数理解析研究所講究録
1025 巻 1998 年 44-65

Stationary Keller-Segel model with the linear sensitivity

Yoshitsugu Kabeya [壁谷 喜継] (Miyazaki Univ.)

Wei-Ming Ni [倪 維明] (University of Minnesota)

1 Introduction

The Keller-Segel models [7], which describes the chemotactic aggregation stage of cellular slime molds, was investigated by many authors, see e.g., Lin, Ni and Takagi [9] and Ni and Takagi [10],[11], [12]. We are interested in the stationary problem of the Keller-Segel system

$$D_1\Delta u-\chi\nabla\cdot(u\nabla\phi(v))=0 \quad \text{in } \Omega, \tag{1.1}$$

$$D_2\Delta v-av+bu=0 \quad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $D_1>0$, $D_2>0$, $a>0$ and $b>0$ are constants, ν is the outer normal unit vector on $\partial\Omega$, ϕ is a smooth function with $\phi'>0$ on $(0,\infty)$ and Ω is a smooth bounded domain in $\mathbf{R}^2$. We will seek a pair of positive solutions (u,v) to (1.1)-(1.3). Biologically, u represents the density of amoebae, v does the concentration of the chemical which amoebae transmit. ϕ represents the sensitivity of amoebae to the chemical.

The logarithmic sensitivity $\phi(v)=\log v$, there are lots of literature, see, e.g., Ni and Takagi [10] and the references therein.

Instead, here we adopt $\phi(v)=v$. In this case, (1.1) is written as

$$\nabla\cdot\left\{D_1u\nabla(\log u-\frac{\chi}{D_1}v)\right\}=0.$$

Then we see that $u = ce^{pv}$ by using (1.3), where $p = \chi/D_1$ and $c > 0$ is a constant. Thus (1.1)-(1.3) is equivalent to.

$$\begin{cases} D_2\Delta v - av + bce^{pv} = 0 \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Now putting $\varepsilon^2 = D_2/a$ and $bc/a = \lambda$, we have

$$\begin{cases} \varepsilon^2\Delta v - v + \lambda e^{pv} = 0 \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Conversely, if w is a positive solution to (1.4), then $u = c_1 e^{pw}$ and $v = c_2 w$ satisfy (1.1)-(1.3) with $c_1 = apD_1\lambda/b\chi$ and $c_2 = pD_1/\chi$.

From now on, we will mainly investigate (1.4) with ε, λ and p being positive parameters.

Before stating our results on (1.4), we first discuss a slightly more general problem:

$$\varepsilon^2\Delta u - cu + h(u) = 0 \quad \text{in } \Omega, \tag{1.5}$$

$$u > 0 \quad \text{in } \Omega, \tag{1.6}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

where $\varepsilon > 0$ and $c > 0$.

We make the following assumptions on h:

(h_1) $h : \mathbf{R} \to \mathbf{R}$ is locally Hölder continuous, $h(z) = 0$ for $z \leq 0$ and $h(z) > 0$ for $z > 0$.

(h_2) $h(z) = o(z)$ as $z \downarrow 0$.

(h_3) $h(z)/z \to \infty$ as $z \to \infty$. Moreover, there exist $\alpha \geq 0$ and $\beta(z)$ with $\beta(z)/z^2 \to 0$ as $z \to \infty$ such that

$$h(z) \leq \alpha \exp \beta(z) \quad \text{for } z > 0.$$

(h_4) Let $H(z) = \int_0^z h(t)\,dt$. There exists $\alpha_1 \geq 0$ and $\theta \in (0, 1/2)$ such that

$$H(z) \leq \theta z h(z) \quad \text{if } z \geq \alpha_1.$$

(h_5) $\gamma = \inf\{cz^2/2 - H(z) \,|\, z \in Z\} > 0$ where $Z = \{z > 0 \,|\, h(z) = cz\}$.

We note that $Z \neq \emptyset$ because of (h_2) and (h_3). If (h_4) holds with $\alpha_1 = 0$, then (h_5) is automatically satisfied. If $\zeta \in Z$, then $u(x) \equiv \zeta$ is a positive solution to (1.5)-(1.7). An example of a function satisifying (h_1)-(h_5) is $h(z) = (e^{pz} - 1 - pz)_+$. Just note that (h_4) is satisfied with $\theta \in [1/3, 1/2)$ and $\alpha_1 = 0$.

