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How to Draw a Simple Closed Curve on a Picture of Riemann Surface

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1 Introduction

Let Σ_g be a Riemann surface of genus g. Let $\pi_1 = \pi_1(\Sigma_g, *)$ be the fundamental group of Σ_g with a base point. Suppose that $g \geq 2$. We fix standard generators ℓ_i, m_i $(i = 1, 2, \cdots, g)$ of π_1 satisfying

$$\pi_1 = \langle \ell_1, \cdots, \ell_g, m_1, \cdots, m_g \mid \Pi_{i=1}^{g} \ell_i m_i \ell_i^{-1} m_i^{-1} = e \rangle.$$

Here we have a simplicity problem on elements of π_1. That is, for an element $\gamma \in \pi_1$, given by a word of ℓ_i's and m_i's, can we determine whether the free homotopy class $[\gamma]$ of γ is represented by a simple closed curve on Σ_g? Lustig [L] and Takarajima [T] give answers of this problem. Lustig uses a hyperbolic metric on Σ_g and geodesics. Takarajima uses *train tracks.*

In this paper we fix a picture of Σ_g as in Figure 1 and we consider an algorithm to draw a 'simple' picture for a given word $\gamma \in \pi_1$, and we show that this algorithm gives an answer of the simplicity problem.

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2 Fundamental group π_1

The fundamental group $\pi_1 = \pi_1(\Sigma_g, *)$ of Riemann surface Σ_g are generated by $(2g)$ elements ℓ_i, m_i $(i = 1, 2, \cdots, g)$. Suppose that they are given by Figure 2.

It is easy to check that ℓ_i's and m_i's have only one relation

$$R = \Pi_{i=1}^{g} \ell_i m_i \ell_i^{-1} m_i^{-1}.$$

That is, we have

$$\pi_1 = \langle \ell_i, m_i \mid R \rangle.$$

For two elements γ_1, γ_2 in π_1, we call they are *free homotopic* if there exists an element σ in π_1 satisfying $\gamma_1 = \sigma\gamma_2\sigma^{-1}$.

3 Partition of Riemann surface

We devide Σ_g to some areas as in Figure 3.We call dotted points *vertices*, we call dotted lines from a vertex to a vertex *internal segments*, and we call solid lines from a vertex to a vertex *external segments*. We consider an orientation on each internal segments.

Remark : In this picture there are *foreside* and *backside* areas, so there also exist foreside interior segments and backside interior segments.

We consider a set C consisted of oriented closed curves drawn in the figure of Σ_g. Let $\{A_1, A_2, \cdots, A_p\}$ be a set of foreside internal segments, and let $\{B_1, B_2, \cdots, B_p\}$ be a set of backside internal segments. Here B_i corresponds to A_i. Let $\{C_1, C_2, \cdots, C_q\}$ be a set of external segments. Let $\{D_1, D_2, \cdots, D_r\}$ be a set of foreside areas, and let $\{E_1, E_2, \cdots, E_r\}$ be a set of corresponding backside areas. Here, $p = 3g - 1$, $q = 6g - 2$, and $r = 2g$.

We consider a cyclic word $X_1X_2\cdots X_n$ of $\{A_i, B_i, C_j, D_k, E_k\}$ satisfying the following conditions.

(a) A length n of the word is even.

(b) Each $X_1, X_3, X_5, \cdots$ is one of $\{A_i, B_i, C_j\}$.

(c) Each $X_2, X_4, X_6, \cdots$ is one of $\{D_k, E_k\}$

(d) For each $\ell = 1, 2, \cdots, n/2$, segments $X_{2\ell-1}$ and $X_{2\ell+1}$ are edges of the area $X_{2\ell}$. (If $\ell = n/2$, $X_{2\ell+1} = X_1$.)

Let C be a set of such cyclic words. For a word w in C, we can consider a corresponding picture of a loop on the figure of Σ_g.

Example :

Let labels be given as in Figure 4.For a given loop γ in Figure 5, the corresponding word w is given by

$$w = A_1D_2C_8E_2B_2E_4B_5E_3C_3D_3A_4D_1.$$

For a word w in C, if w is correspondent to a simple closed curve, then we call w *simple*. The above example is simple.

4 Reducing elements of C

We consider the following 7 operations on C. We call these operations *reducing*.

(1) In the case that there exists an internal (or external) segment a such that the curve intersects transversally a twice in succession, we cancel the two intersections. (See Figure 6.)

Remark : When we have a part as in Figure 7, we call that the curve pass a transversally.

(2) In the case that there exists an internal (or external) segment a such that the curve intersects a once non-tranceversally, we cancel the intersection. (See Figure 8.)

(3) In the case that there exists a connected component of external segments such that a part of the curve is as the left one of Figure 9, we cancel the two intersections of the curves and external segments. (See Figure 9.)

(4) In the case that there exists an internal segment a and an external segment b such that a and b have a intersection, and that the curve pass transversally a, b, a in succession, we slide the intersection of the curve and b. (See Figure 10.)

