数理解析研究所講究録
1025巻 1998年 5-19

NONLINEAR ELLIPTIC PROBLEMS ON ROTATIONALLY SYMMETRIC DOMAINS

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0. Introduction

Consider the following equation:

$$\begin{aligned} \Delta u + u^p &= 0 \quad \text{in} \quad \Omega_R \\ u &= 0 \quad \text{on} \quad \partial\Omega_R \\ u &> 0 \quad \text{in} \quad \Omega_R, \end{aligned} \tag{1}$$

where $\Omega_R \equiv \{x \in \mathbb{R}^N | R-1 < |x| < R+1\}$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3, 1 < p < \infty$ for $N = 2$.

The problem (1) is invariant under the orthogonal coordinate transformation, that is, $O(N)$-symmetric. When $R < 1$ in problem (1), that is, the domain is a ball, we know that the solutions is $O(N)$-symmetric, in other words, radially symmetric. This is an elegant result of Gidas,

Research partially supported by GARC in SNU and KOSEF

Ni and Nirenberg [GNN]. This symmetry result brings about a uniqueness of the solutions [NN]. On the other hand, although annulus have the same symmetric property with balls, Brezis and Nirenberg pointed out in [BN] that there exists a nonradial symmetric solution of problem (1) when $R > 1$, $n \geq 3$ and $(N+2)/(N-2) - p$ is positive and sufficiently small. In fact, they showed that the minimal energy solutions for problem (1) is not radial symmetric in that case. Furthermore, Coffman [Co] proved that, in two-dimensional case, the number of nonradial and nonequivalent solutions of problem (1) goes to ∞ as $R \to \infty$. The same result was obtained by Y.Y. Li for $N \geq 4$ [Li] and by the author for $N = 3$ [By]. In [BN], [Co], [Li], [Lin] and [MS], the nonradial solutions of (1) which have globally minimal energies in some symmetric functions classes have been studied. On the other hand, in [By] the author proved the existence of locally -rather than globally- minimal energy solutions of (1) in certain symmetric functions classes when the space dimension is three; from which it was shown that the number of nonequivalent nonradial positive solutions of (1) goes to infinity as $R \to \infty$. Moreover, when the space dimension is three, it was shown in [By] that via finding only globally minimal energy solutions in the symmetric functions classes, it is impossible to prove that the number of nonequivalent and nonradial positive solutions of (1) goes to ∞ as $R \to \infty$.

It is interesting to note that the $O(N)$-symmetry has two contrasting effects on the structure of the positive solutions; in the case that domain is a ball the symmetry makes the structure of solutions to be simple, on the other hand, in the case that domain is an annuli the symmetry makes the structure of solutions to be complicated. Thus, it is natural to wonder why this contrasting effect of the $O(N)-$ symmetry on the structure of the positive solutions occurs. It is the purpose of this paper to think about this question. Heuristically, we can explain this phenomena, in a variational sense, as follows.

For any closed subgroup G of $O(N)$, we define

$$H_R^G \equiv \left\{u \in H_0^{1,2}(\Omega_R)\middle| u(x) = u(gx) \text{ for any } x \in \Omega_R, g \in G\right\}.$$

Then, from the principle of symmetric criticality [Pal], we see that any critical point of the energy functional

$$\frac{1}{2}\int_{\Omega_R} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\Omega_R} (\max\{u(x), 0\})^{p+1} dx$$

in H_R^G is a solution of problem (1). Considering the group action $G \times \Omega_R \to \Omega_R$ as coordinate transformations, we can imagine that, when the derivative of the energy functional at $u \in H_R^G$ is very close to zero, the energy density

$$\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}(\max\{u(x), 0\})^{p+1}$$

is concentrated around the union of some $G-$orbits. Thus we can expect that, when the energy density of certain functions in H_R^G (where the derivative of energy functional is very close to zero) depend highly on a structure of $G-$orbits, a *critical*(in a sense of magnitude of orbits) $G-$orbit reproduce a critical point of the energy functional. When $R < 1$, that is, Ω_R is a ball, the action $G \times \Omega_R \to \Omega_R$ has only one crtical orbit $\{0\}$. Hence, the energy functional is not much affected, in a variational sense, by the symmetry of problem (1). On the other hand, when $R > 1$, that is, Ω_R is an annulus, as we can see in section 2, the action $G \times \Omega_R \to \Omega_R$ has many *critical* orbits for certain closed subgroup G of $O(N)$. When R is very close to 1, the effect of the critical orbital actions to the energy functional is very small; eventually, their effect is ignored. In fact, there exists a unique solution of problem (1) when R is very close to 1 (refer [Dan]). By way of opposition, as $R \to \infty$, the energy of certain functions depends more highly on a structure of $G-$orbits. Then, as $R \to \infty$, a rich variety of positive solutions due to a structure of $G-$orbits appear.