Let E denote the Hilbert space $W^{1,2}(\Omega)$ endowed with the norm

$$||u|| = \left(\varepsilon^2 \int_\Omega |\nabla u|^2\,dx + c\int_\Omega u^2\,dx\right)^{1/2}.$$

We define a functional J_ε on E by

$$J_\varepsilon(u) = \frac{1}{2}\left(\varepsilon^2 \int_\Omega |\nabla u|^2\,dx + c\int_\Omega u^2\,dx\right) - \int_\Omega H(u)\,dx.$$

Theorem 1.1 *Under assumptions (h_1) through (h_5), there exists a positive nonconstant solution u_ε to (1.5)-(1.7) provided $\varepsilon > 0$ is sufficiently small. Moreover, u_ε satisfies*

$$J_\varepsilon(u_\varepsilon) \leq C_0\varepsilon^2$$

where $C_0 > 0$ depends only on Ω and h.

Corollary 1.1 *In addition to (h_1)-(h_3), assume that (h_4) holds with $\alpha_1 = 0$. Then*

$$\int_\Omega (\varepsilon^2|\nabla u_\varepsilon|^2 + cu_\varepsilon^2)\,dx = \int_\Omega u_\varepsilon h(u_\varepsilon)\,dx \leq \frac{2C_0}{1-2\theta}\varepsilon^2.$$

Now we return to (1.4). First we observe that $t = \lambda e^{pt}$ must have exactly two zeros on $(0, \infty)$ if (1.4) is to have a nonconstant a solution. Indeed, integrating (1.4) gives that $\int_\Omega(-u + e^{pu})\,dx = 0$. Thus $-t + e^{pt}$ must be negative somewhere in $(0, \infty)$, which shows the assertion. Furthermore, let Q be the minimum point of u on $\bar{\Omega}$. Then we have $0 \leq \Delta u(Q) = u(Q) - \lambda e^{pu(Q)}$, which implies that $\min_\Omega u \geq z_\lambda$ where z_λ is the smaller solution of $\lambda e^{pt} - t = 0$.

Let $w = u - z_\lambda$. Then we have

$$\varepsilon^2 \Delta w - w + z_\lambda(e^{pw} - 1) = 0. \tag{1.8}$$

To apply Theorem 1.1, we rewrite as

$$\varepsilon^2\Delta w - (1 - z_\lambda p)w + z_\lambda(e^{pw} - 1 - pw)_+ = 0 \quad \text{in } \Omega, \tag{1.9}$$

$$w > 0 \quad \text{in } \Omega, \tag{1.10}$$

$$\frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.11}$$

From now on, set $c = (1 - z_\lambda p)$. We observe the following fact.

Remark 1.1 If $\lambda e^{pt} = t$ has two solutions, then $c > 0$ holds.

To see this, consider the slope of $\varphi(t) = \lambda e^{pt}$. At $t = z_\lambda$, φ intersects the straight line $y = t$ transversally. This implies that $\varphi'(z_\lambda) = p\lambda e^{pz_\lambda} = pz_\lambda < 1$. The assertion is proved.

Theorem 1.2 *Suppose that $t = \lambda e^{pt}$ has two positive solutions. Then (1.9)-(1.11) has a nonconstant positive solution w_ε which has all the properties that are stated in Theorem 1.1 and Corollary 1.1. Moreover, there exist constants $C_1 > 0$, $C_2 > 0$ and $\gamma > 0$ such that*

$$\sup_\Omega w_\varepsilon \le C_1.$$

Using the proof of Theorem 1.2, we can show that the $||w_\varepsilon|| \sim \varepsilon$ as $\varepsilon \to 0$.

Proposition 1.1 *Suppose that $t = \lambda e^{pt}$ has two positive solutions. Then for the solution w_ε obtained in Theorem 1.2, there exist $K > 0$ and $\varepsilon_0 > 0$ such that*

$$\int_\Omega (\varepsilon^2|\nabla w_\varepsilon|^2 + cw_\varepsilon^2)\, dx \ge K\varepsilon^2$$

for $0 < \varepsilon < \varepsilon_0$.

We also have an upper estimate for $\inf_\Omega w_\varepsilon$.

Theorem 1.3 *Suppose that $t = \lambda e^{pt}$ has two positive solutions. Then for the solution w_ε obtained in Thereom 1.2, there exist $C_2 > 0$, $\gamma > 0$ and $\varepsilon_0 > 0$ such that*

$$\inf_\Omega w_\varepsilon \le C_2 \exp(-\frac{\gamma}{\varepsilon})$$

holds for any $0 < \varepsilon < \varepsilon_0$.