(5) In the case that there exist an internal segment a and two external segment b, c, as in the left below figure and that the curve pass tranversally a, b, c, a, in succession, we slide two intersections of the curve and external segments. (See Figure 11.)

(6) In the case that there exist an internal segment a and an external segment b with one common vertex and the curve pass transversally a, b in succession and the curve pass a in the backside, we slide the intersection of the curve and b. (See Figure 12.)

(7) In the case that the curve and segments are as in the left figures of Figure 13, we slide the intersections of the curve and external segments.

5 Main result

First, we also denote ℓ_i and m_i their representations in C.

Theorem

Let $\gamma \in \pi_1$ be given by a word of ℓ_i's and m_i's. Suppose that the free homotopy class $[\gamma]$ of γ be represented by a simple closed curve. If we regard γ as an element in C, then after operating (1) ,(2), $\cdots$, (7) in the previous section we can reduce γ to a simple word in C.

Remark : If we have a simple word in C, it is easy to make a computer programm to draw a simple closed curve on the figure of Σ_g. So by this algorithm we can know whether a given element of π_1 is represented by a simple closed curve. And we mention that a reduced simple word is not uniquely determined.

Proof : For a given w in C, suppose that the corresponding loop γ is not simple. Then there exist a locally embedded disk D on Σ_g which is bounded by one or two sub-curves of γ as in Figure 14.

Here a disk D is locally embedded in Σ_g if a certain finite partition $D = \cup_i D_i$ satisfies that the restriction on D_i is an embedding. We will show that the operations (1) - (7) allows us to vanish this disk.

Case 0.

In the case that the disk D is a subset of an area, we perturb γ and vanish the disk.

Case 1.

In the case that D does not contain any vertices, we use (1), and (2). See an example in Figure 15.

Remark : On the Riemann surface, a neiborhood of each vertices is as in Figure 16.

And remark that if we consider succesive three vertices A, B, and C as in Figure 17, then $A = C$ on Σ_g.

Case 2.

In the case that the disk D contains only one vertex, we use (1), (3), (4), and (6).

The operation (3) is rewritten as in Figure 18.

The operation (4) is rewritten as in Figure 19.

Hence we reduce the curve as in Figure 20-1.

If we have a part as in Figure 20-2, we use the operation (6). Here areas with slant lines means backside areas.

Case 3.

In the case that the disk D contains two vertices along integral segments, we use (5), (6), and (7). If we have a part as in Figure 18, we can reduce the curve in the case 2. Otherwise we have the following reducing and vanish the disk. See Figure 21.

Case 4.

In the case that the disk D contains more than two successive vertices along internal segments, we use (3), (6) and (7). Here we show that we can decrease the number of vertices in the disk. See Figure 18 and Figure 22.

Case 5.

In the case that D contains some successive vertices along external segments, we use the operation (3), (4) and (6). Here we show that we can decrease the number of vertices in the disk. See Figure 14 and Figure 23.

Case 6.

In the case that D contains some vertices which are not in succession by segments, We divide the disk as in Figure 24.

Case 7.

In the case that D contains some successive vertices but D does not contain any areas, we use (3), [redacted] (6). We divide the disk D into some small disks. See Figure 25. And then we can see that each small pieces contains successive vertices along internal segments.

Case 8.

In the case that D contains some areas, we see a neiborhood of one of these areas. It looks like as in Figure 26.

We use (3) or (5) as in Figure 27, we can reduce the disk in the case the disk does not contain any areas. It follows that we can vanish the disk in any case.

6 Finiteness

When we realize these algorithm in computers, we need to show whether it finishes in finite steps. If γ is represented by a finite word, then the number of disks are finite. In the category C, the number of vertices which are contained in one disk is always finite. For each disks, we can always decrease the number of vertices, so the algorithm finishes finitely.

References:

[L] Lustig M., Paths of geodesics and geometric intersectio n numbers II,Ann. of Math. Studies **111** (1987).

[T] Takarajima I., On intersections of representing curves of elements of the fundamental group of a surface. *preprint.*

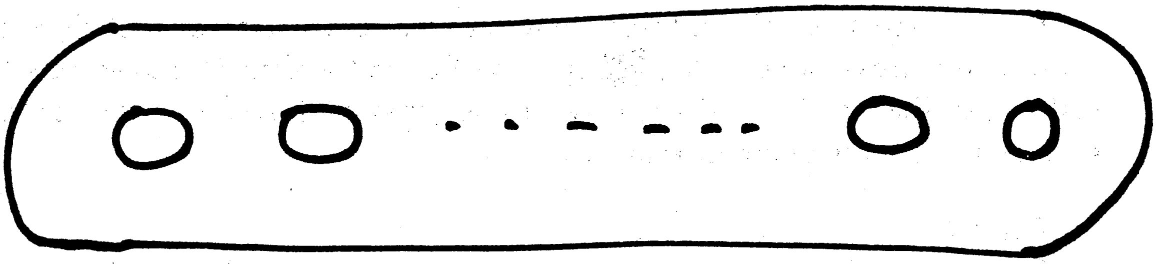


Figure 1.

