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Completeness Proofs for Linear Logic Based on the Proof Search Method
(Preliminary Report)

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Abstract

The proof search method is a traditionally established way to prove the completeness theorem for various logics. The purpose of this paper is to show that this method can be adapted to linear logic.

First we prove the completeness theorem for a certain fragment of intuitionistic linear logic, called *naive linear logic*, with respect to *naive phase semantics*, i.e., phase semantics without any closure condition, using the proof search method in a certain labelled sequent system. Then the completeness of the (rudimentary) classical linear logic can be obtained as a direct corollary by a Kolmogorov-Gödel style double negation interpretation.

To apply the proof search method for the full system of linear logic, we generalize the notion of *branch* in the standard proof search method to that of *OR-tree*, and give a proof of the completeness theorem for intuitionistic (classical, resp.) linear logic with respect to intuitionistic (classical, resp.) phase semantics, based on a generalized form of the proof search method.

1 Introduction

The proof search method is considered to be one of the most natural completeness proof methods and successfully applied to prove the completeness theorems for various logics, typically for classical logic, intuitionistic logic and some of modal logics (cf. Kleene[5], Schütte[9], Takeuti[10]).

This method consists of the following two steps; (i) to prove that for any unprovable formula there exists such a branch in a proof search tree that does not reach an axiom and that contains enough information to refute the formula (called an *open branch*), and (ii) to show the construction of a countermodel of the formula from the open branch obtained by (i). This method could be viewed as a specific way of the Henkin model construction in the classical logic case since the proof search procedure gives a process of construction of a maximal consistent set of formulas.

The usual proof search method uses the structural rules (weakening and contraction) very essentially. Hence it has not been known if or not this method is adaptable to linear logic in which the lack of the structural rules is one of the main features. The purpose of this paper is to give an affirmative answer to this question.

To see how structural rules are essential in the case of the proof search method for traditional logics, let us suppose that we would like to refute (i.e., to find a countermodel of)

$\Gamma \vdash A \vee B$ in classical logic. Our task is to find an open branch in a proof search tree that contains enough information to make every formula in Γ true and both A and B false. A proof search tree of a given sequent is constructed by successively applying the inference rules of classical logic in a bottom-up manner. However, if we use the usual inference rules for $\vee$-right, that is,

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad \text{and} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B},$$

we lose one of A and B (when read bottom-up), hence branches in the resulting proof search tree retain too little information to refute $\Gamma \vdash A \vee B$. The essential trick here is to consider the following derived rule in classical logic which is equivalent to the above two rules, *due to weakening and contraction*;

$$\frac{\Gamma \vdash A, B}{\Gamma \vdash A \vee B.}$$

Thanks to using this equivalent rule we can preserve the information on both A and B. This trick is a part of the reason why an open branch in classical logic suffices to construct a countermodel. One could consider the same trick for intuitionistic logic, which leads to the completeness with respect to the Kripke models (cf. eg. Takeuti[10], CH 1, §8).

In linear logic, however, we cannot use any trick like above because of the lack of structural rules;

- We cannot replace two $\oplus$-right rules by a single rule, contrary to the $\vee$-right case of classical logic.
- We must take all possible context partitions into account when we apply $\otimes$-right and $\multimap$-left rules. For example, $\alpha, \beta \vdash \alpha \otimes \gamma$ has only one non-atomic formula, but we must consider four possibilities when we apply $\otimes$-right rule;

$$\frac{\vdash \alpha \quad \alpha, \beta \vdash \gamma}{\alpha, \beta \vdash \alpha \otimes \gamma} \qquad \frac{\alpha \vdash \alpha \quad \beta \vdash \gamma}{\alpha, \beta \vdash \alpha \otimes \gamma} \qquad \frac{\beta \vdash \alpha \quad \alpha \vdash \gamma}{\alpha, \beta \vdash \alpha \otimes \gamma} \qquad \frac{\alpha, \beta \vdash \alpha \quad \vdash \gamma}{\alpha, \beta \vdash \alpha \otimes \gamma\,.}$$

As a consequence, the standard construction of a countermodel from one open branch is not possible for linear logic. In this paper, we propose two solutions to overcome this difficulty.

1. To introduce labels; we consider a sort of labelled system for the proof search method in which resource information is expressed not in terms of number of formula occurrences in a sequent, but in terms of *labels* attached to formulas. In the system, structural rules can be used freely, hence we can accommodate the standard proof search method to this system. In this way, we prove that a certain fragment of intuitionistic linear logic, which we call *naive linear logic* (**NLL**), is complete with respect to *naive phase semantics*, i.e., phase semantics without any closure condition. Then we show that the multiplicative additive fragment of classical linear logic (**MALL**), sometimes called *rudimentary* linear logic, is encoded into **NLL** via the Kolmogorov-Gödel style double negation translation. Hence, as a direct corollary, we also have a completeness proof of **MALL** based on the proof search method, since the classical phase semantics (in the sense of Girard[3]) is also obtained by the double-negation closure property from our naive phase semantics. The labelling method has been investigated under Gabbay's general framework of *Labelled Deductive Systems* (*LDS*) [2]. Our method could be viewed as an application of LDS (see Kurtonina[6], Venema[12] for other applications of LDS to substructural logics).

2. To generalize the notion of open branch; We generalize the notion of branch in a proof search tree to the notion of *OR-branching tree* or simply *OR-tree*, and show that given an unprovable sequent $\Gamma \vdash C$, one can construct an *open* OR-tree which retains enough information to refute the sequent. Then we describe how to construct an intuitionistic phase model from an open OR-tree of full intuitionistic linear logic (**ILL**), and a classical phase model from that of full classical linear logic (**LL**). In both cases, the closure condition of phase semantics plays an essential role. This gives a completeness proof based on a generalized form of the proof search method both for **ILL** and for **LL**. We also obtain the cut-elimination theorem as a corollary, since we do not use the cut rule during the construction of the required OR-tree and the corresponding countermodel.

In section 2, first we give the definitions of intuitionistic and classical phase semantics. Then naive phase semantics is obtained by dropping the closure condition from intuitionistic phase semantics. In section 3, we define naive linear logic (**NLL**), which consists of connectives ($\otimes$, $\oplus$, $\multimap$) where $\multimap$ is restricted to the form $A \multimap \alpha$ (for any atomic formula α). In order to incorporate with the standard proof search method, we also introduce an alternative formulation of **NLL** using labels, called *labelled naive linear logic* (**LNLL**), and prove the completeness of **NLL** with respect to naive phase semantics. Section 4 is devoted to a completeness proof of both **ILL** and **LL** based on the open OR-branching tree construction. We give the definitions of intuitionistic linear logic and classical (left one-sided) linear logic in Appendix A and B, respectively.

2 Intuitionistic, Classical and Naive Phase Semantics

In this section, we define intuitionistic phase semantics (cf. Abrusci[1], Troelstra[11], Okada[8]), classical phase semantics (cf. Girard[3][4] and Lafont[7]), and naive phase semantics.

Definition 1 An intuitionistic phase space (M, Cl) consists of a commutative monoid M and a function $Cl : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$, called a *closure operator*, satisfying the following;

(C1) $X \subseteq Cl(X)$;

(C2) $X \subseteq Y$ implies $Cl(X) \subseteq Cl(Y)$;

(C3) $Cl(X) = Cl(Cl(X))$;

(C4) $Cl(X)Cl(Y) \subseteq Cl(XY)$;

where XY is defined by $\{xy | x \in X, y \in Y\}$. A set $X \subseteq M$ that satisfies $X = Cl(X)$ is called a *fact*.

Then, we can define $\mathbf{1} = Cl(\{1\})$ (1 stands for the unit element of M), $\top = M$, $\mathbf{0} = Cl(\emptyset)$, and for any facts X, Y,

- $X \multimap Y = \{y | \forall x \in X xy \in Y\}$
- $X \otimes Y = Cl(XY)$;

- $X \,\&\, Y = X \cap Y$;
- $X \oplus Y = Cl(X \cup Y)$.

As easily seen, each constant above is a fact and each operation above produces a fact whenever facts X and Y are given.

