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Geometric superrigidity of (4n+3)-manifolds with quaternionic hyperbolic fundamental groups

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Abstract. Kevin Corlette has shown that *Marugulis' superrigidity* is true for the cases of quaternionic hyperbolic spaces at least two and hyperbolic Cayley plane. We apply this result to prove a *geometric rigidity* for compact $(4n+3)$-dimensional *pseudo-quaternionic flat manifolds* $(n > 1)$. A pseudo-quaternionic flat structure is a geometric structure on a $(4n+3)$-manifold locally modelled on the geometry $(\mathrm{PSp}(n+1,1), S^{4n+3})$. Here $\mathrm{PSp}(n+1,1)$ is the isometry group of the quaternionic hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{H}}$. The space $\mathbb{H}^{n+1}_{\mathbb{H}}$ has a (projective) compactification whose boundary is the sphere S^{4n+3} on which the group $\mathrm{PSp}(n+1,1)$ acts as projective transformations. The pair $(\mathrm{PSp}(n+1,1), S^{4n+3})$ is said to be *pseudo-quaternionic flat geometry.*

Introduction

Margulis has shown that: *Let G be a connected semisimple Lie group with trivial center and has no compact factor. Given an irreducible lattice Γ of G and a homomorphism $\rho : \Gamma \longrightarrow G'$ where G' is a semisimple Lie group with trivial center and without compact factor, ρ extends to a homomorphism from G to G' provided that the real rank of G is least two and $\rho(\Gamma)$ is Zariski dense.* Note that a connected semisimple Lie group with trivial center supports a real algebraic structure. (Compare [18].) This result is called the *Margulis' superrigidity* and the question is left to the rank one semisimple Lie groups, namely the real (resp. complex, quaternionic, Cayley) hyperbolic groups. It is known that the Margulis' superrigidity is false for the real hyperbolic case, for instance, because of the existence of *bending* (= a nontrivial deformation of Fuchsian groups in higher dimensions). On the other hand, Kevin Corlette [3] has proved affirmatively for the cases of quaternionic hyperbolic group $\mathrm{PSp}(n,1)$ $(n \geq 2)$ and the isometry group F_4^{-20}

of the hyperbolic Cayley plane. For our later use, we quote a part of his result. (Compare [12].)

Theorem 1. *Let Γ be a lattice in $\mathrm{PSp}(n,1)$ and G any semisimple Lie group with trivial center and without compact factor. If $\rho : \Gamma \longrightarrow G$ is a homomorphism with Zariski dense image, then ρ extends to a homomorphism $\hat{\rho} : \mathrm{PSp}(n,1) \longrightarrow G$.*

Recall that $\mathrm{Sp}(n+1,1) = \{A \in M(n+2,\mathbb{F}) \mid A^* \mathrm{I}_{1,n+1} A = \mathrm{I}_{1,n+1}\}$ where $\mathbb{F}$ stands for the noncommutative field of quaternions. The center of this group is $\mathbb{Z}/2$ and the quaternionic hyperbolic group $\mathrm{PSp}(n+1,1)$ is the quotient of $\mathrm{Sp}(n+1,1)$ by the center. Then the hyperbolic action of $\mathrm{PSp}(n+1,1)$ on the quaternionic hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{F}}$ extends to a smooth action on the boundary sphere S^{4n+3} of $\mathbb{H}^{n+1}_{\mathbb{F}}$ acting as projective transformations because the compactification $\mathbb{H}^{n+1}_{\mathbb{F}} \cup S^{4n+3}$ sits in the quaternionic projective space $\mathbb{FP}^{n+1}$. Since the action of $\mathrm{PSp}(n+1,1)$ is transitive on S^{4n+3} whose stabilizer at infinity ∞ is isomorphic to the group $\mathrm{Sim}(\mathcal{M})$ of similarity transformations of the $(4n+3)$-dimensional Heisenberg nilpotent Lie group $\mathcal{M}$. Thus we obtain a geometry $(\mathrm{PSp}(n+1,1), S^{4n+3})$ called the *pseudo-quaternionic flat geometry*. Similarly notice that according as the real, complex cases, there correspond the conformally flat geometry $(\mathrm{PO}(n,1), S^n)$, spherical CR-geometry $(\mathrm{PU}(n+1,1), S^{2n+1})$.