Theorem 1.4 *For sufficiently small $\varepsilon > 0$, the solution w_ε obtained in Theorem 1.2 has exactly one local maximum point in $\bar{\Omega}$, which must lie on the boundary $\partial\Omega$.*

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need two lemmas. Since these lemmas are proved in Lin, Ni and Takagi [9] and since these proofs are strightforward calculation, we skip the proofs. Let φ be such that

$$\varphi(x) = \begin{cases} \varepsilon^{-2}(1-\varepsilon^{-1}|x|) & |x| < \varepsilon, \\ 0 & |x| \geq \varepsilon. \end{cases}$$

Lemma 2.1 *For any $s > 0$, there holds*

$$\int_\Omega |\varphi(x)|^s \, dx = K_s \varepsilon^{2(1-s)}, \quad \int_\Omega |\nabla\varphi|^2 \, dx = \pi\varepsilon^{-4}$$

where

$$K_s = 2\pi \int_0^1 (1-\rho)^s \rho \, d\rho.$$

Now let $g(t) := J_\varepsilon(t\varphi)$ for $t \geq 0$. We investigate the property of $g(t)$.

Lemma 2.2 *There exist t_1, t_2 with $0 < t_1 < t_2$ such that*

(a) $g'(t) < 0$ *for* $t > t_1$.

(b) $g(t) < 0$ *for* $t > t_2$.

As for a proof, see [9] (pp.11-12, Lemma 2.4).

Proof of Theorem 1.1. Step 1. First we remark that any critical point of J_ε is a classical solution to (1.5)-(1.7). In fact, a critical point of that is a generalized solution in $W^{1,2}(\Omega)$. The elliptic regularity theorem yields that it is a classical solution(note that $h(u) \in L^q(\Omega)$ for $q \geq 1$ by (h_3)).

Next, we verify that any nonconstant critical point of J_ε is positive everywhere in Ω. This fact is proved exactly the same way as before, see p.9 in [9].

Step 2. To obtain nonconstant critical points of J_ε, we shall make use of the mountain pass theorem. Clearly, $J_\varepsilon : W^{1,2}(\Omega) \to \mathbf{R}$ is a C^1-mapping and $J_\varepsilon(0) = 0$. We must check

(i) J_ε satisfies the Palais-Smale condition.

(ii) There exist $\rho > 0$ and $\beta > 0$ such that $J_\varepsilon(u) > 0$ if $0 < ||u|| < \rho$ and $J_\varepsilon(u) \geq \beta > 0$ if $||u|| = \rho$.

(iii) For sufficiently small $\varepsilon > 0$, there exist a nonnegative function $\varphi \in H^1(\Omega)$ and positive constants C_0 and t_0 such that $J_\varepsilon(t_0\varphi) = 0$ and $J_\varepsilon(t\varphi) \leq C_0\varepsilon^2$

The checking will be done by following the argument of [9] with some modification. After verifying these conditions, we can apply the mountain pass theorem as follows: Let $e = t_0\varphi$ and

$$\Gamma = \{l \in C([0,1]; H^1(\Omega)) \,|\, l(0) = 0,\, l(1) = e\}.$$

Then

$$c := \inf_{l \in \Gamma} \sup_{s \in [0,1]} J_\varepsilon(l(s)) \tag{2.1}$$

is a critical value of J_ε with $0 < \beta \leq c < \infty$.

In general, $J_\varepsilon^{-1}(c)$ may consists only of constant functions. We must deny this possibility. By (h_5), the infimum of the energy of constant solution $\bar{z}$ is

$$\inf_{\bar{z} \in Z} \left\{\frac{1}{2}c \int_\Omega \bar{z}^2\, dx - \int_\Omega H(\bar{z})\, dx\right\} = \inf_{\bar{z} \in Z} (\frac{1}{2}c\bar{z}^2 - H(\bar{z}))|\Omega| = \gamma|\Omega| > 0.$$

So we obtain a nonconstant critical point by taking $\varepsilon > 0$ as $C_0\varepsilon^2 < \gamma|\Omega|$ and using (iii). □

Proof of Corollary1.1. Since u_ε is a solution to (1.5)-(1.7), we obtain

$$\int_\Omega (\varepsilon^2|\nabla u_\varepsilon|^2 + c|u_\varepsilon|^2)\, dx = \int_\Omega u_\varepsilon h(u_\varepsilon)\, dx.$$

On the other hand, from (h_4) with $\alpha_1 = 0$, we have

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u_\varepsilon|^2 + c|u_\varepsilon|^2)\, dx - \int_\Omega H(u_\varepsilon)\, dx \\ &= \frac{1}{2}\int_\Omega u_\varepsilon h(u_\varepsilon)\, dx - \int_\Omega H(u_\varepsilon)\, dx \\ &\geq (\frac{1}{2} - \theta)\int_\Omega u_\varepsilon h(u_\varepsilon)\, dx. \end{aligned}$$

Thus we obtain the desired inequality. □

3 Proofs of Theorem 1.2 and Proposition 1.1

To prove Theorem 1.2, we need to investigate the dependence of the Sobolev constant on the exponent of the target L^q space of the embedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$. We invoke an inequality in the proof of Theorem 2.9.1 in Ziemer [13].