If M is an intuitionistic phase space, then $J(M) = \{x \in \mathbf{1} | x \in Cl(\{xx\})\}$ is a submonoid of M. An *enriched intuitionistic phase space* is an intuitionistic phase space M endowed with a submonoid K of $J(M)$ (not necessary to be a fact).

For any fact X of intuitionistic phase space, define

- $!X = Cl(X \cap K)$.

An intuitionistic phase model $M = (M, Cl, K, v)$ is given by an intuitionistic phase space $M = (M, Cl, K)$ and an *interpretation* v which maps each atomic formula α to a fact $v(\alpha)$ of M, which is also denoted by α^*. Then each formula A is interpreted by a fact A^* along the above definitions, and $\Gamma \equiv A_1, \ldots, A_n$ is interpreted by $\Gamma^* = A_1^* \otimes \cdots \otimes A_n^*$. We say that A is *satisfied* in M if $1 \in A^*$, and that $\Gamma \vdash C$ is *satisfied* in M if $\Gamma^* \subseteq C^*$.

A classical phase space $(M, \perp, K)$ is an intuitionistic phase space (M, Cl, K) with a subset $\perp$ of M in which the closure operator Cl is defined by double negation, i.e., $Cl(X) = X^{\perp\perp} = (X \multimap \perp) \multimap \perp$. In classical phase spaces, we have two additional operations;

- $X \parr Y = (X^{\perp} Y^{\perp})^{\perp}$;
- $?X = (X^{\perp} \cap K)^{\perp}$.

A *naive phase space* M is an intuitionistic phase space (M, Cl) where Cl is the identity function. In a naive phase space M, any subset of M is a fact, $X \otimes Y = XY$ and $X \oplus Y = X \cup Y$. Since $J(M)$ is degenerate $(= \{1\})$, we do not consider enriched naive phase spaces at all. The following examples indicate that naive phase semantics is a natural generalization of traditional semantics such as classical 2-valued semantics and Kripke semantics.

1. Consider a naive phase model whose underlying monoid is the singleton $\{1\}$. Write F, T to denote $\phi, \{1\}$, respectively. Then $A^* \otimes B^* = T$ iff $A^* \,\&\, B^* = T$ iff $A^* = T$ and $B^* = T$; $A^* \oplus B^* = T$ iff $A^* = T$ or $B^* = T$; and $A^* \multimap B^* = T$ iff $A^* = T$ implies $B^* = T$. Hence, this model is a usual 2-valued model for classical logic.

2. Consider (M, v) where M is *idempotent*, i.e., $xx = x$ for any $x \in M$ and v maps each atom to an *ideal* of M, i.e. a subset satisfying $XM \subseteq X$. M can be seen as the set of *possible worlds* with accessibility relation $\leq$ defined by $x \leq y \Leftrightarrow y = xz$ for some $z \in M$. Write $x \models A$ if $x \in A^*$. Then $x \models A \otimes B$ iff $x \models A \,\&\, B$ iff $x \models A$ and $x \models B$; $x \models A \oplus B$ iff $x \models A$ or $x \models B$; and $x \models A \multimap B$ iff for every $y \geq x$, $y \models A$ implies $y \models B$. Hence, (M, v) is a usual Kripke model for intuitionistic logic.

3 Naive Linear Logic and its Completeness with respect to Naive Phase Semantics

Naive linear logic (**NLL**) is a fragment of intuitionistic linear logic with connectives $(\otimes, \multimap, \oplus)$ such that $\multimap$ is restricted to the form $X \multimap \alpha$. More precisely, given a set V of propositional variables, the set L of **NLL** formulas is defined as follows;

$$L ::= V|L \otimes L|L \oplus L|L \multimap V.$$

NLL does not have any constant. However, we will see in the end of this section that the multiplicative additive fragment of classical linear logic (**MALL**) including constants **1** and $\perp$ can be encoded into this simple fragment.

Now we introduce *labelled naive linear logic* (**LNLL**). We presuppose that a countable alphabet Σ is given. The free commutative monoid generated by Σ is denoted by $Com(\Sigma)$. $a, b, c, \ldots \in Com(\Sigma)$ are called *labels* and in particular $x, y, z, \ldots \in \Sigma$ are called *simple labels.* A label a is said to be *linear* if it contains no repetition; for example, xyx is not linear while xyz is linear where $x, y, z \in \Sigma$. The set of linear labels is denoted by $Lin(\Sigma)$. In particular, the unit 1 of $Com(\Sigma)$ is in $Lin(\Sigma)$. We write $a \leq b$ if $b = ac$ for some $c \in Com(\Sigma)$. A *labelled formula* is of the form $a\!:\!A$ where a is a *linear* label and A is a formula of **NLL**. A *labelled sequent* is of the form $\Gamma \vdash \Delta$, where Γ and Δ are finite *multisets* of labelled formulas. Let Γ be $\{a_1\!:\!A_1, \ldots, a_n\!:\!A_n\}$. Then we also write $a_1 \cdots a_n\!:\!\Gamma \vdash \Delta$ to indicate the total amount of labels occuring in Γ (in particular, we write $1\!:\!\Gamma' \vdash \Delta$ if $\Gamma' = \phi$). We say that b *occurs in* Γ if $b \leq a_i$ for some i, and also say that b is *unique in* Γ if b occurs in $\{a_i\!:\!A_i\}$ for exactly one i. The similar conventions also apply to Δ.

A labelled sequent $a_1\!:\!A_1, \ldots, a_n\!:\!A_n \vdash c\!:\!C$ is said to be *strict* if $c = a_1 \cdots a_n$ and $a_i \neq 1$ for any i. In particular, a sequent of the form $\vdash c\!:\!C$ is strict iff $c = 1$.

The formulas and the sequents of **LNLL** are labelled formulas and labelled sequents, respectively. The inference rules of **LNLL** are those in Figure 1.

It is assumed that each inference rule should preserve the linearity of the labels; for example, the following inference is not allowed in **LNLL**;

$$\frac{\Gamma_1 \vdash a\!:\!A \quad \Gamma_2 \vdash a\!:\!B}{\Gamma_1, \Gamma_2 \vdash aa\!:\!A \otimes B} .$$

Lemma 1 *Given a proof π of* **LNLL** *sequent $\Gamma \vdash \Delta$, one can construct a proof π' of strict* **LNLL** *sequent $c\!:\!\Gamma_0 \vdash c\!:\!C_0$, where $c\!:\!\Gamma_0 \subseteq \Gamma$ and $c\!:\!C_0 \in \Delta$, such that each sequent occuring in π' is strict.*

Proof. By induction on the length of π. ∎

Proposition 1 *If a strict* **LNLL** *sequent $c\!:\!\Gamma \vdash c\!:\!C$ is provable in* **LNLL**, *then $\Gamma \vdash C$ is provable in* **NLL**.

Proof. Given a proof π of strict **LNLL** sequent $c\!:\!\Gamma \vdash c\!:\!C$, we can obtain another proof π' of the same sequent in which each sequent is strict by Lemma 1. Such a proof is easily transformed into a proof of $\Gamma \vdash C$ in **NLL** by dropping all the labels occurring in it. ∎

Next, we enrich the labelled sequents with *tags* which express additional information needed to define a suitable proof search procedure. A *tagged sequent* is of the form $< (\Gamma \vdash \Delta), \Sigma^1, \Sigma^2, \Sigma^3 >$ where $\Gamma \vdash \Delta$ is a labelled sequent, Σ^1 is a finite multiset of *atomic* labelled formulas such that $\Sigma^1 \subseteq \Gamma$, and Σ^2 and Σ^3 are finite multisets of labelled formulas. $< (\Gamma \vdash \Delta), \Sigma^1, \Sigma^2, \Sigma^3 >$ is often denoted by $< (\Gamma \vdash \Delta), \overline{\Sigma} >$. If Π is a multiset of labelled formulas such that $\Sigma^1 \subseteq \Pi$, the difference of Π and Σ^1 is denoted by Π^-.