Denote by $\mathrm{GL}(n+1,\mathbb{F})$ the group of invertible $n \times n$-matrices with quaternion entries of the quaternion number space $\mathbb{F}^n$ acting on the left and $\mathbb{F}^* = \mathrm{GL}(1,\mathbb{F})$ acting as the scalar multiplications of the vector space $\mathbb{F}^n$ from the right. The vector space $\mathbb{F}^{n+1}$ is endowed with the Hermitian pairing over $\mathbb{F}$:

$$b(z,w) = -\bar{z}_0 w_0 + \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n.$$

By the definition, $\mathrm{Sp}(n,1)$ is the subgroup of $\mathrm{GL}(n+1,\mathbb{F})$ whose elements preserve the Hermitian pairing b. Consider the quadric

$$V^{4n+3}_{-1} = \{z \in \mathbb{F}^{n+1} - \{0\} \mid b(z,z) = -1\},$$

which is left invariant under $\mathrm{Sp}(n+1,1)$. We have the following equivariant principal bundles over the quaternionic projective space $\mathbb{FP}^n$.

$$\begin{array}{ccccc}
(\mathbb{R}^*, \mathbb{F}^*) & \longrightarrow & (\mathrm{GL}(n+1,\mathbb{F}) \cdot \mathbb{F}^*, \mathbb{F}^{n+1} - \{0\}) & \xrightarrow{P} & (\mathrm{P\,GL}(n+1,\mathbb{F}), \mathbb{FP}^n) \\
\uparrow & & \uparrow & & \uparrow \\
(\mathbb{Z}/2, \mathrm{Sp}(1)) & \longrightarrow & (\mathrm{Sp}(n,1) \cdot \mathrm{Sp}(1), V^{4n+3}_{-1}) & \xrightarrow{P} & (\mathrm{PSp}(n,1), \mathbb{H}^n_{\mathbb{F}}).
\end{array}$$

Let Γ be a torsionfree discrete uniform subgroup of $\mathrm{Sp}(n,1) \cdot \mathrm{Sp}(1)$. From the above sequence, P maps isomorphically onto a torsionfree discrete uniform subgroup $\hat{\Gamma}$ of $\mathrm{PSp}(n,1)$. Since $\mathrm{Sp}(1)$ is compact, Γ acts properly discontinuously and freely

on V_{-1}^{4n+3}. Moreover, there is a principal bundle over the compact quaternionic hyperbolic manifold

$$\mathrm{Sp}(1) \to V_{-1}^{4n+3}/\Gamma \longrightarrow \mathbb{H}_{\mathbb{F}}^n/\hat{\Gamma}.$$

Put $M_0 = V_{-1}^{4n+3}/\Gamma$. Then it is known that M_0 is a compact (geodesically) complete semi-Riemannian manifold of type $(3, 4n)$ with constant curvature -1 (cf. [15]). Then it is shown that M_0 admits a canonical pseudo-quaternionic flat structure whose developing image is the sphere complement $S^{4n+3} - S^{4n-1}$. (Compare §1.) We prove the following rigidity.

Theorem 2. *Let M be a compact pseudo-quaternionic flat $(4n+3)$-manifold. Suppose that the fundamental group $\pi_1(M)$ is isomorphic to that of a compact quaternionic hyperbolic $4n$-manifold. Then M is pseudo-quaternionically isomorphic to $M_0 = V_{-1}^{4n+3}/\Gamma$.*

Let $\mathcal{T}(M_0)$ be the deformation space of $(\mathrm{PSp}(n+1,1), S^{4n+3})$-structures (*i.e.*, pseudo-quaternionic flat structures) on marked manifolds homeomorphic to M_0. There is the natural map hol : $\mathcal{T}(M_0) \to \mathrm{Hom}(\Gamma, \mathrm{PSp}(n+1,1))/\mathrm{PSp}(n+1,1)$ which assigns to a marked structure its holonomy representation.