Only in this section, for $u \in W^{1,2}(\Omega)$, we define $||u||_{W^{1,2}(\Omega)}$ by

$$||u||_{W^{1,2}(\Omega)} = \Big\{ \int_\Omega (|\nabla u|^2 + |u|^2)\,dx \Big\}^{1/2}.$$

The following is a key lemma to obtain the upper estimate of the solution. The proof is done by the combination of an inequality in Lemma 7. 12 of Gilbarg and Trudinger [6] and Lemma 5.14 in Adams [1], so we omit it.

Lemma 3.1 *For any $u \in W^{1,2}(\Omega)$, there exists a positive constant K_1 independent of $|\Omega|$ but on the cone property of Ω such that*

$$||u||_{L^q(\Omega)} \le K_1 \pi^{1/2} (\frac{q+2}{2})^{(q+2)/2q} ||u||_{W^{1,2}(\Omega)} \tag{3.1}$$

holds.

From Lemma 3.1, we have

$$||u||^q_{L^q(\Omega)} \le (\frac{q+2}{2})^{(q+2)/2} K_2^q \Big(\int_\Omega (|\nabla u|^2 + cu^2)\,dx \Big)^{q/2}$$

with positive constant $K_2 > 0$. As is the same way in (2.19) of [9], we obtain for a solution u_ε in Theorem 1.1

$$\begin{aligned} \Big(\int_\Omega |u|^q\,dx \Big)^{2/q} &\le K_2^2 \Big(\frac{q+2}{2}\Big)^{(q+2)/q} \varepsilon^{-2+4/q} \int_\Omega \Big(\varepsilon^2 |\nabla u|^2 + c|u|^2 \Big)\,dx \\ &\le K_2^2 \Big(\frac{q+2}{2}\Big)^{(q+2)/q} \varepsilon^{4/q} \end{aligned} \tag{3.2}$$

by Corollary 1.1. The inequality (3.2) is proved as follows: Let $u_\varepsilon(x) = v(y)$ with $x = \varepsilon y$ and $\Omega_\varepsilon = \{y \,|\, \varepsilon y \in \Omega\}$. Then we have

$$\begin{aligned} \int_\Omega (\varepsilon^2 |\nabla u_\varepsilon|^2 + c|u_\varepsilon|^2)\,dx &= \varepsilon^2 \int_{\Omega_\varepsilon} \Big(|\nabla v|^2 + c|v|^2 \Big)\,dy \\ &\ge K_2^{-2} \Big(\frac{q+2}{2}\Big)^{-(q+2)/q} \varepsilon^2 \Big(\int_{\Omega_\varepsilon} |v|^q\,dy \Big)^{2/q} \\ &\ge K_2^{-2} \Big(\frac{q+2}{q}\Big)^{-(q+2)/q} \varepsilon^{2-4/q} \Big(\int_\Omega |u_\varepsilon|^q\,dx \Big)^{2/q}. \end{aligned}$$

Now we denote various constants independent of ε by C.

Proof of Theorem 1.2. Step 1. In the case $h(u) = z_\lambda(e^{pu} - 1 - pu)_+$, we can take $A > 0$ sufficiently large such that

$$h(u) \le \frac{1}{2}(u + Auh(u)) \tag{3.3}$$

for any $u \ge 0$. Integrating (1.1) over Ω, we have

$$\int_\Omega u\,dx = \int_\Omega h\,dx \le \frac{1}{2}\int_\Omega (u + Auh(u))\,dx$$

by taking the boundary condition into account. It follows from Corollary 1.1 that

$$\int_\Omega u\,dx \le A\int_\Omega uh(u)\,dx \le C\varepsilon^2. \tag{3.4}$$

Thus we obtain

$$\int_\Omega h\,dx \le C\varepsilon^2.$$

Now we prove that

$$z_\lambda^{-2}\int_\Omega h^2\,dx = \int_\Omega (e^{pu} - 1 - pu)^2\,dx \le C\varepsilon^2. \tag{3.5}$$

Expanding in the Taylor series, we have

$$z_\lambda^{-2}\int_\Omega h^2\,dx = \sum_{k=2}^{\infty}\int_\Omega (2^k - 2pu - 2)\frac{(pu)^k}{k!}\,dx.$$