$< (\Gamma \vdash \Delta), \Sigma^1, \Sigma^2, \Sigma^3 >$ is *regular* if

Identity and Cut:

$$\frac{}{a{:}A \vdash a{:}A}\ \textit{Identity} \qquad \frac{\Gamma_1 \vdash \Delta_1, a{:}A \quad a{:}A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textit{Cut}$$

Structural Rules:

$$\frac{\Gamma \vdash \Delta}{a{:}A, \Gamma \vdash \Delta}\ Wl \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, a{:}A}\ Wr \qquad \frac{a{:}A, a{:}A, \Gamma \vdash \Delta}{a{:}A, \Gamma \vdash \Delta}\ Cl \qquad \frac{\Gamma \vdash \Delta, a{:}A, a{:}A}{\Gamma \vdash \Delta, a{:}A}\ Cr$$

Loose Rules:

$$\frac{\Gamma_1 \vdash \Delta_1, a{:}A \quad \Gamma_2 \vdash \Delta_2, b{:}B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, ab{:}A \otimes B}\ \otimes r \qquad \frac{\Gamma_1 \vdash \Delta_1, a{:}A \quad ab{:}\alpha, \Gamma_2 \vdash \Delta_2}{b{:}A \multimap \alpha, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ {\multimap} l$$

$$\frac{\Gamma \vdash \Delta, a{:}A}{\Gamma \vdash \Delta, a{:}A \oplus B}\ \oplus r_1 \qquad \frac{\Gamma \vdash \Delta, a{:}B}{\Gamma \vdash \Delta, a{:}A \oplus B}\ \oplus r_2$$

Strict Rules:

$$\frac{a{:}A, \Gamma \vdash \Delta, ab{:}\alpha}{\Gamma \vdash \Delta, b{:}A \multimap \alpha}\ {\multimap} r$$

where a does not occur in Γ and Δ.

$$\frac{a{:}A, b{:}B, \Gamma \vdash \Delta[x := ab]}{x{:}A \otimes B, \Gamma \vdash \Delta}\ \otimes l$$

where a and b do not occur in Γ and Δ, simple label x does not occur in Γ and is unique in Δ. $[x := ab]$ means the usual substitution operation.

$$\frac{a{:}A, \Gamma \vdash \Delta \quad a{:}B, \Gamma \vdash \Delta}{a{:}A \oplus B, \Gamma \vdash \Delta}\ \oplus l$$

where a does not occur in Γ and is unique in Δ. Moreover, each formula in Γ is labelled with a simple label distinct from each other.

Figure 1: Inference Rules of Labelled Naive Linear Logic

($\theta 1$) For each $a:\alpha \in \Sigma^1$, there are $a_1:A \in \Sigma^2$ and $a_2:A \multimap \alpha \in \Gamma^-$ such that $a = a_1a_2$.

($\theta 2$) Each formula in Γ^- is labelled with a simple label distinct from each other.

($\theta 3$) If Γ contains a formula of the form $a:A \otimes B$ or $a:A \oplus B$, then a is unique in Γ and also unique in Δ.

($\theta 4$) $\Gamma^- \vdash \Delta, c:C$ is provable for any $c:C \in \Sigma^2$.

Lemma 2 *Let* $< (\Gamma \vdash \Delta), \Sigma^1, \Sigma^2, \Sigma^3 >$ *be a regular tagged sequent. Then* $\Gamma^- \vdash \Delta$ *is derivable from* $\Gamma \vdash \Delta$.

Proof. $\Gamma \vdash \Delta$ is of the form $\Sigma^1, \Gamma^- \vdash \Delta$ and $\Sigma^1 = \{a_1:\alpha_1, \ldots, a_n:\alpha_n\}$. By ($\theta 1$), there are $a_{11}:A_1 \in \Sigma^2$ and $a_{12}:A_1 \multimap \alpha_1 \in \Gamma^-$ such that $a_1 = a_{11}a_{12}$. By ($\theta 4$), $\Gamma^- \vdash \Delta, a_{11}:A_1$ is provable. Hence,

$$\dfrac{\dfrac{\Gamma^- \vdash \Delta, a_{11}:A_1 \quad a_1:\alpha_1, \ldots, a_n:\alpha_n, \Gamma^- \vdash \Delta}{a_{12}:A_1 \multimap \alpha_1, a_2:\alpha_2, \ldots, a_n:\alpha_n, \Gamma^-, \Gamma^- \vdash \Delta, \Delta}}{a_2:\alpha_2, \ldots, a_n:\alpha_n, \Gamma^- \vdash \Delta}$$

By repeating this process n times, we can eliminate all $a_i:\alpha_i$'s. ■

Definition 2 We assume a fixed well-ordering $\prec_L$ on the labels and also assume a fixed well-ordering $\prec_F$ on the labelled formulas. A labelled formula of the form $a:A \multimap \alpha$ is called a $\multimap$-*formula*. $\otimes$-*formulas* and $\oplus$-*formulas* are defined similarly.

Let σ be a function which maps a tagged sequent to either a finite set of tagged sequents or symbol $*$, defined as follows;

(i) If $< S, \overline{\Sigma} >$ is irregular, then $\sigma(< S, \overline{\Sigma} >) = *$;

(ii) else if $S \equiv a:A, \Gamma \vdash \Delta, a:A$, then $\sigma(< S, \overline{\Sigma} >) = *$;

(iii) else if $S \equiv \Gamma \vdash \Delta, d:A \multimap \alpha$ and no $\multimap$-formula $\prec_F$-smaller than $d:A \multimap \alpha$ is in Δ, then $\sigma(< S, \overline{\Sigma} >) = \{< (x:A, \Gamma \vdash \Delta, xd:\alpha), \overline{\Sigma} >\}$, where x is the $\prec_L$-smallest simple label not occuring in Γ and Δ;

(iv) else if $S \equiv d:A \otimes B, \Gamma \vdash \Delta$ and no $\otimes$-formula $\prec_F$-smaller than $d:A \otimes B$ is in Γ, then d is simple (otherwise $< S, \overline{\Sigma} >$ would be irregular by ($\theta 3$)), so let $\sigma(< S, \Sigma^1, \Sigma^2, \Sigma^3 >) = \{< (x:A, y:B, \Gamma \vdash \Delta[d := xy]), \Sigma^1, \Sigma^2, \Sigma^3[d := xy] >\}$, where x and y are the two $\prec_L$-smallest simple labels not occurring in Γ and Δ;

(v) else if $S \equiv a:A \oplus B, \Gamma \vdash \Delta$ and no $\oplus$-formula $\prec_F$-smaller than $a:A \oplus B$ is in Γ, then $\sigma(< S, \overline{\Sigma} >) = \{< (a:A, \Gamma \vdash \Delta), \overline{\Sigma} >, < (a:B, \Gamma \vdash \Delta), \overline{\Sigma} >\}$;

(vi) else if (1) $S \equiv d:A \multimap \alpha, \Gamma \vdash \Delta$, (2) ad occurs in Δ, (3) $ad:\alpha \notin \Gamma$ and $a:A \notin \Sigma^3$, and ($d:A \multimap \alpha$, a) is the smallest pair (according to $\prec_F$ and $\prec_L$) satisfying conditions (1)–(3), then

$$\begin{aligned} \sigma(< S, \Sigma^1, \Sigma^2, \Sigma^3 >) \quad = \quad & \{< (d:A \multimap \alpha, \Gamma \vdash \Delta, a:A), \Sigma^1, \Sigma^2, \Sigma^3 \cup \{a:A\} >, \\ & < (ad:\alpha, d:A \multimap \alpha, \Gamma \vdash \Delta), \Sigma^1 \cup \{ad:\alpha\}, \Sigma^2 \cup \{a:A\}, \Sigma^3 >\}; \end{aligned}$$

(vii) else if (1) $S \equiv \Gamma \vdash \Delta, d\!:\!A \otimes B$, (2) $d = ab$, (3) $a\!:\!A \notin \Sigma^3$ and $b\!:\!B \notin \Sigma^3$, and $(d\!:\!A \otimes B, a)$ is the smallest pair (according to $\prec_F$ and $\prec_L$) satisfying conditions (1)–(3), then

$$\begin{aligned}\sigma(< S, \Sigma^1, \Sigma^2, \Sigma^3 >) &= \{< (\Gamma \vdash \Delta, ab\!:\!A \otimes B, a\!:\!A), \Sigma^1, \Sigma^2, \Sigma^3 \cup \{a\!:\!A\} >, \\ &\quad < (\Gamma \vdash \Delta, ab\!:\!A \otimes B, b\!:\!B), \Sigma^1, \Sigma^2, \Sigma^3 \cup \{b\!:\!B\} >\};\end{aligned}$$

(viii) else if $S \equiv \Gamma \vdash \Delta, a\!:\!A \oplus B$ and no $\oplus$-formula $\prec_L$-smaller than $a\!:\!A \oplus B$ is in Δ, then $\sigma(< S, \overline{\Sigma} >) = \{< (\Gamma \vdash \Delta, a\!:\!A, a\!:\!B), \overline{\Sigma} >\}$;

(ix) else $\sigma(< S, \overline{\Sigma} >) = \phi$.