Theorem 3. *The map hol maps $\mathcal{T}(M_0)$ homeomorphically onto a connected component in* $\mathrm{Hom}(\Gamma, \mathrm{PSp}(n+1,1))/\mathrm{PSp}(n+1,1)$. *Moreover, the connected component is diffeomorphic to* $\mathrm{Hom}(\Gamma, \mathrm{Sp}(1))/\mathrm{Sp}(1)$.

1. Examples

We give examples of compact pseudo-quaternionic flat manifolds. Let S^{4n+3} be the sphere with the pseudo-quaternionic flat structure with the automorphism group $\mathrm{PSp}(n+1,1)$ The sphere with one point removed $S^{4n+3} - \{\infty\}$ is identified pseudo-quaternionically with the Heisenberg nilpotent Lie group $\mathcal{M}$ with the automorphism group $\mathrm{Sim}(\mathcal{M}) = \mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+)$. $\mathcal{M}$ lies in the central extension $1 \to \mathbb{R}^3 \to \mathcal{M} \to \mathbb{F}^n \to 1$ where $\mathbb{R}^3 = \{z \in \mathbb{F} | \mathrm{Re}(z) = 0\}$. (See §4.) Choosing a torsionfree discrete cocompact subgroup Γ from $\mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1))$, we have a principal fibration of an infranilmanifold as a compact pseudo-quaternionic flat manifold;

$$(i) \qquad T^3 \to \mathcal{M}/\Gamma \longrightarrow \mathbb{F}^n/\Gamma^*$$

where T^3 is the 3-torus and $\mathbb{F}^n/\Gamma^*$ is the quaternionic euclidean flat orbifold.

Let $\mathcal{M} - \{\infty\}(= S^{4n+3} - \{0,\infty\}) \approx \mathbb{R}^+ \times S^{4n+2}$. Choosing a torsion free discrete cocompact subgroup Δ of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+$ we obtain an infra-Hopf manifold

$$(ii) \qquad \mathbb{R}^+ \times S^{2n}/\Delta \approx S^1 \times S^{4n+2}/G$$

where G is a finite group. In particular, the Hopf manifold $S^{4n+2}\times S^1$ is a pseudo-quaternionic flat manifold.

Let $S^{4n+3} - S^{4m-1}$ be the sphere complement. Then $\mathrm{Aut}_{\mathrm{PSp}}(S^{4n+3} - S^{4m-1})$ is isomorphic to $P(\mathrm{Sp}(m,1)\times\mathrm{Sp}(n-m+1)) = \mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)$ by chasing the equivariant principal bundle:

$$\begin{array}{ccc} & \mathrm{Sp}(1) & \\ & \downarrow & \\ V_{-1}^{4m+3}\times S^{4(n-m)+3} & & \\ \downarrow P & \searrow & \\ P(V_{-1}^{4m+3}\times S^{4(n-m)+3}) & \xrightarrow{\approx} & S^{4n+3} - S^{4m-1} \subset \mathbb{FP}^{n+1}, \end{array}$$

where the automorphism group has the following group extension corresponding to the vertical sequence:

$$\begin{array}{ccccc} \mathrm{Sp}(1) & \longrightarrow & T & \xrightarrow{P} & P(\mathrm{Sp}(m,1)\times\mathrm{Sp}(n-m+1)) \\ \uparrow & & \uparrow & & \| \\ \mathbb{Z}/2 & \longrightarrow & S & \xrightarrow{P} & \mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1), \end{array}$$