In this paper we will see that the rich structure of the space of orbits under the action of closed subgroups of $O(N)$ on S^{N-1} brings about a rich variety of positive solutions of (1) as $R \to \infty$. We should note that almost all solutions found in this paper never have been found in the literature. In fact, we will show that the problem (1) has various solutions with different shapes, which are related to *locally minimal orbital sets* under the action of the closed subgroups of $O(N)$ on sphere S^{N-1}. Roughly speaking, for any closed subgroup G of $O(N)$, we find solutions of problem (1) which are concentrated around each *locally minimal* $G-$ *orbit* on a sphere $\{x||x|=R\}$.

This paper is orginized as follows. In section 1, we state basic assumptions and prepare some necessary results. In section 2, we study the structure of orbits under the action of the closed subgroups of $O(N)$ on sphere. The statement of our main theorem will be given in section 3.

1. Preliminary

We consider the following problem:

$$\begin{aligned} \Delta u + hu + f(u) = 0 \quad & \text{in} \quad \Omega_R \\ u = 0 \quad & \text{on} \quad \partial\Omega_R \\ u > 0 \quad & \text{in} \quad \Omega_R, \end{aligned} \tag{2}$$

where $\Omega_R \equiv \{x \in \mathbb{R}^N | R-1 < |x| < R+1\}$. We assume that the function f and the constant h satisfy the following conditions:

(A1) $h < \pi^2/4$;

(A2) f is continuousely differentiable on $\mathbb{R}$;

(A3) $f(t) = 0$ for $t \le 0$ and there exist a constant $\theta \in (0,1)$ such that $0 < f(t) < \theta f'(t)t$ for all $t > 0$.

Since the first eigenfunction of $-\Delta$ on Ω_R with Dirichlet condition zero is radially symmetric, we easily deduce that the corresponding first eigenvalue goes to $\pi^2/4$ as $R \to \infty$. In condition (A1), the restriction $h < \pi^2/4$ is related to this fact.

Let G be a closed subgroup of $O(N)$. Then, for any $x \in S^{N-1}$, the orbit xG is a closed submanifold of S^{N-1}. Denote $d(xG)$ the dimension of the manifold xG. Define

$$N_G \equiv N - \min_{x \in S^{N-1}} \{d(xG)\}.$$

For any closed subgroup G of $O(N)$, we define (G_f) the G-growth condition of f as follows:

(G_f) $|f(t)| + |f'(t)t| \le C|t|^p$ for some positive constant C, where $p \in (1, (N_G + 2)/(N_G - 2))$ in case $N_G \ge 3$ and $p \in (1, \infty)$ in case $N_G = 1$ or 2.

For any subgroup G of $O(N)$, we denote

$$H_R^G \equiv \{u \in H_0^{1,2}(\Omega_R) \mid u(g\cdot) = u(\cdot),\ g \in G\}.$$

For any $u \in H_R^I = H_0^{1,2}(\Omega_R)$, we define its energy

$$\Gamma(u) \equiv \frac{1}{2}\int_{\Omega_R} |\nabla u|^2 - hu^2 dx - \int_{\Omega_R} F(u)dx,$$

where $F(u) \equiv \int_0^u f(t)dt$ is a primitive of f.

The Sobolev imbedding theorem says that the space $H_R^I = H_0^{1,2}(\Omega_R)$ is continuously imbedded into L^q for $q \in [1, 2N/(N-2)]$, and the imbedding is compact for $q \in [1, 2N/(N-2))$. We note that, when $R > 1$, the space $H_R^{O(N)}$ is compactly imbedded into L^q for any $q > 1$. Thus, we expect that for a closed subgroup G of $O(N)$, the functions space H_R^G may be imbedded into L^q for some $q > 2N/(N-2)$. In fact, there are some results about this expectation when the symmetry group G is $O(l) \otimes O(N-l), l \geq 2$. (Refer to [Din] and [Li].) Here we give a general imbedding result about H_R^G for any closed subgroup G of $O(N)$.

Proposition 1.1. *Let G be a closed subgroup of $O(N)$. Then, if $R > 1$, the subspace H_R^G of $H_0^{1,2}(\Omega_R)$ is continuousely imbedded into $L^q(\Omega_R)$ for $q \in [1, 2N_G/(N_G - 2)]$. Moreover, the imbedding is compact for $q \in [1, 2N_G/(N_G - 2))$.*

Let G be a closed subgroup of $O(N)$. We consider the following elliptic problem on an infinite strip-like domain:

$$\begin{aligned} \Delta u + hu + f(u) &= 0 \quad \text{in} \quad (-1,1) \times \mathbb{R}^{N_G - 1} \\ u &= 0 \quad \text{on} \quad \{-1,1\} \times \mathbb{R}^{N_G - 1} \\ u &> 0 \quad \text{in} \quad (-1,1) \times \mathbb{R}^{N_G - 1}. \end{aligned} \tag{3}$$

Then we have the following result.