From Lemma 3.1 and (3.2), we get

$$\begin{aligned}\int_\Omega h^2\,dx \ &\le \sum_{k=2}^{\infty}(2^k - 2)\frac{1}{k!}\left(\frac{k+2}{2}\right)^{(k+2)/2}K_2^k p^k \varepsilon^2 \\ &\quad -2\sum_{k=2}^{\infty}\frac{1}{k!}\left(\frac{k+3}{2}\right)^{(k+3)/2}K_2^{k+1}p^{k+1}\varepsilon^2.\end{aligned}$$

It suffice to show that the power series are convergent. Let

$$a_k := \frac{2^k - 2}{k!}\left(\frac{k+2}{2}\right)^{(k+2)/2}K_2^k p^k, \quad b_k := \frac{1}{k!}\left(\frac{k+3}{2}\right)^{(k+3)/2}K_2^{(k+1)/2}p^{k+1}.$$

We have $\lim_{k\to\infty} a_{k+1}/a_k = 0$ and $\lim_{k\to\infty} b_{k+1}/b_k = 0$. Hence we have proved (3.5).

Step 2. This step is similar to Step 2 in the proof of Corollary 2.1 of [9]. As we have seen in Introduction, $h(u) = z_\lambda(e^{pu} - 1 - pu)_+$ satisfies (h_1)-(h_5). Thus there exists a solution w_ε by Theorem 1.1.

Multiplying the both sides of (1.7) by u^{2s-1}, with $s \geq 1$, we have

$$\frac{2s-1}{s^2}\varepsilon^2 \int_\Omega |\nabla u^s|^2\, dx + c\int_\Omega u^{2s}\, dx = \int_\Omega h(u)u^{2s-1}\, dx. \tag{3.6}$$

By the Schwarz inequality, the right-hand side is estimated as

$$\int_\Omega h(u)u^{2s-1}\, dx \leq \Big(\int_\Omega (h(u))^2\, dx\Big)^{1/2}\Big(\int_\Omega u^{4s-2}\, dx\Big)^{1/2}. \tag{3.7}$$

Since we have already had (3.5), we obtain

$$\Big(\int_\Omega u^{s\mu}\, dx\Big)^{2/\mu} \leq C(\mu)s\varepsilon^{(4/\mu)-1}\Big(\int_\Omega u^{4s-2}\, dx\Big)^{1/2} \tag{3.8}$$

by (3.6), (3.7) and the Sobolev inequality (3.1) with $\mu > 4$. Here we do not need to have an exact embedding constant, so we just denote the constant by $C(\mu)$. Now we define two sequences $\{s_j\}$ and $\{M_j\}$ by

$$\begin{aligned} 4s_0 - 2 &= \mu \\ 4s_{j+1} - 2 &= \mu s_j \quad \text{for } j = 0, 1, 2, \ldots \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} M_0 &= C(\mu)^{\mu/2} \\ M_{j+1} &= (C(\mu)s_j)^{\mu/2}(M_j)^{\mu/4} \quad \text{for } j = 0, 1, 2, \ldots \end{aligned} \tag{3.10}$$

We note that s_j is explicitly given by

$$s_j = (\frac{\mu}{4})^j(s_0 + \frac{2}{\mu-4}) - \frac{2}{\mu-4}. \tag{3.11}$$

Since we have chosen $\mu > 4$, $s_j \to \infty$ as $j \to \infty$. We shall show

$$\int_\Omega u^{4s_j-2}\, dx \leq M_j\varepsilon^2 \tag{3.12}$$

for $j \geq 0$ and

$$M_j \leq e^{ms_{j-1}} \tag{3.13}$$

for some constant $m > 0$. Verfying these inequalities is done by induction. So we omit the detail. Hence we have

$$||u||_{L^{\mu s_{j-1}}(\Omega)} \leq (e^{ms_{j-1}}\varepsilon^2)^{1/\mu s_{j-1}} = e^{m/\mu}\varepsilon^{2/\mu s_{j-1}}. \tag{3.14}$$

Letting $j \to \infty$, we obtain

$$||u||_{L^\infty(\Omega)} \leq e^{m/\mu}.$$

□

Proof of Proposition 1.1. Suppose to the contrary that there exist a sequence $\{\varepsilon_k\}_{k=1}^\infty$ and a sequence of positive solutions $\{w_k\}$ to (1.9)–(1.11) with $\varepsilon = \varepsilon_k$ such that

$$\eta_k := \varepsilon_k^{-2}\Big(\int_\Omega (\varepsilon_k^2|\nabla w_k|^2 + cw_k^2)\,dx\Big) \to 0$$

as $k \to \infty$. As in the proof of Theorem 1.2, we define $\{s_j\}$ and $\{M_j\}$ by (3.9) and (3.10) with $C(\mu)$ replaced by η_k. Similar to the proof of Theorem 1.2, we have (3.8) with $w = w_k$ and $\varepsilon = \varepsilon_k$ for $k \geq 1$. Using the argument in the proof of Theorem 1.2, we obtain

