non-elementary finitely generated discrete groups converging algebraically to a non-elementary non-discrete group G. Since $ord(h_m) \to \infty$ as $m \to \infty$, the sequence $\{G_m\}$ has not uniformly bounded torsion. So Theorem 1 cannot be extended directly to several dimensional case .

Now we consider the three dimensional case. Let G_0, θ_m, θ be same as those in the four dimensional case. We embed B^2 in B^3 by the map $(x, y) \mapsto (x, y, 0)$. We set

$$h_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_m & -\sin\theta_m \\ 0 & \sin\theta_m & \cos\theta_m \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and $G_m = < G_0, h_m > (m = 1, 2, \cdots), G = < G_0, h >$. For any m there exists a hyperbolic element $g \in G_m$, so that g and $h_m g h_m^{-1}$ have distinct fixed points. So G_m is non-elementary for $m = 1, 2, \cdots$. We can easily see that for arbitrary small $\varepsilon > 0$ there exist an integer m_0 and $f_m \in < h_m >$ such that $D(f_m, Id) < \varepsilon$ for every $m \geq m_0$. So we can deduce that there exist $\tilde{f}_m \in < h_m >$ and a hyperbolic element $g_m \in G_m$ so that

$$\mid tr[\tilde{f}_m, g_m] - 2 \mid + \mid tr^2(\tilde{f}_m) - 4 \mid \ < \ 1$$

for any sufficiently large integer m. Note that hyperbolic elements $g_m, \tilde{f}_m g_m \tilde{f}_m^{-1}$ are contained in $< \tilde{f}_m, g_m >$ and have distinct fixed points. Hence $< \tilde{f}_m, g_m >$ is a non-elementary group. So Jørgensen's inequality yields that $< \tilde{f}_m, g_m >$ is non-discrete for any sufficiently large m and so is G_m.

Another point to the above example, we can arrange that the elliptic elements converges to the identity.

EXAMPLE 2. In the first place we consider the four dimensional (several dimensional) case. Let G_0 be a surface group and

$$h_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\pi/m) & -\sin(\pi/m) \\ 0 & 0 & \sin(\pi/m) & \cos(\pi/m) \end{pmatrix},$$

and $h = E_4$, the four dimensional unit matrix. We set $G_m =< G_0, h_m > (m = 1, 2, \cdots)$ and $G =< G_0, h >= G_0$. We can conclude that $G_m (m = 1, 2, \cdots), G$ are non-elementary discrete groups and G_m converges algebraically to G. Obviously we can see that $\{G_m\}$ has not uniformly bounded torsion. In this case however the correspondence from generators of G to G_m cannot be extended to a homomorphism of G onto G_m for any m.

In the case $n = 3$, a sequence of non-elementary groups $\{G_m\}$ converges to a non-elementary discrete group G. But for any sufficiently large m, G_m is not discrete.

References

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