Proposition 1.2. *Suppose that the function f satisfies the conditions $(A1-3)$ and G_f. Then, there exists a minimal energy solution V_{N_G} of problem (3). Moreover, the solution has the following properties:*

(i) *$V_{N_G}(x_1, \cdot)$ is radially symmetric up to an translation*

(ii) *$V_{N_G}(x_1, x_2, \cdots, x_{N_G}) \le C \exp(-c(x_2^2 + \cdots + x_{N_G}^2)^{1/2})$ for some constants $c, C > 0$.*

2. Structure of orbits space

In this section we study a structure of orbits space. Let G be a closed subgroup of $O(N)$. Then the group G acts on S^{N-1} as linear transformations. We denote the action by $g \cdot x$ for $g \in G$ and $x \in S^{N-1}$. Denote the G-orbit of x by $xG = \{g \cdot x | g \in G\}$ for $x \in S^{N-1}$. We know that xG is a closed submanifold of S^{N-1}. Define $d(xG)$ the dimension of the manifold xG and $m(xG)$ the $d(xG)-$dimensional Housdorff measure. We, then, give a partial order $\prec$ on the space $\{xG | x \in S^{N-1}\}$ as follows:

$$xG \prec yG$$

if and ony if

$$d(xG) < d(yG),$$

or

$$d(xG) = d(yG) = 0 \text{ and } m(xG) < m(yG).$$

The relation $\prec$ is a criterion of dimensional magnitude of orbits.

Definition 2.1. A set $M \subset S^{N-1}$ is called a locally minimal orbital set under the action of G if M is invariant under the action of G and a minimal set satisfying the following conditions:

(i) for any $x, y \in M$, $d(xG) = d(yG)$, and $m(xG) = m(yG)$ in the case that $d(xG) = d(yG) = 0$.

(ii) there exists a positive constant $\delta_0 > 0$ such that for any $y \in \{x \in S^{N-1} |\ dist(x, M) \leq \delta_0\} \setminus M$ and $x \in M$, it holds that $xG \prec yG$.

In particular, a G-invariant set $M \subset S^{N-1}$ is called the globally minimal orbital set under the action of $G \subset O(N)$ if above properties (i) and (ii) hold with $\delta_0 > 2$.

We call a set M a minimal orbital set when M is the globally minimal orbital set or a locally minimal orbital set.

We note that the existence of the globally minimal orbital set under the action of closed subgroup of $O(N)$ is obvious. Moreover, the globally minimal orbital set is unique for each closed subgroup G of $O(N)$. We investigate the structure of minimal orbital sets.

Lemma 2.2. *For any $x \in S^{N-1}$, there exists a constant $\delta > 0$ such that*

$$d(xG) \leq d(yG) \quad \text{for any } y \in B(x,\delta) \cap S^{N-1}.$$

Proposition 2.3. *If $M \subset S^{N-1}$ is a minimal orbital set under the action of G, then M is closed and any component of M is a totally geodesic closed submanifold of S^{N-1}.*

Corollary 2.4. *Suppose that M' is a component of a minimal orbital set M under the action of G, and that its dimension m is larger than 1. Then, there exist $\xi_1, \cdots, \xi_{N-1-m}$ such that*

$$M' = \{x \in S^{N-1} \big| < x, \xi_i >= 0, i = 1, \cdots, N-1-m\}.$$

Corollary 2.5. *If M is a minimal orbital set under the action of G, then the number of components of M is finite.*

Corollary 2.6. *The globally minimal orbital set M under the action of $G \subset O(N)$ is a finite disjoint union of locally minimal orbital sets.*

It seems that there exists a systematic method to find locally minimal obital sets under the action of $G \subset O(N)$ from the globally minimal orbital set under the action of $G \subset O(N)$ when the globally minimal orbital set has a finite number of elements. Here we will see a conjecture, which is true in the three dimensional case and for some closed subgroup G of $O(N)$.