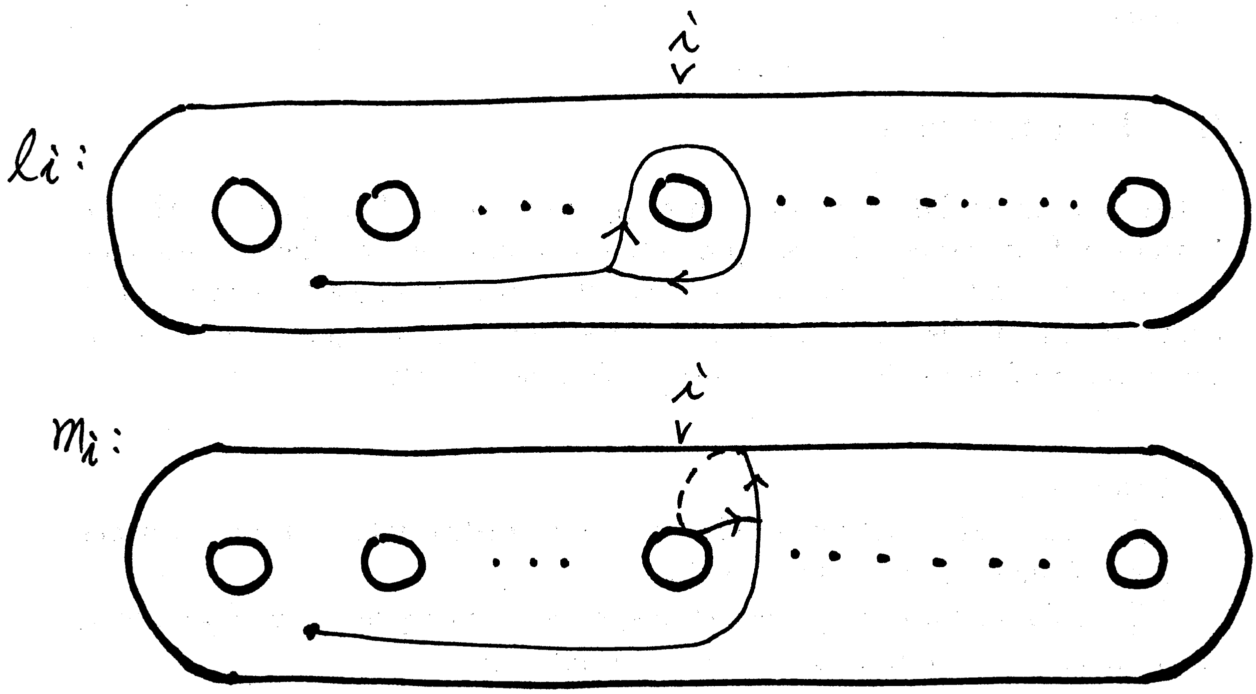


Figure 2.

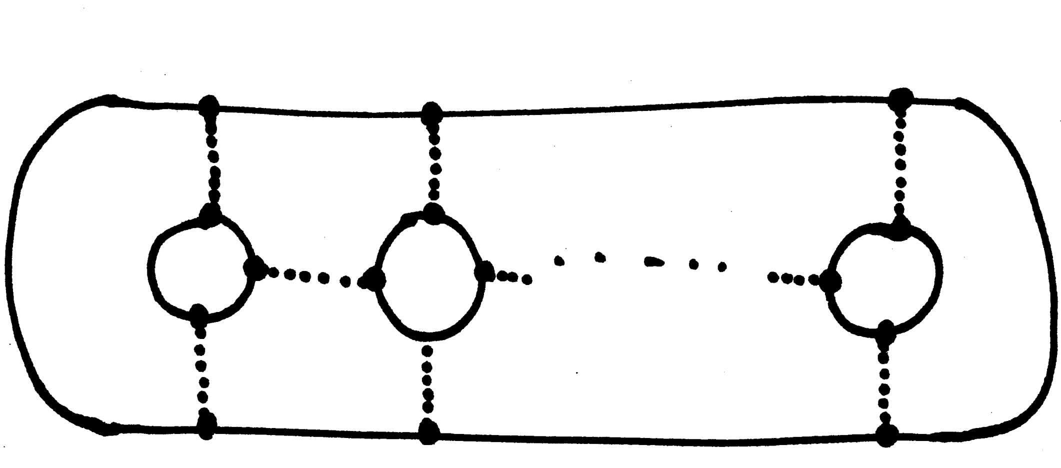
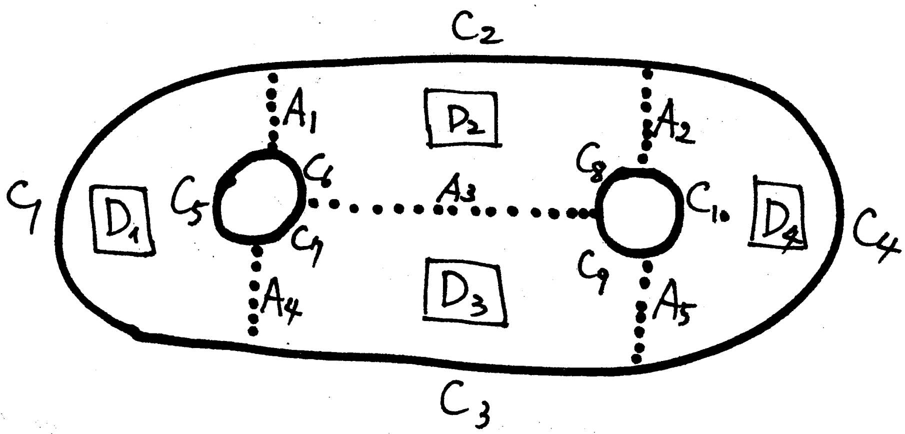