where $T = (\mathrm{Sp}(m,1)\times\mathrm{Sp}(n-m+1))\cdot\mathrm{Sp}(1)$, and $S = \mathrm{Sp}(m,1)\times\mathrm{Sp}(n-m+1)$. Then $S^{4(n-m)+3}\to(V_{-1}^{4m+3}\times S^{4(n-m)+3})/\mathrm{Sp}(1)\longrightarrow V_{-1}^{4m+3}/\mathrm{Sp}(1) = \mathbb{H}_{\mathbb{F}}^m$ is the fiber bundle over the quaternionic hyperbolic space $\mathbb{H}_{\mathbb{F}}^m$. Since this bundle is topologically product, letting $\mathrm{Sp}(1) = P(\mathbb{Z}/2\times\mathrm{Sp}(1)) \subset \mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)$, the quotient $\mathrm{Sp}(1)\backslash S^{4n+3} - S^{4m-1}$ is homeomorphic to the product $\mathbb{H}_{\mathbb{F}}^m\times\mathbb{FP}^{n-m}$ which is thus the base space of the following equivariant principal bundle:

$$\mathrm{Sp}(1)\to(\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(1), S^{4n+3} - S^{4m-1})\longrightarrow(\mathrm{PSp}(m,1), \mathbb{H}_{\mathbb{F}}^m\times\mathbb{FP}^{n-m})$$

where $\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(1)$ is a subgroup of $\mathrm{Aut}_{\mathrm{PSp}}(S^{4n+3} - S^{4m-1})$.
Let Γ be a torsionfree discrete uniform subgroup of $\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(1)$ which commutes with $\mathrm{Sp}(1) = P(\mathbb{Z}/2\times\mathrm{Sp}(1))$. Then we have a principal bundle of a compact pseudo-quaternionic flat manifold;

$$(iii) \qquad \mathrm{Sp}(1)\to S^{4n+3} - S^{4m-1}/\Gamma\longrightarrow\mathbb{H}_{\mathbb{F}}^m\times\mathbb{FP}^{n-m}/\hat{\Gamma}$$

where $\hat{\Gamma} \subset \mathrm{PSp}(m,1)$. In particular when $m = n$,

$$\begin{aligned}(\mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1), V_{-1}^{4n+3}) &= (P(\mathrm{Sp}(n,1)\times\mathrm{Sp}(1)), V_{-1}^{4n+3}\times S^3/\mathrm{Sp}(1)) \\ &= (\mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1), S^{4n+3} - S^{4n-1}).\end{aligned}$$

Proposition 4. *$M_0 = V_{-1}^{4n+3}/\Gamma$ is a pseudo-quaternionic flat manifold which is a fibration over the quaternionic hyperbolic space form:* $\mathrm{Sp}(1)\to M_0 \longrightarrow \mathbb{H}^n_{\mathbf{F}}/\hat{\Gamma}$.

Similar to the conformally flat, spherical CR case (cf. [1],[14]), we can show that

Proposition 5. *Let M_1, M_2 be a (compact) pseudo-quaternionic flat manifold. Then, the connected sum $M_1\#M_2$ also admits a pseudo-quaternionic flat structure.*

2. Deformation space

Recall that a geometric structure on a smooth n-manifold is a maximal collection of charts modeled on a simply connected n-dimensional homogeneous space X of a Lie group $\mathcal{G}$ whose coordinate changes are restrictions of transformations from $\mathcal{G}$. We call such a structure a $(\mathcal{G}, X)$-structure. In particular, a $(\mathrm{PSp}(n+1,1), S^{4n+3})$-structure is said to be the pseudo-quaternionic flat structure as before. A manifold equipped with a $(\mathcal{G}, X)$-structure is called a $(\mathcal{G}, X)$-manifold.
Suppose that a smooth connected n-manifold M admits a $(\mathcal{G}, X)$-structure. Then there exists a developing pair (ρ, dev), where $\mathrm{dev} : \tilde{M}\to X$ is a $(\mathcal{G}, X)$-*structure preserving* immersion and $\rho : \pi_1(M) \to \mathcal{G}$ is a homomorphism (both unique up to conjugacy by an element of $\mathcal{G}$). The group $\Gamma = \rho(\pi_1(M))$ is called the holonomy group for M.