For any $< S_0, \overline{\Sigma_0} >$, σ induces a rooted tree $\mathcal{T}_\sigma(< S_0, \overline{\Sigma_0} >)$ (called a *proof search tree*) each node of which is labelled with a tagged sequent, constructed as follows;

1. The root of $\mathcal{T}_\sigma(< S_0, \overline{\Sigma_0} >)$ is labelled with $< S_0, \overline{\Sigma_0} >$;
2. If a node x is labelled with $< S, \overline{\Sigma} >$ and $\sigma(< S, \overline{\Sigma} >) = *$, then x has a child node which is labelled with $*$ and is a leaf of $\mathcal{T}_\sigma(< S_0, \overline{\Sigma_0} >)$;
3. If a node x is labelled with $< S, \overline{\Sigma} >$ and $\sigma(< S, \overline{\Sigma} >) = \{< S_1, \overline{\Sigma^1} >, \ldots, < S_k, \overline{\Sigma_k} >\}$ (where k is 0 or 1 or 2), then x has k children nodes $x_1, \ldots, x_k$ and each x_i is labelled with $< S_i, \overline{\Sigma_i} >$ (in particular, x is a leaf of $\mathcal{T}_\sigma(< S_0, \overline{\Sigma_0} >)$ if $k = 0$).

A *branch* of a proof search tree is either a path from the root to a leaf or an infinite sequence of nodes in the tree such that every initial segment of it is a path from the root. A branch of a proof search tree is said to be *closed* if it is a finite path $x_0, \ldots, x_n$ and x_n is labelled with $*$; otherwise a branch is said to be *open*.

Lemma 3 *$\mathcal{T}_\sigma(< S_0, \overline{\Sigma_0} >)$ is finite for any $< S_0, \overline{\Sigma_0} >$.*

Lemma 4 *Let $< S_0, \overline{\Sigma_0} >$ be regular. If S_0 is unprovable, then $\mathcal{T}_\sigma(< S_0, \overline{\Sigma_0} >)$ has an open branch.*

Proof. Let S be unprovable and $< S, \overline{\Sigma} >$ be regular. Then $\sigma(< S, \overline{\Sigma} >) \neq *$. Hence it suffices to show that if $\sigma(< S, \overline{\Sigma} >) = \{< S_1, \overline{\Sigma^1} >, \ldots, < S_k, \overline{\Sigma_k} >\}$, then $< S_i, \overline{\Sigma_i} >$ is regular and S_i is unprovable for some i.

It is easily shown that (θ1)–(θ3) hold for every $< S_i, \overline{\Sigma_i} >$. Hence it suffices to show that $< S_i, \overline{\Sigma_i} >$ satisfies (θ4) and S_i is unprovable for some i. We only prove the two essential cases;

(v) $S \equiv a\!:\!A \oplus B, \Gamma \vdash \Delta$ and $\sigma(< S, \overline{\Sigma} >) = \{< S_1, \overline{\Sigma} >, < S_2, \overline{\Sigma} >\}$, where $S_1 \equiv a\!:\!A, \Gamma \vdash \Delta$ and $S_2 \equiv a\!:\!B, \Gamma \vdash \Delta$. First we prove that both $< S_1, \overline{\Sigma} >$ and $< S_2, \overline{\Sigma} >$ satisfies (θ4). Let $c\!:\!C \in \Sigma^2$. Then by the assumption $a\!:\!A \oplus B, \Gamma^- \vdash \Delta, c\!:\!C$ is provable. Hence,

$$\dfrac{\dfrac{a\!:\!A \vdash a\!:\!A}{a\!:\!A \vdash a\!:\!A \oplus B} \quad a\!:\!A \oplus B, \Gamma^- \vdash \Delta, c\!:\!C}{a\!:\!A, \Gamma^- \vdash \Delta, c\!:\!C}.$$

The same holds for $a : B, \Gamma^- \vdash \Delta, c : C$. Hence both $< S_1, \overline{\Sigma} >$ and $< S_2, \overline{\Sigma} >$ are regular.

Now we prove that either S_1 or S_2 is unprovable. Suppose that both S_1 and S_2 are provable. Then by the regularity of $< S_1, \overline{\Sigma} >$ and $< S_2, \overline{\Sigma} >$ and by Lemma 2,

$$\dfrac{\dfrac{\dfrac{a:A,\Gamma\vdash\Delta}{a:A,\Gamma^-\vdash\Delta}\quad\dfrac{a:B,\Gamma\vdash\Delta}{a:B,\Gamma^-\vdash\Delta}}{a:A\oplus B,\Gamma^-\vdash\Delta}}{a:A\oplus B,\Gamma\vdash\Delta}\quad ,$$

which contradicts the assumption.

(vi) $S\equiv d:A\multimap\alpha,\Gamma\vdash\Delta$ and

$$\begin{aligned}\sigma(<S,\Sigma^1,\Sigma^2,\Sigma^3>) &= \{<S_1,\Sigma^1,\Sigma^2,\Sigma^3\cup\{a:A\}>,\\ &\quad <S_2,\Sigma^1\cup\{ad:\alpha\},\Sigma^2\cup\{a:A\},\Sigma^3>\},\end{aligned}$$

where $S_1\equiv d:A\multimap\alpha,\Gamma\vdash\Delta,a:A$ and $S_2\equiv ad:\alpha,d:A\multimap\alpha,\Gamma\vdash\Delta$.

$<S_1,\Sigma^1,\Sigma^2,\Sigma^3\cup\{a:A\}>$ satisfies ($\theta 4$) by Wr rule. Hence if S_1 is unprovable, our claim holds. Suppose that S_1 is provable. Then S_2 should be unprovable by the assumption, $\multimap l$ rule and Cl rule. Moreover, $<S_2,\Sigma^1\cup\{ad:\alpha\},\Sigma^2\cup\{a:A\},\Sigma^3>$ satisfies ($\theta 4$); $d:A\multimap\alpha,\Gamma^-\vdash\Delta,c:C$ is provable by the assumption for $c:C\in\Sigma^2$, and $d:A\multimap\alpha,\Gamma^-\vdash\Delta,a:A$ is also provable because S_1 is provable and $<S_1,\Sigma^1,\Sigma^2,\Sigma^3\cup\{a:A\}>$ is regular, hence

$$\dfrac{d:A\multimap\alpha,\Gamma\vdash\Delta,a:A}{d:A\multimap\alpha,\Gamma^-\vdash\Delta,a:A}$$

by Lemma 2.

■

Let $\mathcal{R}$ be a open branch. By Lemma 3, $\mathcal{R}$ is a finite path, say, of length n. $\mathcal{R}$ can be represented as;

$$\mathcal{R}\equiv<S_0,\Sigma^1_0,\Sigma^2_0,\Sigma^3_0>,<S_1,\Sigma^1_1,\Sigma^2_1,\Sigma^3_1>,\ldots,<S_n,\Sigma^1_n,\Sigma^2_n,\Sigma^3_n>.$$

From now on, we will not use the second and the third component (Σ^1_i and Σ^2_i) of each tagged sequent (the fourth component Σ^3_i will be used to prove the next lemma). Hence we consider the following sequence

$$\mathcal{R}'=<(\Gamma_0\vdash\Delta_0),\Sigma_0>,<(\Gamma_1\vdash\Delta_1),\Sigma_1>,\ldots,<(\Gamma_n\vdash\Delta_n),\Sigma_n>,$$

where $\Gamma_i\vdash\Delta_i\equiv S_i$ and $\Sigma_i\equiv\Sigma^3_i$.