Figure 3.



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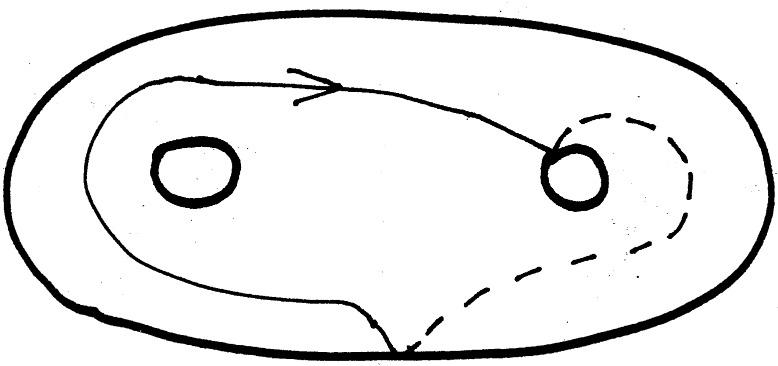

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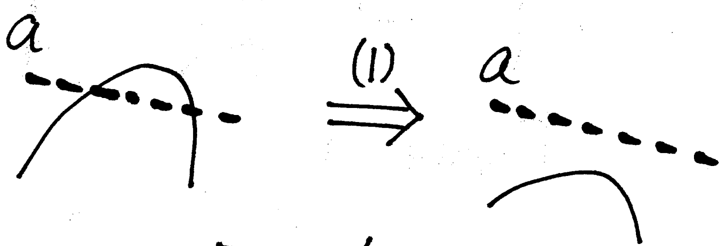
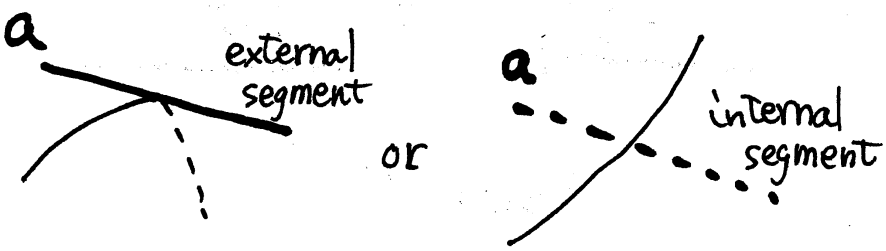


Figure 6.



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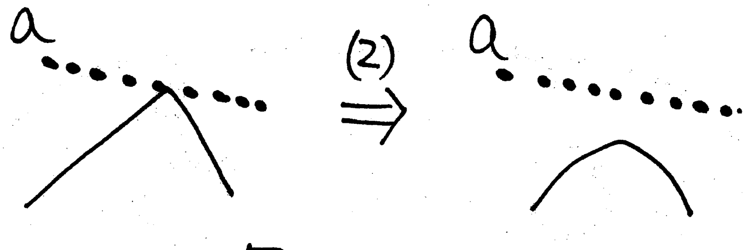

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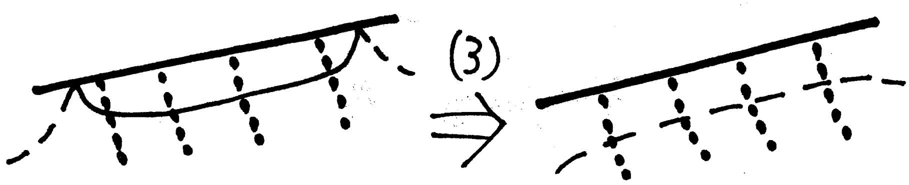

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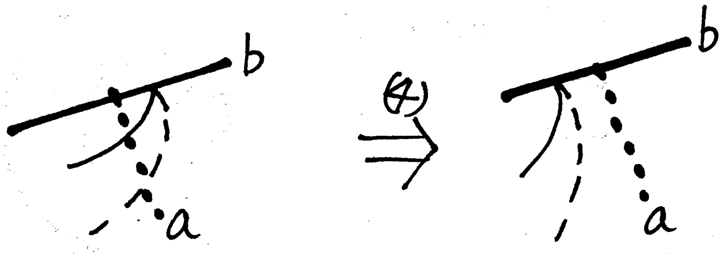