Let M_0 be a (compact) $(\mathcal{G}, X)$-manifold and put $\pi_1(M_0) = \Gamma$. The deformation space $\mathcal{T}(M_0)$ is the space of $(\mathcal{G}, X)$ -structures on marked manifolds homeomorphic to M_0. $\mathcal{T}(M_0)$ consists of equivalence classes of diffeomorphisms $f : M_0 \to M$ from M_0 to $(\mathcal{G}, X)$-manifolds M. Two such diffeomorphisms $f_i : M_0 \to M_i$ $(i = 1, 2)$ are equivalent if and only if there is an isomorphism (i.e., $(\mathcal{G}, X)$-*structure preserving diffeomorphism*) $h : M_1\to M_2$ such that $h \circ f_1$ is isotopic to f_2.

$$\begin{array}{ccc} M_0 & \xrightarrow{f_1} & M_1 \\ {\scriptstyle f_2}\searrow & \simeq & \downarrow h \\ & & M_2 \end{array}$$

Let $\widehat{\Omega}(M_0)$ be the space consisting of all possible developing pairs (ρ, dev). Let $\mathrm{Diff}(M_0)$ be the group of all diffeomorphisms of M_0 onto itself. Denote $\mathrm{Diff}^0(M_0)$

the subgroup of $\mathrm{Diff}(M_0)$ each element of which is *isotopic* to the identity map. Consider the following exact sequences of the diffeomorphism groups.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & N_{\mathrm{Diff}(\tilde{M}_0)}(\Gamma) & \xrightarrow{\eta} & \mathrm{Diff}(M_0) & \longrightarrow & 1 \\ & & & & \uparrow & & \uparrow & & \\ & & & & C_{\mathrm{Diff}(\tilde{M}_0)}(\Gamma) & \longrightarrow & \mathrm{Diff}^0(M_0), & & \end{array}$$

where $N_{\mathrm{Diff}(\tilde{M}_0)}(\Gamma)$ (resp. $C_{\mathrm{Diff}(\tilde{M}_0)}(\Gamma)$) is the normalizer (resp. centralizer) of Γ in $\mathrm{Diff}(\tilde{M}_0)$. Put $\widehat{\mathrm{Diff}}(M_0) = N_{\mathrm{Diff}(\tilde{M}_0)}(\Gamma)$. The natural right action of $\widehat{\mathrm{Diff}}(M_0)$ and the natural left action of $\mathcal{G}$ on $\hat{\Omega}(M_0)$ are defined respectively:

$$\begin{aligned} (\rho, \mathrm{dev}) \circ \tilde{f} &= (\rho \circ \mu(\tilde{f}), \mathrm{dev} \circ \tilde{f}) \\ g \circ (\rho, \mathrm{dev}) &= (g \circ \rho \circ g^{-1}, g \circ \mathrm{dev}) \end{aligned}$$

where $\mu(\tilde{f}) : \Gamma \longrightarrow \Gamma$ is an isomorphism defined by the conjugate $\mu(\tilde{f})(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$.