$\mathcal{R}'$ may contain the following subsequence;

$$\ldots,<(z:A\otimes B,\Gamma\vdash\Delta),\Sigma>,<(x:A,y:B,\Gamma\vdash\Delta[z:=xy]),\Sigma[z:=xy]>,\ldots.$$

To correlate the resource information in the first tagged sequent above with that in the second one, we would like to make a relabelling operation on the labels occurring in $\mathcal{R}'$ to obtain the following sequence;

$$\ldots,<(xy:A\otimes B,\Gamma\vdash\Delta[z:=xy]),\Sigma[z:=xy]>,<(x:A,y:B,\Gamma\vdash\Delta[z:=xy]),\Sigma[z:=xy]>,\ldots.$$

The process of relabelling is described below.

For each $0 \leq j \leq n$, we define a finite sequence $\mathcal{R}^j$ of the form $< (\Gamma_0^j \vdash \Delta_0^j), \Sigma_0^j >, \ldots, < (\Gamma_j^j \vdash \Delta_j^j), \Sigma_j^j >$ of length $j+1$, as follows;

- $\mathcal{R}^0 \equiv < (\Gamma_0 \vdash \Delta_0), \Sigma_0 >$;
- If $S_j \equiv z{:}A \otimes B, \Gamma \vdash \Delta$ and $S_{j+1} \equiv a{:}A, b{:}B \vdash \Delta[z := xy]$, then $\mathcal{R}^{j+1} \equiv < (\Gamma_0^j \vdash \Delta_0^j[z := xy]), \Sigma_0^j[z := xy] >, \ldots, < (\Gamma_j^j \vdash \Delta_j^j[z := xy]), \Sigma_j^j[z := xy] >, < (\Gamma_{j+1} \vdash \Delta_{j+1}), \Sigma_{j+1} >$;
- otherwise $\mathcal{R}^{j+1} \equiv < (\Gamma_0^j \vdash \Delta_0^j), \Sigma_0^j >, \ldots, < (\Gamma_j^j \vdash \Delta_j^j), \Sigma_j^j >, < (\Gamma_{j+1} \vdash \Delta_{j+1}), \Sigma_{j+1} >$.

Now we have sequence $\mathcal{R}^n$ of length n. Let $\Gamma_{\mathcal{R}}$ be $\bigcup_{0 \leq j \leq n} \Gamma_j^n$ and $\Delta_{\mathcal{R}}$ be $\bigcup_{0 \leq j \leq n} \Delta_j^n$. The following lemma is checked by induction on the construction of $\mathcal{R}^j$;

Lemma 5

(1) $\Gamma_{\mathcal{R}}$ and $\Delta_{\mathcal{R}}$ are disjoint;

(2) If $a : A \otimes B \in \Gamma_{\mathcal{R}}$, then there are some b, c such that $b : A \in \Gamma_{\mathcal{R}}$ and $c : B \in \Gamma_{\mathcal{R}}$ and $a = bc$;

(3) If $a{:}A \otimes B \in \Delta_{\mathcal{R}}$, then for any b, c such that $a = bc$, either $b{:}A \in \Delta_{\mathcal{R}}$ or $c{:}B \in \Delta_{\mathcal{R}}$;

(4) If $a : A \multimap \alpha \in \Gamma_{\mathcal{R}}$, then for any b such that ab occurs in $\Delta_{\mathcal{R}}$, either $b : A \in \Delta_{\mathcal{R}}$ or $ab{:}\alpha \in \Gamma_{\mathcal{R}}$;

(5) If $a{:}A \multimap \alpha \in \Delta_{\mathcal{R}}$, then there is b such that $b{:}A \in \Gamma_{\mathcal{R}}$ and $ab{:}\alpha \in \Delta_{\mathcal{R}}$;

(6) If $a{:}A \oplus B \in \Gamma_{\mathcal{R}}$, then either $a{:}A \in \Gamma_{\mathcal{R}}$ or $a{:}B \in \Gamma_{\mathcal{R}}$;

(7) If $a{:}A \oplus B \in \Delta_{\mathcal{R}}$, then $a{:}A \in \Delta_{\mathcal{R}}$ and $a{:}B \in \Delta_{\mathcal{R}}$.

We define naive phase model $\mathcal{M}_{\mathcal{R}} = (M, v)$ by

- $M = Com(\Sigma)$;
- $v(\alpha) = \{a | a{:}\alpha \notin \Delta_{\mathcal{R}}\}$.

Proposition 2 *For any* **NLL** *formula A, the following hold;*
(a) If $a{:}A \in \Gamma_{\mathcal{R}}$, then $a \in A^*$ in $\mathcal{M}_{\mathcal{R}}$;
(b) If $a{:}A \in \Delta_{\mathcal{R}}$, then $a \notin A^*$ in $\mathcal{M}_{\mathcal{R}}$.

Proof. By induction on the complexity of A.

(Case 1) A is an atomic formula α. (b) is by definition. As for (a), suppose that $a{:}\alpha \in \Gamma_{\mathcal{R}}$ and $a \notin \alpha^*$. The latter means that $a{:}\alpha \in \Delta_{\mathcal{R}}$, which is impossible by Lemma 5(1).

(Case 2) A is of the form $B \otimes C$. To show (a), suppose that $a : B \otimes C \in \Gamma_R$. Then for some b and c, $b{:}B \in \Gamma_{\mathcal{R}}$ and $c{:}C \in \Gamma_{\mathcal{R}}$ and $a = bc$ by Lemma 5(2), hence by IH, $b \in B^*$ and $c \in C^*$, thus $a = bc \in B^* \otimes C^*$.

As for (b), note that $a \notin B^* \otimes C^*$ iff for any b and c such that $bc = a$, either $b \notin B^*$ or $c \notin C^*$. Suppose that $a{:}B \otimes C \in \Delta_{\mathcal{R}}$ and $a = bc$. Then either $b{:}B \in \Delta_R$ or $c{:}C \in \Delta_R$ by

Lemma 5(3). Hence by IH, either $b \notin B^*$ or $c \notin C^*$, so the claim holds.

(Case 3) A is of the form $B \multimap \alpha$. To show (a), suppose that $a{:}B \multimap \alpha \in \Gamma_R$ and $b \in B^*$. If ab does not occur in $\Delta_{\mathcal{R}}$, then $ab \in \alpha^*$ by definition. Otherwise, ab occurs in $\Delta_{\mathcal{R}}$ and by Lemma 5(4), either $b{:}B \in \Delta_{\mathcal{R}}$ or $ab{:}\alpha \in \Gamma_{\mathcal{R}}$. However, the former is impossible by IH(b), hence $ab \in \alpha^*$, so the claim holds.

As for (b), if $a{:}B \multimap \alpha \in \Delta_{\mathcal{R}}$ then $b{:}B \in \Gamma_{\mathcal{R}}$ and $ab{:}\alpha \in \Delta_{\mathcal{R}}$ for some b by Lemma 5(5). By IH, $b \in B^*$ and $ab \notin \alpha^*$. Therefore, $a \notin B^* \multimap \alpha^*$.

(Case 4) A is of the form $B \oplus C$. Similarly shown using Lemma 5(6) and (7). ■

Theorem 1 *For any* **NLL** *sequent* $\Gamma \vdash C$*, the following are equivalent;*

1. $\Gamma \vdash C$ *is provable in* **NLL***;*
2. $\Gamma \vdash C$ *is satisfied in all naive phase models;*
3. $a{:}\Gamma \vdash a{:}C$ *is provable in* **LNLL** *for some linear label* a*.*

Proof. 1 implies 2 by the usual soundness argument. 3 implies 1 by Proposition 1. To show that 2 implies 3, suppose that $a_1 : A_1, \ldots, a_l : A_l \vdash a_1 \cdots a_l : C$ is unprovable for any $a_1, \ldots, a_l$ where $\Gamma \equiv A_1, \ldots, A_l$. Let $x_1, \ldots, x_l$ be distinct simple labels. Then $S_0 \equiv x_1 : A_1, \ldots, x_l : A_n \vdash x_1 \cdots x_l : C$ is unprovable and $< S_0, \phi, \phi, \phi >$ is regular. Hence by Lemma 4, $\mathcal{T}_\sigma(< S_0, \phi, \phi, \phi >)$ has an open branch $\mathcal{R}$. By the construction described before, we get sequence

$$\mathcal{R}^n \equiv < (b_1{:}A_1, \ldots, b_l{:}A_l \vdash b_1 \cdots b_l{:}C), \Sigma_0^n >, \ldots,$$

from which naive phase model $\mathcal{M}_{\mathcal{R}}$ is constructed. By Proposition 2, $b_i \in A_i^*$ for $1 \leq i \leq l$ and $b_1 \cdots b_l \notin C^*$, i.e., $\Gamma \vdash C$ is not satisfied in $\mathcal{M}_{\mathcal{R}}$, that contradicts 2. ■

The multiplicative additive fragment of classical linear logic (**MALL**) can be encoded into **NLL** by the following Kolmogorov-Gödel style double negation interpretation.