Assume that M is the globally minimal orbital set with finite elements $\{x_1, \cdots, x_m\}$ and $m > 2$. Then, it is obvious that there exist $\{z_1, \cdots, z_k\} \subset M$ such that

$$z_i G \cap z_j G = \emptyset, \qquad i \neq j$$

and

$$M = \cup_{i=1}^{k} z_i G.$$

Then each $z_i G, i = 1, \cdots, k$, is a locally minimal orbital set. For each $i \in \{1, \cdots, k\}$, we denote $< z_i G >$ the smallest subspace of $\mathbb{R}^N$ containing $z_i G$. Then, it follows easily that the space $< z_i G >$ is invariant under the action of G, and that for any $i, j \in \{1, \cdots, k\}$,

$$< z_i G > \cap < z_j G >= \{0\} \quad \text{or} \quad < z_i G >=< z_j G > .$$

For each $i \in \{1, \cdots, k\}$, let $\{x_1^i, \cdots, x_l^i\}$ be the elements of $z_i G$. ¿From the definition of the globally minimal orbital set, we see that the number of elements of $z_i G$ is independant of i. For each $i \in \{1, \cdots, k\}$, define the polytope generated from $z_i G$ by

$$C(z_i G) \equiv \{t_1 x_1^i + \cdots + t_l x_l^i \big| t_1 + \cdots + t_l \leq 1 \text{ and } t_j \geq 0, j = 1, \cdots, l\},$$

and

$$B(z_iG) = C(z_iG) \setminus \mathrm{int}(C(z_iG)),$$

where $\mathrm{int}(C(z_iG))$ is the set of interior points of $C(z_iG)$ in $< z_iG >$. We easily see that $B(z_iG)$ is invariant under the action of G. For any nonnegative integer n, we denote

$$(z_iG)(n) = \{ \text{ the } n-\text{dimesional facets of } B(z_iG)\}.$$

Then the group G acts on $(z_iG)(n)$. Denote the dimension of $B(z_iG)$ by d_i. For $i \in \{1, \cdots, k\}$ and $n \in \{0, 1, \cdots, d_i\}$, define

$$(z_iG)_n \equiv \left\{ \frac{x}{|x|} \;\middle|\; x \text{ is the center of } A \in (z_iG)(n) \right\} \subset S^{N-1}.$$

We see that the group G acts on $(z_iG)_n$. Then we conjecture the followings.

Conjecture. *The* $\{(z_iG)_n | i \in \{1, \cdots, k\}, n \in \{0, \cdots, d_i\}$ *is the set of locally minimal orbital sets under the action of* G.

3. Statement of result

Let M be a locally minimal orbital set under the action of a closed subgroup G on S^{N-1}. Then we have the following theorem.

Theorem 3.1. *Suppose that the function f satisfies conditions $(A1-3)$ and G_f. Then, there exists a solution $u_R \in H_R^G$ of problem (2) such that*

(i) *for some $x_R \in \{Rx | x \in M\}$, $u_R(x) \to 0$ as $dist(x, x_R G) \to \infty$; and*

(ii) *for any $x \in M$,*

$$\lim_{R\to\infty} \Gamma(u_R)/R^{d(xG)} = m(xG)\Gamma_\infty^{N_G},$$

where $\Gamma_\infty^{N_G}$ is the energy of the solution V_{N_G} for problem (3).

In [BN], [Co], [Li] and [Lin], the globally minimal energy solutions of (1) in H_R^G have been investigated when the closed subgroup G of of $O(N)$ is one of forms, $G_k \otimes O(N-2), k = 2, 3, \cdots$ and $O(l) \otimes O(N-l), l = 2, \cdots, N-1$ (here, the G_k is a subgroup of $O(2)$ generated by rotation through an angle of $\frac{2\pi}{k}$). In [MS], when the space dimension is three, using the complete classification of closed subgroups of $O(3)$, they investigated the globally minimal energy solutions in H_R^G for all closed subgroups of $O(3)$. Here we give a result about globally minimal energy solutions in H_R^G for all closed subgroup G of $O(N)$, which can be obtained simply from Theorem 3.1. We should note that, although we may not know the closed subgroups of $O(N)$ completely, we can characterize the property of glabally minimal energy solutions of (2) in H_R^G in terms of intrinsic property of group action $G \times S^{N-1} \to S^{N-1}$.

Proposition 3.2. *Assume that G is a closed subgroup of $O(N)$. Let u_R be a (globally) minimal energy solution of (2) in H_R^G and M the globally minimal orbital set under the action of G. Then there exist $\{x_R\} \subset \{Rx | x \in M\}$ such that*

(i) *$u_R(x) \to 0$ as $dist(x, x_R G) \to \infty$, and*

(ii) *for any $x \in M$,*

$$\lim_{R\to\infty} \Gamma(u_R)/R^{d(xG)} = m(xG)\Gamma_\infty^{N_G},$$

where $\Gamma_\infty^{N_G}$ is the energy of the solution V_{N_G} for problem (3).

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