It is noted that two developing pairs (ρ_i, dev_i) $(i = 1, 2)$ represent the same structure on M_0 if and only if there exists an element $g \in \mathcal{G}$ such that $g \circ \mathrm{dev}_1 = \mathrm{dev}_2$. Put

$$\Omega(M_0) = \hat{\Omega}(M_0)/\widehat{\mathrm{Diff}}^0(M_0).$$

Since both actions of $\widehat{\mathrm{Diff}}(M_0)$ and and $\mathcal{G}$ on $\hat{\Omega}(M_0)$ commute, the action of $\mathcal{G}$ induces an action of $\Omega(M_0)$. Then it follows that (cf. [9])

Lemma 6. *The elements of $\mathcal{T}(M_0)$ are in one-to-one correspondence with the orbits of $\mathcal{G} \setminus \Omega(M_0)$.*

If $f : M_0 \to M$ is a representative element of $\mathcal{T}(M_0)$, there is a developing pair $(\rho, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \to (\mathcal{G}, X)$ as above. We have the holonomy representation $\rho \circ f_\sharp : \Gamma \to \mathcal{G}$ up to conjugacy. There is a map $\mathrm{hol} : \mathcal{T}(M_0) \to \mathrm{Hom}(\Gamma, \mathcal{G})/\mathcal{G}$ which assigns to a marked structure its holonomy representation. By the definition hol lifts to a map $\widehat{\mathrm{hol}} : \Omega(M_0) \to \mathrm{Hom}(\Gamma, \mathcal{G})$ for which the following diagram is commutative

$$\begin{array}{ccc} \Omega(M_0) & \xrightarrow{\widehat{\mathrm{hol}}} & \mathrm{Hom}(\Gamma, \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{T}(M_0) & \xrightarrow{\mathrm{hol}} & \mathrm{Hom}(\Gamma, \mathcal{G})/\mathcal{G}. \end{array}$$

The following is proved by Thurston [17].

Theorem 7 (Holonomy Theorem). *$\widehat{hol}$ is a local homeomorphism.*

3. Geometric superrigidity on M_0

Let $M_0 = V_{-1}^{4n+3}/\Gamma$ be a compact pseudo-quaternionic flat $(4n+3)$-manifold as before. When we take $(\mathcal{G}, X) = (\mathrm{PSp}(n+1,1), S^{4n+3})$, the deformation space $\mathcal{T}(M_0)$ is the space of all possible pseudo-quaternionic flat structures on M_0. In this section we prove the following rigidity.

Theorem 8. *Let M be a compact pseudo-quaternionic flat $(4n+3)$-manifold. Suppose that the fundamental group $\pi_1(M)$ is isomorphic to that of a compact quaternionic hyperbolic $4n$-manifold. Then, M is pseudo-quaternionically isomorphic to $M_0 = V_{-1}^{4n+3}/\Gamma$.*

Let $\mathrm{hol} : \mathcal{T}(M_0) \to \mathrm{Hom}(\Gamma, \mathrm{PSp}(n+1,1))/\mathrm{PSp}(n+1,1)$ be the map as before.

Theorem 9. *The map hol maps $\mathcal{T}(M_0)$ homeomorphically onto a connected component diffeomorphic to* $\mathrm{Hom}(\Gamma, \mathrm{Sp}(1))/\mathrm{Sp}(1)$.

A representation $\rho : \Gamma \to \mathrm{PSp}(n+1,1)$ in $\mathrm{Hom}(\Gamma, \mathrm{PSp}(n+1,1))$ is said to be *amenable* if its closure of the image $\rho(\Gamma)$ lies in the maximal amenable Lie subgroup of $\mathrm{PSp}(n+1,1)$. A maximal amenable Lie group in $\mathrm{PSp}(n+1,1)$ is conjugate to the compact subgroup $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ or to the group of similarity transformations $\mathrm{Sim}(\mathcal{M})$. A Fuchsian representation $\rho : \Gamma \to \mathrm{PSp}(n+1,1)$ in $\mathrm{Hom}(\Gamma, \mathrm{PSp}(n+1,1))$ is a discrete faithful representation whose image $\rho(\Gamma)$ leaves a totally geodesic $4n$-subspace in $\mathbb{H}^n_{\mathbf{F}}$. (Compare [6], [2].)
Let $\mathcal{S}(0,\infty)$ be the set of amenable representations in $\mathrm{Hom}(\Gamma, \mathrm{PSp}(n+1,1))$ and $\mathcal{S}(-1)$ the set of non-amenable representations in $\mathrm{Hom}(\Gamma, \mathrm{PSp}(n+1,1))$. Then the disjoint union $\mathcal{S}(0,\infty) \cup \mathcal{S}(-1)$ constitutes $\mathrm{Hom}(\Gamma, \mathrm{PSp}(n+1,1))$. Applying Theorem 1 to the set $\mathcal{S}(-1)$, we can prove that

Lemma 10.