$$||w_k||_{L^\infty(\Omega)} \leq \tilde{C}\exp(b_1\rho_0)$$

where $\tilde{C} > 0$ and $b_1 > 0$ are constants independent of k and $\rho_0 = (\mu/2)\log(C(\mu)\varepsilon_k)$. Since $\rho \to -\infty$, we have

$$||w_k||_{L^\infty(\Omega)} \to 0 \tag{3.15}$$

as $k \to \infty$. Hence if k is sufficiently large,

$$\varepsilon_k^2\Delta w_k = cw_k - z_\lambda(e^{pw_k} - 1 - pw_k) > 0$$

on Ω by (3.15). This contradicts the Neumann boundary condition. The assertion is proved. □

4 Proof of Theorem 1.3

To prove Theorem 1.3, we need the Harnack inequality due to [9]. We also need some estimates on L^q norm of w_ε.

Lemma 4.1 *Let w_ε be the solution obtained in Theorem 1.2. Then there hold*

$$m(q)\varepsilon^2 \leq \int_\Omega w_\varepsilon^q\,dx \leq M(q)\varepsilon^2 \quad \text{if } 1 \leq q < \infty, \tag{4.1}$$

$$m(q)\varepsilon^2 \leq \int_\Omega w_\varepsilon^q\,dx \leq M(q)\varepsilon^{2q} \quad \text{if } 0 < q < 1. \tag{4.2}$$

where $m(q)$ and $M(q)$ are positive constants such that $m(q) < M(q)$ and are independent of ε.

To prove Theorem 1.3, the following proposition is useful. As in [9], we define a family of cubes. For $K = (k_1, k_2) \in \mathbf{Z}^2$ and $l > 0$, we define

$$C[K,l] := \{(x_1,x_2) \in \mathbf{R}^2 \,|\, |x_j - lk_j| \leq \frac{l}{2}, j = 1,2\}.$$

Clearly, $\mathbf{R}^2 = \bigcup_{K\in\mathbf{Z}^2} C[K,l]$ and the intersection of two such cubes is either empty or a line segment(face).

Proposition 4.1 *Let w_ε be the solution obtained in Thereom 1.2. For $\eta > 0$, let $\Omega_\eta := \{x \in \Omega \,|\, w_\varepsilon(x) > \eta\}$. Then there exist a positive integer $m > 0$ independent of $\varepsilon > 0$ such that Ω_η is covered by at most m of the $C[K,l]$'s provided ε is sufficiently small.*

To prove Proposition 4.1, we need the Harnack inequality valid for the boundary, which was introduced by [9].

Lemma 4.2 *Let w be a positive solution to*

$$\varepsilon^2 \Delta w + c(x) w = 0 \quad \text{in } \Omega$$

with $\partial w/\partial\nu = 0$ on $\partial\Omega$, where $c(x) \in C(\bar\Omega)$. Then there exists a positive constant $C_3 = C_3(\Omega, R\sqrt{\|c\|_{L^\infty}/\varepsilon})$ such that

$$\sup_{B(z,R)\cap\Omega} w \leq C_3 \inf_{B(z,R)\cap\Omega} w$$

for any ball $B(z,R)$ with radius R and centered at $z \in \bar\Omega$.

For a proof, see that of Lemma 4.3 of [9].

Using Lemma 4.2, we prove Proposition 4.1.

Proof of Proposition 4.1. Just follow the argument in [9] with Lemma 4.2 above. □

Now we are in a position to prove Theorem 1.3. *Proof of Theroem 1.3.*

First we will show that for any $\eta > 0$, there exist $x_0 \in \Omega$ and $r_0 > 0$ independent of $\varepsilon > 0$ such that $B(x_0, r_0) \subset \Omega \backslash \Omega_\eta$. If this statement is not true, then there exist $\eta > 0$ and two sequences $\{r_j\}$ ($r_j \to 0$) and $\{\varepsilon_j\}$ ($\varepsilon_j \to 0$) such that

$$B(x, r_j) \cap \Omega_{\eta,j} \neq \emptyset$$

for any $x \in \Omega$, where

$$\Omega_{\eta,j} = \{x \in \Omega \,|\, u_{\varepsilon_j} > \eta\}.$$

Hence any point $x \in \Omega$ belongs to the r_j-neighborhood of $\Omega_{\eta,j}$. Since $\Omega_{\eta,j}$ is covered by at most m cubes with its segment length ε_j by Proposition 4.1, $|\Omega| \to 0$ as $j \to \infty$. This is absurd.