Definition 3

1. Let us fix an atomic formula α_0 and assume that no **MALL** formula contains α_0. Then, given an **MALL** formula A, an **NLL** formula A° is defined as follows;

$$\begin{array}{rcl}
1^\circ & = & \alpha_0 \\
\bot^\circ & = & \alpha_0 \multimap \alpha_0 \\
\beta^\circ & = & \beta \multimap \alpha_0 \\
(\beta^\perp)^\circ & = & \beta \\
(B \otimes C)^\circ & = & ((B^\circ \multimap \alpha_0) \otimes (C^\circ \multimap \alpha_0)) \multimap \alpha_0 \\
(B \parr C)^\circ & = & B^\circ \otimes C^\circ \\
(B \oplus C)^\circ & = & ((B^\circ \multimap \alpha_0) \oplus (C^\circ \multimap \alpha_0)) \multimap \alpha_0 \\
(B \,\&\, C)^\circ & = & B^\circ \oplus C^\circ
\end{array}$$

2. $A^\bullet$ is defined to be $A^\circ \multimap \alpha_0$.

Proposition 3 *A is provable in* **MALL** *iff $A^{\bullet}$ is provable in* **NLL**.

Proof. The Only-if-part is shown by induction on the length of proof. The If-part is by induction on the complexity of A. ■

Classical phase models are obtained from naive phase models by the double negation closure condition, which precisely corresponds to the above syntactic double negation translation. Hence the following proposition is almost immediate.

Proposition 4 *A is satisfied in all classical phase models iff $A^{\bullet}$ is satisfied in all naive phase models.*

As a direct corollary of Theorem 1, Proposition 3 and Proposition 4, we have

Corollary 1 **MALL** *is complete with respect to classical phase models.*

4 A Completeness Proof for Full Intuitionistic and Classical Linear Logics Based on the Proof Search Method

Now we move on to the problem if or not the proof search method can be extended to the full systems of intuitionistic linear logic (**ILL**) and classical linear logic (**LL**). As discussed in section 1, the standard countermodel construction from one open branch does not work for **ILL** and **LL**. Hence, in this section, we generalize the notion of branch in a proof search tree to the notion of *OR-branching tree* or simply *OR-tree*, and show that given an unprovable sequent $\Gamma \vdash C$, one can always find an *open* OR-tree, which is considered to retain enough information to refute the sequent (in §4.1). Then we describe how to construct an intuitionistic phase model from an open OR-tree of **ILL** (in §4.2) and a classical phase model from that of **LL** (in §4.3). These countermodel constructions give the completeness theorem (and the cut-elimination theorem as a corollary) both for **ILL** and for **LL**. In both cases, the closure condition of phase semantics plays an essential role.

4.1 OR-branching trees

Let **L** be an arbitrary sequent-based inference system of a logic, and S_0 be an **L**-sequent. An *OR-branching tree* (or, simply *OR-tree*) of S_0 in **L** is a rooted tree each node of which is labelled with an **L**-sequent, satisfying the following;

(1) The root is labelled with S_0;

(2) If a node x is labelled with S, and

(i) if no rule can be applied (bottom-up) to S, then x is a leaf of $\mathcal{R}$;

(ii) otherwise, let

$$\frac{S_1^1,\ldots,S_{m_1}^1}{S}, \frac{S_1^2,\ldots,S_{m_2}^2}{S}, \ldots, \frac{S_1^i,\ldots,S_{m_i}^i}{S}, \ldots$$

be the enumeration of all instances of inference rules of **L** that can be applied (bottom-up) to S. Then, x has children nodes $x^1, x^2, \ldots$ and each x^i is labelled with S_j^i for some $1 \leq j \leq m_i$.

An OR-tree is *open* if no node in it is labelled with an axiom.

Proposition 5 *S_0 is provable in* **L** *if and only if there is no open OR-tree of S_0.*

Proof. Assume that S_0 has a proof π in **L**. We show that if $\mathcal{R}$ is an OR-tree of S_0, then $\mathcal{R}$ contains at least one axiom by induction on length of π. If S_0 is an axiom, then the claim is trivial. Suppose that the last part of π is of the form

$$\frac{S_1, \ldots, S_n}{S_0} \quad (n \geq 1).$$

Since $\mathcal{R}$ is an OR-tree, $\mathcal{R}$ should contain some S_i $(1 \leq i \leq n)$ as a child of S_0. But by IH the sub-OR-tree $\mathcal{R}$' of which the root is S_i contains an axiom, hence so does $\mathcal{R}$.

To show the reverse, observe that if S' is not provable in **L** and

$$\frac{S_1, \ldots, S_n}{S'} \quad (n \geq 1)$$

is an instance of an inference rule of **L**, then at least one of S_i $(1 \leq i \leq n)$ is unprovable. Therefore, by choosing such an unprovable sequent at each stage of OR-tree construction (2ii), we can obtain an OR-tree in which each node is labelled with an unprovable sequent. In particular, such an OR-tree contains no axiom. ■

If $\mathcal{R}$ is an OR-tree, let $|\mathcal{R}|$ be the set $\{S | S \text{ is a label of a node in } \mathcal{R}\}$, and $\mathcal{R}_l^*$ be $\{\Delta | \Delta, \Pi \vdash C \in |\mathcal{R}| \text{ for some } \Pi \text{ and } C\}$.

4.2 Countermodel Construction for Intuitionistic Linear Logic

In this subsection, we consider the case of **ILL** and describe how to construct a intuitionistic phase model from a given open OR-tree.

Let $\mathcal{R}$ be an open OR-tree in *cut-free* **ILL**. Based on $\mathcal{R}$, we define an intuitionistic phase model $\mathcal{M}_{\mathcal{R}} = (M, Cl, K, v)$ as follows.

- $M = \mathcal{R}_l^* \cup \{\surd\}$, where $\surd$ is a distinguished formula not occurring in $\mathcal{R}$. Note that the empty sequence $\emptyset$ is always in M.

- For each $\Gamma \in M$ and $\Delta \in M$,
  $\Gamma \cdot \Delta = \begin{cases} \Gamma, \Delta & \text{if } \Gamma, \Delta \in \mathcal{R}_l^* \\ \surd & \text{otherwise.} \end{cases}$
  In particular, $\surd \cdot \Gamma = \surd$ for any $\Gamma \in M$. It is immediate that $< M, \cdot, \emptyset >$ forms a commutative monoid.

- Let $[\![\Gamma \vdash C]\!]$ be $\{\Sigma \in M \mid \Sigma, \Gamma \vdash C \notin |\mathcal{R}|\}$. We write $[\![C]\!]$ to denote $[\![\vdash C]\!]$.

- For $X \subseteq M$, $Cl(X) = \bigcap\{[\![\Gamma \vdash C]\!] \mid X \subseteq [\![\Gamma \vdash C]\!], \Gamma \vdash C \text{ is a sequent of } \mathbf{ILL}\}$. Then clearly $Cl([\![\Gamma \vdash C]\!]) = [\![\Gamma \vdash C]\!]$. These facts are called *base facts* of $\mathcal{M}_{\mathcal{R}}$.

- $K = \{!\Delta \mid !\Delta \in M\} \cup \{\emptyset, \surd\}$.

- $v(\alpha) = [\![\alpha]\!]$ for each atomic formula α.