(i) *The set $\mathcal{S}(-1)$ coincides with the set of discrete faithful representations of Γ.*

(ii) *The set of discrete faithful representations coincides with the set of Fuchsian representations of Γ.*

Recall that $\hat{\Gamma}$ is a uniform lattice in $\mathrm{PSp}(n,1)$. If we put

$$\mathcal{R}(\hat{\Gamma}) = \{\rho \in \mathrm{Hom}(\hat{\Gamma}, \mathrm{PSp}(n,1)) \mid \rho \text{ is a discrete faithful representation}\},$$

then by the well known Mostow rigidity, the orbit space $\mathcal{R}(\hat{\Gamma})/\mathrm{PSp}(n,1)$ is a single point. Let $\mathcal{R}(\Gamma, \mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1))$ be the set of discrete faithful representations in $\mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1)$. Then it is easy to see that

{The set of discrete faithful representations of Γ in $\mathrm{PSp}(n+1,1)\}/\mathrm{PSp}(n+1,1)$ is in one-to-one correspondence with $\mathcal{R}(\Gamma, \mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1))/\mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1)$. As there is the fibration:

$$\begin{aligned}\mathrm{Hom}(\Gamma,\mathrm{Sp}(1))/\mathrm{Sp}(1) &\to \mathrm{Hom}(\Gamma,\mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1))/\mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1)\\ &\longrightarrow \mathrm{Hom}(\hat{\Gamma},\mathrm{PSp}(n,1))/\mathrm{PSp}(n,1),\end{aligned}$$

it follows that

$$\mathcal{R}(\Gamma,\mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1))/\mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1)\approx\mathrm{Hom}(\Gamma,\mathrm{Sp}(1))/\mathrm{Sp}(1).$$

Since the set of discrete faithful representations of Γ is a closed subset in $\mathrm{Hom}(\Gamma,\mathrm{PSp}(n+1,1))$ by Lemma 1.2 [7], we have

Corollary 11. *The set of Fuchsian representations is a component of* $\mathrm{Hom}(\Gamma,\mathrm{PSp}(n+1,1))$. *Moreover, the set of Fuchsian representations is diffeomorphic to the space* $\mathrm{PSp}(n+1,1)\times\mathrm{Hom}(\Gamma,\mathrm{Sp}(1))/\mathrm{Sp}(1)$.

4. Amenable holonomy groups

Recall that an amenable representation $\rho:\Gamma\to\mathrm{PSp}(n+1,1)$ is a representation whose closure of the image $\overline{\rho(\Gamma)}$ in $\mathrm{PSp}(n+1,1)$ lies in the maximal amenable Lie subgroup of $\mathrm{PSp}(n+1,1)$. As the first step to prove Theorem 2, we must show that

Theorem 12. *Let M be a compact pseudo-quaternionic flat $(4n+3)$-manifold. If the holonomy group is amenable, then M is finitely covered by the sphere S^{4n+3}, a Hopf manifold $S^1\times S^{4n+2}$ or a nilmanifold $\mathcal{M}/\Gamma$.*

We examine *quaternionic Heisenberg geometry.* As usual, we write $\mathrm{Aut}_{\mathrm{PSp}}(S^{4n+3})=\mathrm{PSp}(n+1,1)$. Recall $S^{4n+3}-\{\infty\}=\mathcal{M}$ and let $\mathrm{Aut}_{\mathrm{PSp}}(\mathcal{M})$ be the subgroup of $\mathrm{Aut}_{\mathrm{PSp}}(S^{4n+3})$ which stabilizes the point at infinity $\{\infty\}$.
Then the geometry $(\mathrm{Aut}_{\mathrm{PSp}}(\mathcal{M}),\mathcal{M})$ is called quaternionic Heisenberg geometry.