Now we take $\eta > 0$ such that

$$z_\lambda(e^{pu} - 1 - pu) - (1 - z_\lambda p)u < 0$$

for $0 < u \leq \eta$ and let

$$\gamma_0 := \inf_{0<u\leq\eta} \sqrt{-\frac{z_\lambda(e^{pu} - 1 - pu)}{u} + (1 - z_\lambda p)}.$$

Let w be the solution of the linear Dirichlet problem

$$\varepsilon^2 \Delta w - \gamma_0^2 w = 0 \quad \text{in } B(x_0, r_0) \tag{4.3}$$

$$w = \eta \quad \text{on } \partial B(x_0, r_0). \tag{4.4}$$

Since

$$\varepsilon^2 \Delta(w_\varepsilon - w) - \gamma_0^2 (w_\varepsilon - w) \geq 0$$

in $B(r_0, r_0)$ and $w_\varepsilon - w \leq 0$ on $\partial B(r_0, x_0)$, we have

$$w_\varepsilon(x) \leq w(x) \quad \text{in } B(x_0, r_0). \tag{4.5}$$

Put $r := |x - x_0|$. Then w is given by

$$w(r) = W^* I_0(\frac{\gamma}{\varepsilon} r) \tag{4.6}$$

where $W^* = \eta / I_0(\gamma_0 r_0/\varepsilon)$, $I_0(z)$ is the modified Bessel function of the first kind of order 0. Making use of the asymptotic formula $I_0(r) \sim e^r/\sqrt{2\pi r}$ as $r \to \infty$, we see that

$$\inf w_\varepsilon \leq C_* \varepsilon^{-1/2} \exp(-\gamma_0 r_0/\varepsilon)$$

by (4.5) with $C_* = C_*(\eta, \gamma_0 r_0) > 0$. Choosing smaller γ, we obtain the desired estimate. □

5 Preliminaries to a Proof of Theorem 1.4.

To show that the maximum point is on the boundary, we efficiently use the minimax value (2.1) of w_ε. In this section, let $u_\varepsilon := w_\varepsilon$ be the solution to (1.9)–(1.11) obtained in Thereom 1.2. Since the proof of Theorem 1.4 is lengthy, we collect technical lemmas here.

First we show an important characterization of the minimax value. For $v \in W^{1,2}(\Omega)$, put

$$M[v] := \sup_{t \geq 0} J_\varepsilon(tv).$$

Recall the Mountain Pass critical value

$$c_\varepsilon := \inf_{l \in \Gamma} \sup_{s \in [0,1]} J_\varepsilon(l(s)).$$

The following lemma is almost identical to Lemma 3.1 of [10] so we omit the proof.

Lemma 5.1 *Let c_ε as above. Then c_ε does not depend on the choice of $e \in W^{1,2}(\Omega)$ such that $e \geq 0$, $e \not\equiv 0$ and $J_\varepsilon(e) = 0$. More precisely, c_ε is the least positive critical value of J_ε and is given by*

$$c_\varepsilon = \inf\{M[v] \,|\, v \in W^{1,2}(\Omega)\, v \not\equiv 0 \text{ and } v \geq 0 \text{ in } \Omega\}.$$

As in [9] and [10], we use a diffeomorphism which straightens a boundary portion near $P \in \partial\Omega$. Since the space dimension is two in this case, it is much easier to understand the nature of the diffeomorphism than that in [9] and [10].

Through translation and rotation, we may assume $P \in \partial\Omega$ is the origin and the inner normal to $\partial\Omega$ is pointing in the direction of the positive x_2 axis. In this situation, we can take a smooth function $\psi(x_1)$ defined in $(-\delta_0, \delta_0)$ such that

(i) $\psi(0) = 0$ and $\psi'(0) = 0$,

(ii) $\partial\Omega \cap \mathcal{N} = \{(x_1, x_2) \mid x_2 = \psi(x_1)\}$,

(iii) $\mathcal{N} \cap \Omega = \{(x_1, x_2) \mid x_2 > \psi(x_1)\}$,

where $\mathcal{N}$ is a neighborhood of $P = (0,0)$.