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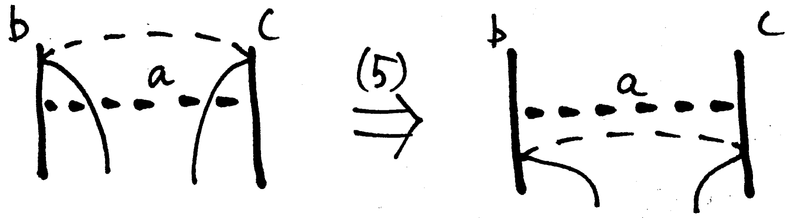

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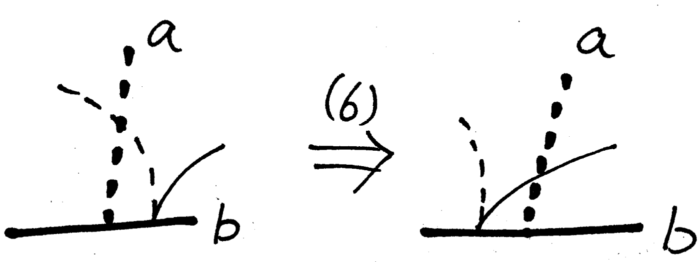

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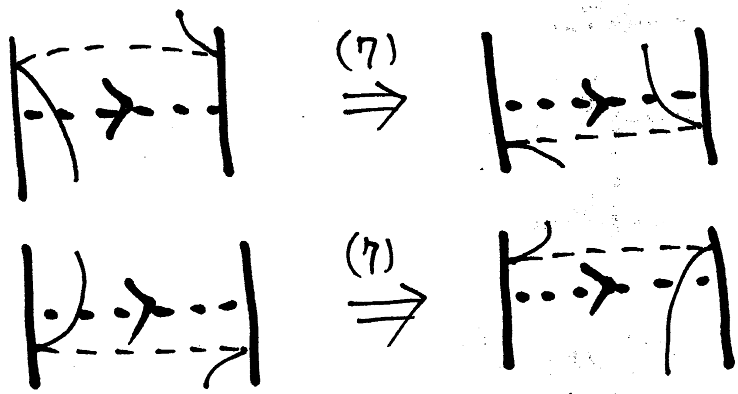

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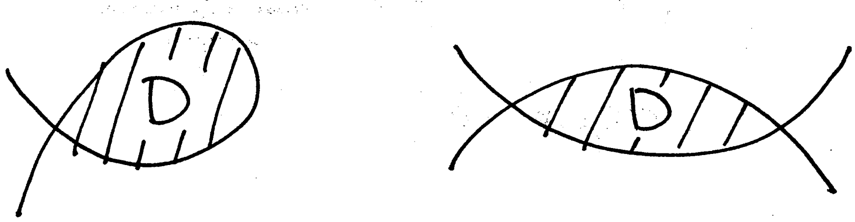

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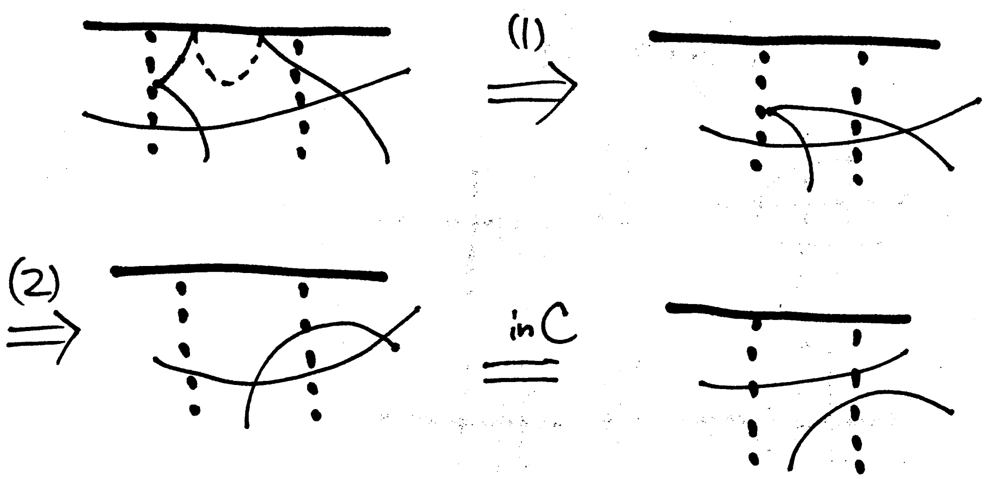


Figure 15.