Lemma 6 *The operator Cl defined above is actually a closure operator.*

Lemma 7 *Each fact $X = Cl(X)$ satisfies the follwing properties;*

(i) $\surd \in X$;

(ii) if $A, B, \Gamma \in X$ *and* $A \otimes B, \Gamma \in M$, *then* $A \otimes B, \Gamma \in X$;

(iii) if $A, \Gamma \in X$, $B, \Gamma \in X$, *and* $A \oplus B, \Gamma \in M$, *then* $A \oplus B, \Gamma \in X$;

(iii)' if $A, \Gamma \in X$, $B, \Gamma \notin M$, *and* $A \oplus B, \Gamma \in M$, *then* $A \oplus B, \Gamma \in X$;

(iii)" if $A, \Gamma \notin M$, $B, \Gamma \in X$, *and* $A \oplus B, \Gamma \in M$, *then* $A \oplus B, \Gamma \in X$;

(iv) if either $A, \Gamma \in X$ *or* $B, \Gamma \in X$, *and* $A \& B, \Gamma \in M$, *then* $A \& B, \Gamma \in X$;

(v) if $B, \Gamma \in X$, $\Delta \vdash A \notin |\mathcal{R}|$, *and* $\Delta, A \multimap B, \Gamma \in M$, *then* $\Delta, A \multimap B, \Gamma \in X$;

(vi) if $A, \Gamma \in X$ *and* $!A, \Gamma \in M$, *then* $!A, \Gamma \in X$;

(vii) if $!A, !A, \Gamma \in X$, *then* $!A, \Gamma \in X$;

(viii) if $\Gamma \in X$ *and* $!A, \Gamma \in M$, *then* $!A, \Gamma \in X$.

Proof. It suffices to show that the properties hold for each base fact $[\![\Delta \vdash C]\!]$, since the above properties are preserved by arbitrary intersection.

As for (ii), for example, suppose $A, B, \Gamma \in [\![\Delta \vdash C]\!]$, that means $A, B, \Gamma, \Delta \vdash C \notin |\mathcal{R}|$. Since

$$\frac{A, B, \Gamma, \Delta \vdash C}{A \otimes B, \Gamma, \Delta \vdash C}$$

is an instance of an inference rule of *cut-free* **ILL**, $A \otimes B, \Gamma, \Delta \vdash C$ is not in $|\mathcal{R}|$ by the definition of OR-trees, hence $A \otimes B, \Gamma \in [\![\Delta \vdash C]\!]$. The other properties are shown similarly.■

Proposition 6 *In $\mathcal{M}_{\mathcal{R}}$, the following hold;*

(a) if A is in $\mathcal{R}_l^*$, then $A \in A^*$;

(b) if $\Gamma \vdash A \in |\mathcal{R}|$, then $\Gamma \notin A^*$.

Proof. We prove the following equivalent form (b') instead of (b);

(b') for any A, $A^* \subseteq [\![A]\!]$.

The proof is carried out by induction on the complexity of A.

(Case 1) A is an atomic formula. $A \in [\![A]\!] = A^*$ since $|\mathcal{R}|$ contains no axiom. (b') is by definition.

(Case 2) A is of the form $B \otimes C$. As for (a), $B \in B^*$ and $C \in C^*$ by IH (induction hypotheses). Hence $B, C \in B^*C^*$. Therefore, by Lemma 7(ii), $B \otimes C \in B^* \otimes C^*$.

As for (b'), $B^* \subseteq [\![B]\!]$ and $C^* \subseteq [\![C]\!]$ by IH, hence $B^*C^* \subseteq [\![B]\!][\![C]\!]$. To show $[\![B]\!][\![C]\!] \subseteq [\![B \otimes C]\!]$, suppose that $\Gamma_1 \in [\![B]\!]$ and $\Gamma_2 \in [\![C]\!]$, that mean $\Gamma_1 \vdash B \notin |\mathcal{R}|$ and $\Gamma_2 \vdash C \notin |\mathcal{R}|$. Since

$$\frac{\Gamma_1 \vdash B \quad \Gamma_2 \vdash C}{\Gamma_1, \Gamma_2 \vdash B \otimes C}$$

is an instance of $\otimes r$ rule of *cut-free* **ILL**, $\Gamma_1, \Gamma_2 \vdash B \otimes C \notin |\mathcal{R}|$ by the definition of the OR-trees. Consequently, $B^*C^* \subseteq [\![B \otimes C]\!]$ and we conclude that $B^* \otimes C^* = Cl(B^*C^*) \subseteq [\![B \otimes C]\!]$.

(Case 3) A is of the form $B \multimap C$. As for (a), it suffices to show that for any $\Delta \in B^*$, $\Delta \cdot B \multimap C \in C^*$. If $\Delta \cdot B \multimap C = \surd$, then $\surd \in C^*$ by Lemma 7(i). Hence we may assume that $\Delta, B \multimap C$ is in $\mathcal{R}_l^*$, i.e., $\Delta, B \multimap C, \Pi \vdash E \in |\mathcal{R}|$ for some Π and E. Since

$$\frac{\Delta \vdash B \quad C, \Pi \vdash E}{\Delta, B \multimap C, \Pi \vdash E}$$

is an instance of $\multimap l$ rule of *cut-free* **ILL**, either $\Delta \vdash B \in |\mathcal{R}|$ or $C, \Pi \vdash E \in |\mathcal{R}|$. However, the former is impossible by IH(b'). Hence the latter holds, and by IH(a), $C \in C^*$. Therefore by Lemma 7(v), $\Delta, B \multimap C \in C^*$.

As for (b'), assume that $\Gamma \in B^* \multimap C^*$. It suffices to show that $\Gamma \vdash B \multimap C \notin |\mathcal{R}|$. If $\Gamma \vdash B \multimap C \in |\mathcal{R}|$, then $\Gamma, B \vdash C$ would be also in $|\mathcal{R}|$, thus $\Gamma, B \notin [\![C]\!]$. But it is impossible because $B \in B^*$ by IH(a), $\Gamma \in B^* \multimap C^*$ by assumption, and $C^* \subseteq [\![C]\!]$ by IH(b'). Hence $\Gamma \vdash B \multimap C \notin |\mathcal{R}|$.

(Case 4) A is of the form $B \mathbin{\&} C$. As for (a), since both B and C are in $\mathcal{R}_l^*$, $B \in B^*$ and $C \in C^*$ by IH. Hence $B \mathbin{\&} C \in B^*$ and $B \mathbin{\&} C \in C^*$ by Lemma 7(iv). Thus $B \mathbin{\&} C \in B^* \mathbin{\&} C^*$.

As for (b'), assume that $\Gamma \in A^* \mathbin{\&} B^*$. Then $\Gamma \in [\![B]\!]$ and $\Gamma \in [\![C]\!]$ by IH. It is immediate from the definition of the OR-trees that $\Gamma \vdash B \mathbin{\&} C \notin |\mathcal{R}|$.

(Case 5) A is of the form $B \oplus C$. This is essentially the reverse of (Case 4). (a) is shown by using (iii), (iii)' and (iii)" of Lemma 7. As for (b'), show that $A^* \cup B^* \subseteq [\![A \oplus B]\!]$.

(Case 6) A is of the form $!B$. As for (a), $B \in B^*$ by IH (since $B \in \mathcal{R}_l^*$), hence $!B \in B^*$ by Lemma 7(vi). On the other hand, $!B \in K$ by definition. Therefore $!B \in Cl(B^* \cap K) = !B^*$.

As for (b'), we show that $B^* \cap K \subseteq [\![!B]\!]$. Assume that $\Gamma \in B^* \cap K$. If $\Gamma \equiv \surd$, then by Lemma 7(i). Otherwise Γ is of the form $!\Delta$ (the case that Γ is the empty sequece is shown in the same way). If $!\Delta \vdash !B \in |\mathcal{R}|$, then $!\Delta \vdash B$ would be also in $|\mathcal{R}|$. But it is impossible because $!\Delta \in [\![B]\!]$ by IH. Therefore $!\Delta \vdash !B \notin |\mathcal{R}|$, and $!\Delta \in [\![!B]\!]$.