A maximal amenable group G of $Sp(n+1,1)$ is isomorphic to the semidirect product $\mathcal{M}\rtimes(Sp(n)\times\mathbb{F}^*)$ where $\mathcal{M}$ is the quaternionic Heisenberg group. It lies in the following exact sequence: $1\longrightarrow\mathbb{R}^3\longrightarrow\mathcal{M}\longrightarrow\mathbb{F}^n\longrightarrow 1$. For the point $\{\infty\}$ of S^{4n+3}, as we identify $S^{4n+3}-\{\infty\}$ with $\mathcal{M}$, $\mathrm{Aut}_{\mathrm{PSp}}(\mathcal{M})$ is the stabilizer in $\mathrm{PSp}(n+1,1)$ of the point $\{\infty\}$. Then $\mathrm{Aut}_{\mathrm{PSp}}(\mathcal{M})$ is a maximal amenable subgroup of $\mathrm{PSp}(n+1,1)$.
Let $\mathbb{Z}/2\to\mathrm{Sp}(n+1,1)\xrightarrow{P}\mathrm{PSp}(n+1,1)$ be the projection. Since G is as above, PG is isomorphic to $\mathrm{Aut}_{\mathrm{PSp}}(\mathcal{M})=\mathcal{M}\rtimes(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+)$.

The Heisenberg Lie group $\mathcal{M}$ is the product $\mathbb{R}^3 \times \mathbb{F}^n$ with group law

$$(a,y)\cdot(b,y') = (a+b+\mathrm{Im} < y,y' >, y+y').$$

The group $\mathcal{M}$ is nilpotent because $[\mathcal{M},\mathcal{M}] = \mathbb{R}^3$ which is the center consisting of the form $(a,0)$. As above, $\mathcal{M} \rtimes (\mathrm{Sp}(n) \times \mathbb{F}^*)$ is the semidirect product for which the action of $\mathrm{Sp}(n) \times \mathbb{F}^*$ on $\mathcal{M}$ is given by

$$(*) \qquad (A,\nu)\circ(a,y) = (|\nu|^2 \nu a \nu^{-1}, Ay\nu^{-1}).$$

Since $\mathrm{Aut}_{\mathrm{PSp}}(\mathcal{M}) = PG$, PG is isomorphic to $\mathcal{M} \rtimes (\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) \times \mathbb{R}^+)$, for which the action of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) \times \mathbb{R}^+$ on $\mathcal{M}$ is given as follows: if $\nu = (g,t)$, then

$$(A,(g,t))\circ(a,y) = (t^2 \cdot gag^{-1}, t\cdot Ayg^{-1}).$$

Thus the Heisenberg dilation D^t with scale factor $t \in \mathbb{R}^+$ is

$$D^t(a,y) = (t^2 a, ty).$$

A gauge on $\mathcal{M}$ is defined by

$$|(a,y)|_{\mathcal{M}} = (4|a|^2 + |y|^4)^{\frac{1}{4}}.$$

A left invariant metric $d_{\mathcal{M}}$ is given by

$$d((a,x),(b,y))_{\mathcal{M}} = |(b,y)^{-1}\cdot(a,x)|_{\mathcal{M}},$$

where $(b,y)^{-1} = (-b,-y)$.
Given a (geodesically) incomplete similarity manifold M, we can find a Γ-invariant vector subspace I in $\mathcal{M}$, which is called the invisible set according to the result of Fried [4] (also, [16]). Using this invariant set, we can prove the above theorem. We refer to [11] for the detail of this proof.

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