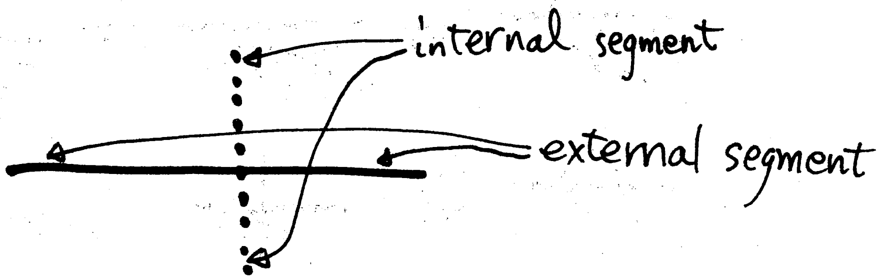
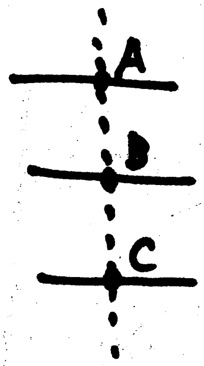


Figure 16



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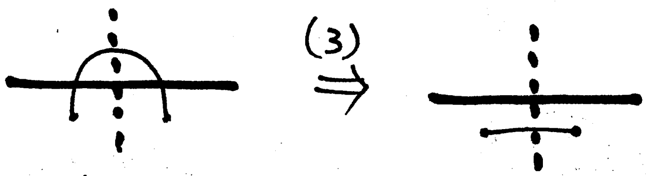
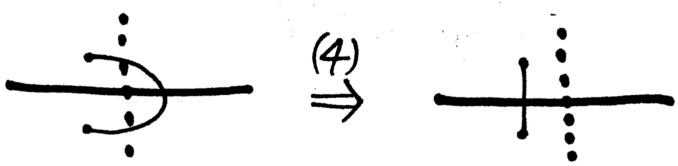


Figure 18



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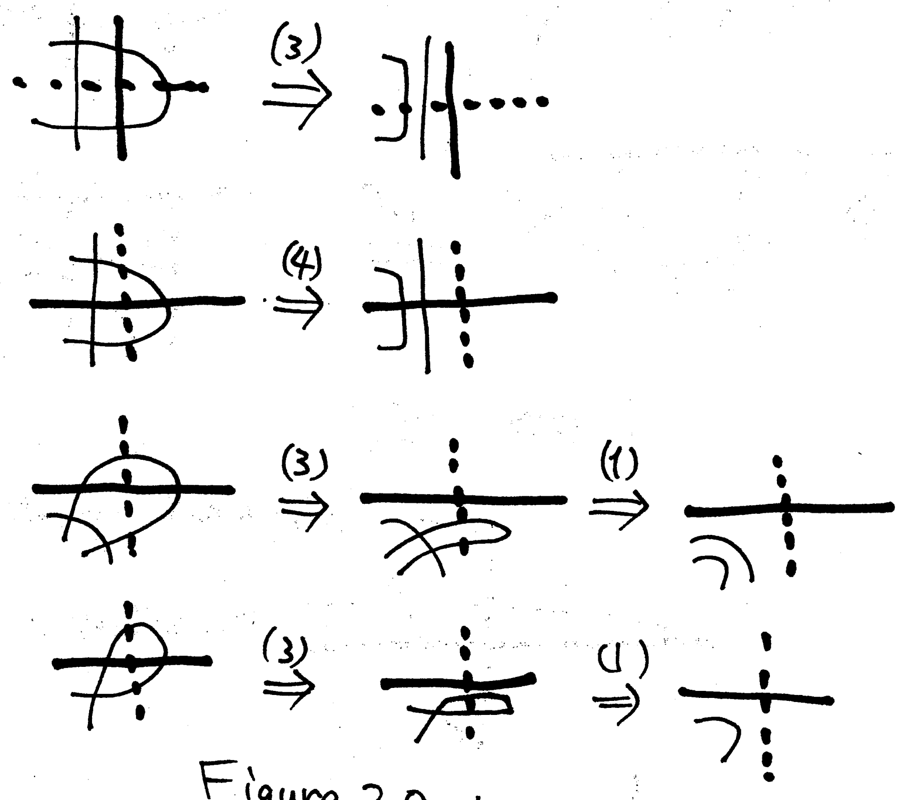


Figure 20-1

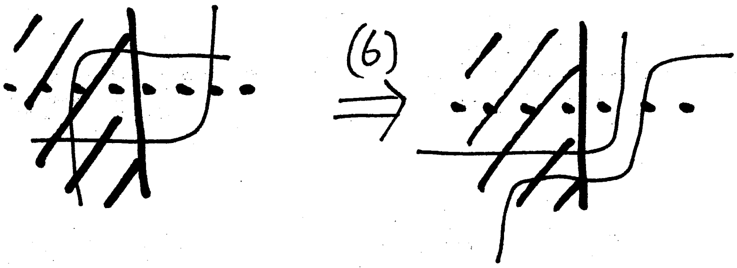


Figure 20-2.