(Case 7) A is a logical constant. Immediate. ∎

Theorem 2 (Completeness and Cut-Elimination) *Let S_0 be a sequent of* **ILL**. *Then the following are equivalent;*

1. *S_0 is satisfied in every intuitionistic phase model;*
2. *S_0 is cut-free provable in* **ILL**;
3. *S_0 is provable in* **ILL**.

Proof. 2 implies 3 trivially. 3 implies 1 by the usual soundness argument. Here we prove that 1 implies 2.

Suppose that $S_0 \equiv A_1, \ldots, A_n \vdash B$ is not provable in *cut-free* **ILL**. Then by Proposition 5, there is an open OR-tree $\mathcal{R}$ of S_0, from which we can construct an intuitionistic phase model $\mathcal{M}_{\mathcal{R}}$. By Proposition 6, $A_i \in A_i^*$ for each $1 \leq i \leq n$ and $A_1, \ldots, A_n \notin B^*$ in $\mathcal{M}_{\mathcal{R}}$. Hence $A_1^* \otimes \cdots \otimes A_n^* \not\subseteq B^*$. Thus $\mathcal{M}_{\mathcal{R}}$ is a countermodel of S_0, but it is impossible by the assumption. ∎

4.3 Countermodel Construction for Classical Linear Logic

In this subsection, we sketch the countermodel construction in the case of classical logic. For technical reasons, we employ the left one-sided formulation of classical linear logic (see Appendix B). $\Gamma \vdash$ is *satisfied* in a classical phase model $(M, \perp, K, v)$ if $\Gamma^* \subseteq \perp$. Note that

- $\Gamma \vdash$ is provable in left one-sided linear logic iff $\vdash \Gamma^\perp$ is provable in right one-sided linear logic, where $\Gamma^\perp$ denotes $A_1^\perp, \ldots, A_n^\perp$ when $\Gamma \equiv A_1, \ldots, A_n$;
- $\Gamma^* \subseteq \perp$ iff $1 \in \Gamma^{*\perp}$.

Let $\mathcal{R}$ be an open OR-tree in *cut-free* **LL**. Based on $\mathcal{R}$, we define an enriched classical phase model $\mathcal{M}_{\mathcal{R}} = (M, \perp, K, v)$ as follows;

- M is defined by $\mathcal{R}_l^* \cup \{\surd\}$ as before.
- $\perp = \{\Sigma \in M \mid \Sigma \vdash\notin |\mathcal{R}|\}$.
- $K = \{!\Delta \mid !\Delta \in M\} \cup \{\emptyset, \surd\}$.
- $v(\alpha) = \{\alpha^\perp\}^\perp = \{\Sigma \in M \mid \Sigma, \alpha \vdash\notin |\mathcal{R}|\}$.

Proposition 7 *In $\mathcal{M}_{\mathcal{R}}$, if A is in $\mathcal{R}_l^*$, then $A \in A^*$.*

Proof. By induction on the complexity of A. ∎

Theorem 3 (Completeness and Cut-Elimination) *Let S_0 be a sequent of* **LL**. *Then the following are equivalent;*

1. *S_0 is satisfied in every classical phase model;*
2. *S_0 is cut-free provable in* **LL**;
3. *S_0 is provable in* **LL**.

Proof. Similar to Theorem 2, using Proposition 7. ∎

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A Syntax of Intuitionistic Linear Logic

Roman capitals $A, B, \ldots$ stand for formulas. The constants and the connectives of intuitionistic linear logic are classified into three groups;

- Multiplicatives: $\mathbf{1}$, $A \otimes B$, $A \multimap B$;
- Additives: $\top$, $\mathbf{0}$, $A \,\&\, B$, $A \oplus B$;
- Modality (Exponential): $!A$.

Greek capitals $\Gamma_1, \Gamma_2, \Delta, \ldots$ stand for finite multisets of formulas. A sequent of **ILL** is of the form $\Gamma \vdash C$. The inference rules of **ILL** are as follows;

Identity and Cut:

$$\frac{}{A \vdash A}\ Identity \qquad \frac{\Gamma \vdash A \quad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ Cut$$

Multiplicatives:

$$\frac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C} \otimes l \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes r \quad \frac{\Gamma \vdash C}{\mathbf{1}, \Gamma \vdash C} \mathbf{1}l \quad \frac{}{\vdash \mathbf{1}} \mathbf{1}r$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C} \multimap l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B} \multimap r$$

Additives:

$$\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \oplus B, \Gamma \vdash C} \oplus l \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \oplus r_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \oplus r_2 \qquad \frac{}{\mathbf{0}, \Gamma \vdash C} \mathbf{0}l$$

$$\frac{A, \Gamma \vdash C}{A \,\&\, B, \Gamma \vdash C} \&l_1 \qquad \frac{B, \Gamma \vdash C}{A \,\&\, B, \Gamma \vdash C} \&l_2 \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \&r \qquad \frac{}{\Gamma \vdash \top} \top r$$

Modality (Exponential):

$$\frac{A, \Gamma \vdash C}{!A, \Gamma \vdash C} !D \quad \frac{!A, !A, \Gamma \vdash C}{!A, \Gamma \vdash C} !C \quad \frac{\Gamma \vdash C}{!A, \Gamma \vdash C} !W \quad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} !r$$

Here $!\Gamma$ stands for a multiset of the form $!A_1, \ldots, !A_n$.

B Syntax of Left One-sided Classical Linear Logic

Each atomic formula of classical linear logic (**LL**) is either a positive literal α or a negative literal $\alpha^\perp$. The connectives and constants of **LL** are as follows;

- Multiplicatives: $\mathbf{1}$, $\perp$, $A \otimes B$, $A \parr B$;
- Additives: $\top$, $\mathbf{0}$, $A \,\&\, B$, $A \oplus B$;
- Exponentials: $!A$, $?A$.

The negation $A^\perp$ of a formula A is defined as follows;

- $(\alpha)^\perp = \alpha^\perp$; $(\alpha^\perp)^\perp = \alpha$;
- $(\mathbf{1})^\perp = \perp$; $(\perp)^\perp = \mathbf{1}$;
- $(A \otimes B)^\perp = A^\perp \parr B^\perp$; $(A \parr B)^\perp = A^\perp \otimes B^\perp$;
- $(\top)^\perp = \mathbf{0}$; $(\mathbf{0})^\perp = \top$;
- $(A \,\&\, B)^\perp = A^\perp \oplus B^\perp$; $(A \oplus B)^\perp = A^\perp \,\&\, B^\perp$;
- $(!A)^\perp = ?(A^\perp)$; $(?A)^\perp = !(A^\perp)$.

A sequent of left one-sided **LL** is of the form $\Gamma \vdash$ where Γ is a multiset of **LL** formulas. Listed below are the inference rules of left one-sided **LL**;

$$\frac{}{A, A^{\perp} \vdash}\ \mathit{Identity} \qquad \frac{\Gamma, A \vdash \quad A^{\perp}, \Delta \vdash}{\Gamma, \Delta \vdash}\ \mathit{Cut}$$

$$\frac{A, B, \Gamma \vdash}{A \otimes B, \Gamma \vdash}\ \otimes \qquad \frac{\Gamma, A \vdash \quad B, \Delta \vdash}{\Gamma, A \parr B, \Delta \vdash}\ \parr \qquad \frac{\Gamma \vdash}{\mathbf{1}, \Gamma \vdash}\ \mathbf{1} \qquad \frac{}{\perp \vdash}\ \perp$$

$$\frac{A, \Gamma \vdash \quad B, \Gamma \vdash}{A \oplus B, \Gamma \vdash}\ \oplus \qquad \frac{A, \Gamma \vdash}{A \,\&\, B, \Gamma \vdash}\ \&1 \qquad \frac{B, \Gamma \vdash}{A \,\&\, B, \Gamma \vdash}\ \&2 \qquad \frac{}{\mathbf{0}, \Gamma \vdash}\ \mathbf{0}$$

$$\frac{A, \Gamma \vdash}{!A, \Gamma \vdash}\ !D \qquad \frac{!A, !A, \Gamma \vdash}{!A, \Gamma \vdash}\ !C \qquad \frac{\Gamma \vdash}{!A, \Gamma \vdash}\ !W \qquad \frac{A, !\Gamma \vdash}{?A, !\Gamma \vdash}\ ?$$