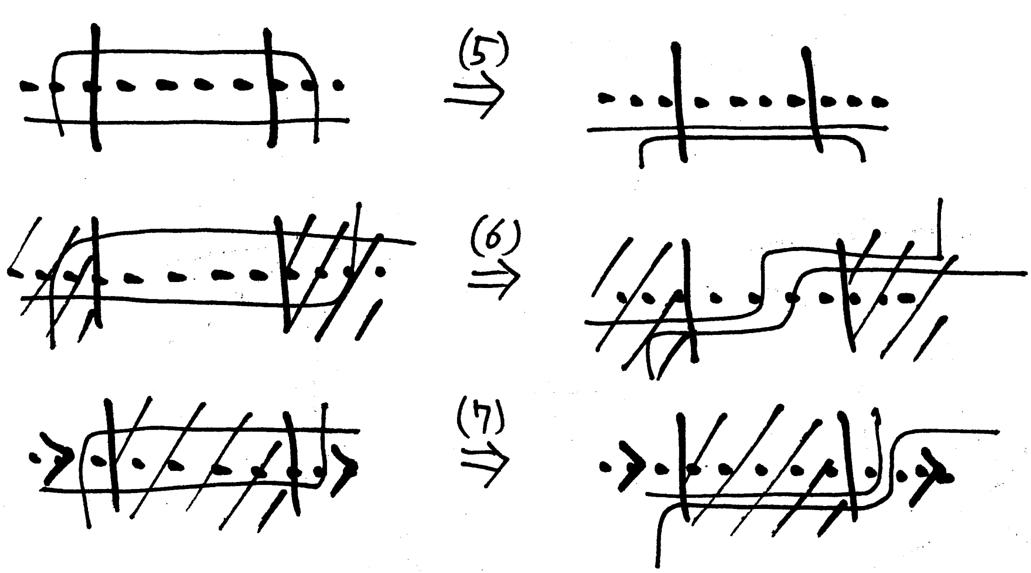
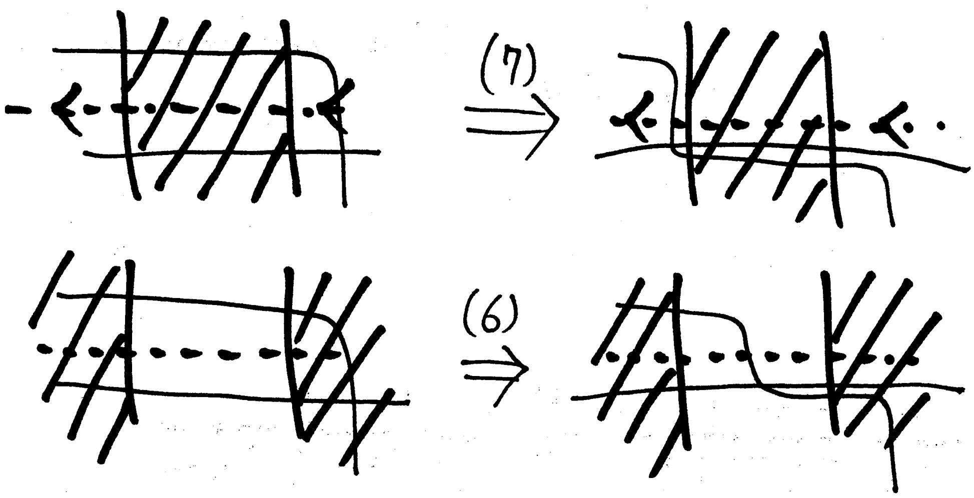


Figure 21.



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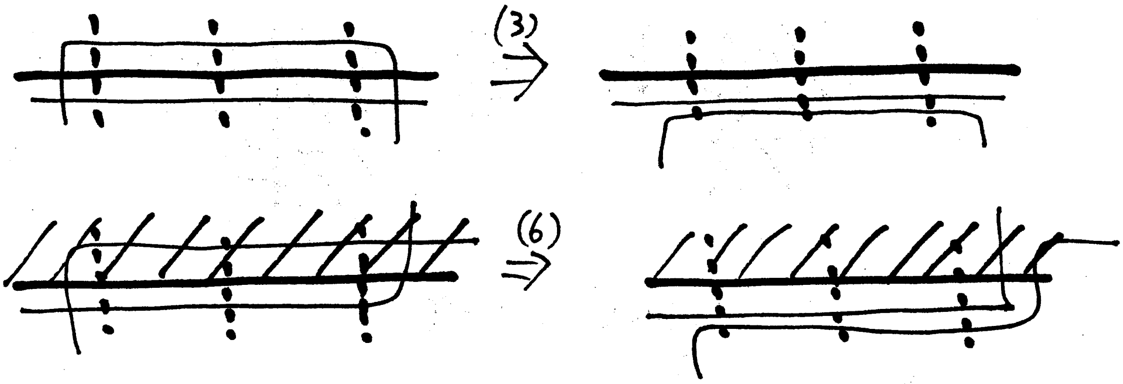

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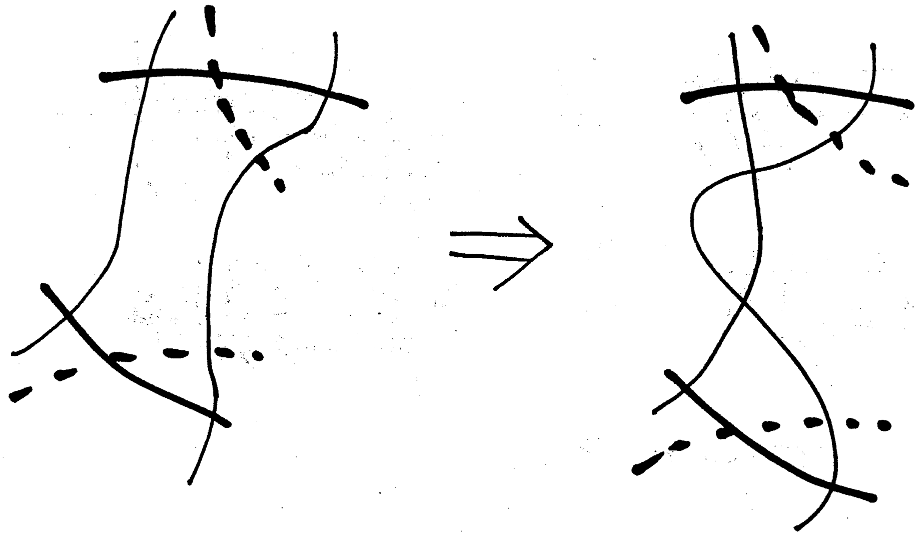
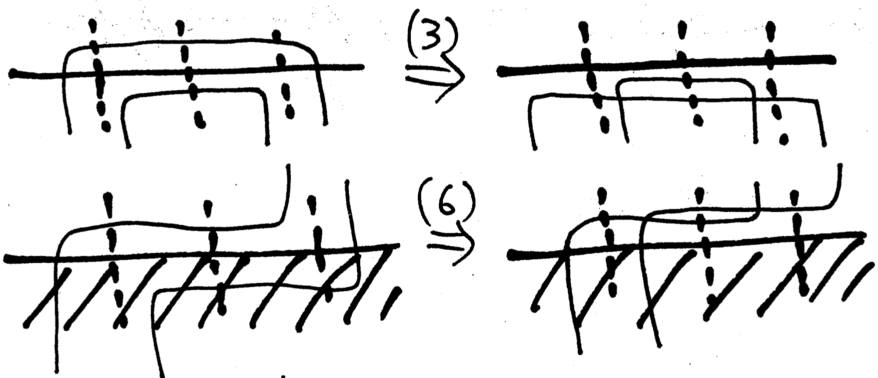


(the same one in C)

Figure 24.



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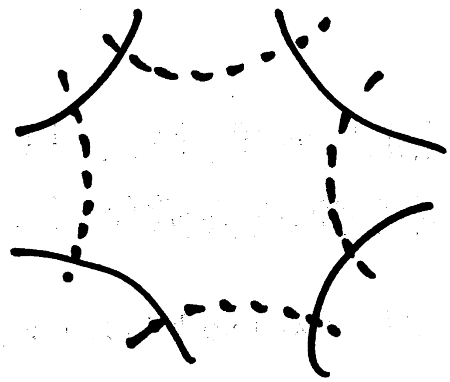

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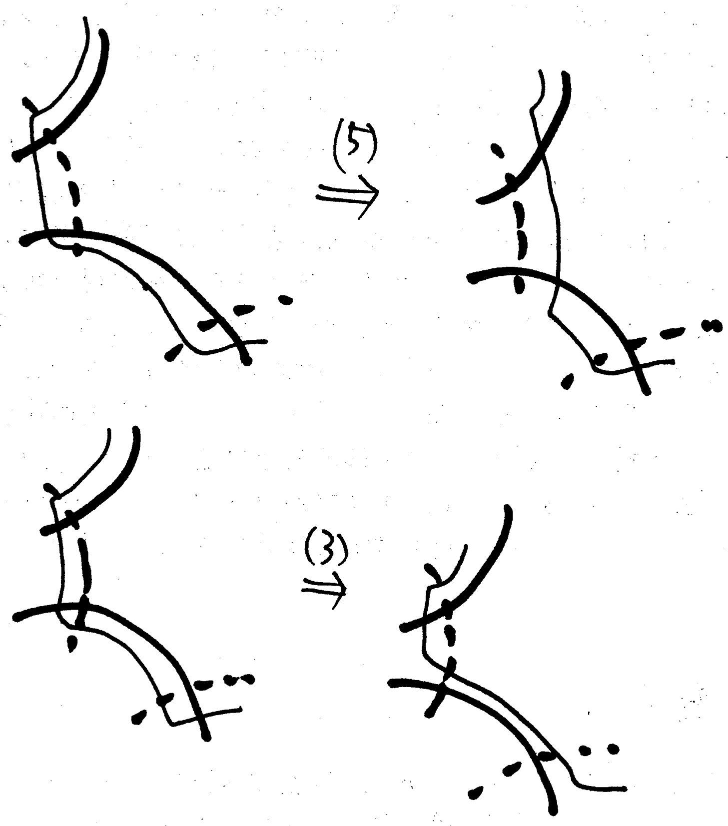

Figure 27.