For $y = (y_1, y_2) \in \mathbf{R}^2$ with $|y|$ sufficiently small, we define a mapping $x = \Phi(y) = (\Phi_1(y), \Phi(y))$ by

$$\begin{aligned} \Phi_1(y) &= y_1 - y_2\psi'(y_1), \\ \Phi_2(y) &= y_2 + \psi(y_1). \end{aligned} \tag{5.1}$$

Since

$$D\Phi = \begin{pmatrix} \dfrac{\partial\Phi_1}{\partial y_1} & \dfrac{\partial\Phi_1}{\partial y_2} \\ \dfrac{\partial\Phi_2}{\partial y_1} & \dfrac{\partial\Phi_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 1 - y_2\psi''(y_1) & -\psi'(y_1) \\ \psi'(y_1) & 1 \end{pmatrix},$$

$\det D\Phi = 1 - y_2\psi''(y_1) + (\psi'(y_1))^2$. Thus $\det D\Phi(0,0) = 1$. Hence Φ has the inverse mapping $y = \Phi^{-1}(x) = \Psi(x)$ for $|x| < \delta'$. We write

$$\Psi(x) = (\Psi_1(x), \Psi_2(x)).$$

As we will see later that by a suitable transformation involving $\Psi(x)$ and a scaling, the information on positive solutions to

$$\Delta w - (1 - z_\lambda p)w + z_\lambda(e^{pw} - 1 - pw) = 0 \quad \text{in } \mathbf{R}^2 \tag{5.2}$$

is required. We enumerate properties of positive solutions of (5.2). Let

$$I(u) := \frac{1}{2}\int_{\mathbf{R}^2} (|\nabla u|^2 + (1 - z_\lambda p)u^2)\, dx - z_\lambda \int_{\mathbf{R}^2} \left\{\frac{1}{p}(e^{pu} - 1) - u - \frac{1}{2}pu^2\right\} dx.$$

Proposition 5.1 *(5.2) has a solution w satisfying*

(i) $w \in C^2(\mathbf{R}^2) \cap W^{1,2}(\mathbf{R}^2)$ *and* $w > 0$ *in* $\mathbf{R}^2$.

(ii) w *is spherically symmetric, i.e.,* $w(z) = w(r)$ *with* $r = |z|$ *and* $dw/dr < 0$ *for* $r > 0$.

(iii) *There exist constants* $C > 0$ *and* $\mu > 0$ *such that*

$$|D^\alpha w| \leq Ce^{-\mu|z|} \quad \text{for } z \in \mathbf{R}^2.$$

with $|\alpha| \leq 1$.

(iv) *For any nonnegative solution* $u \in C^2(\mathbf{R}^2) \cap W^{1,2}(\mathbf{R}^2)$ *of (5.2),* $0 < I(w) \leq I(u)$ *holds unless* $u \equiv 0$.

Such w is called a ground state solution of (5.2). For a proof, see Berestycki-Gallouët-Kavian [3].

Now we introduce a new function φ_ε constructed from the diffeomorphism Ψ which straightens a portion of the boundary. Recall the definition of ψ, Φ and Ψ. We assume that $x = \Phi(y)$ is defined in $\omega \supset \bar{B}_{3\kappa}$ where $\kappa > 0$ and B_r is the open ball centered at the origin with radius $r > 0$.

For $\rho > 0$, define a cut-off function $\zeta_\rho : [0, \infty) \mapsto \mathbf{R}$ by

$$\zeta_\rho(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \rho, \\ 2 - \dfrac{t}{\rho} & \text{if } \rho < t \leq 2\rho, \\ 0 & \text{if } 2\rho < t. \end{cases}$$

Let $w = w(z)$ be a ground state solution to (5.2) given by Proposition 5.1, and set

$$w_*(z) := \zeta_{\kappa/\varepsilon}(|z|)w(z).$$

Moreover, put $D_1 := \Phi(B_\kappa^+)$ and $D_2 := \Phi(B_{2\kappa}^+)$, where $B_r^+ = B_r \cap \mathbf{R}_+^2$. Note that $D_1 \subset D_2 \subset \Omega$. We define a comparison function φ_ε as

$$\varphi_\varepsilon(x) := \begin{cases} w_*(\psi(x)/\varepsilon) & x \in D_2, \\ 0 & x \in \Omega \backslash D_2. \end{cases} \tag{5.3}$$

Now we are in a position to state an asymptotic behavior of $M[\varphi_\varepsilon]$.

Proposition 5.2 *As $\varepsilon \downarrow 0$, the asymptotic expansion*

$$M[\varphi_\varepsilon] = \varepsilon^2\{\frac{1}{2}I(w) - \psi''(0)\gamma\varepsilon + o(\varepsilon)\}$$

holds, where

$$\gamma := \frac{1}{3}\int_{\mathbf{R}^2_+}(w'(|z|))^2 z_2\, dz.$$

To prove Proposition 5.2, we need three lemmas. The proofs of these Lemmas are almost identical to those in [10] in Appendix pp. 844–849. So we omit here.

Lemma 5.2 *There hold*

$$\int_{\mathbf{R}^2_+}(\frac{\partial w}{\partial z_2})^2 z_2\, dz_1\, dz_2 = 2\gamma$$

and

$$\int_{\mathbf{R}^2_+}\Big[\frac{1}{2}\Big\{|\nabla w|^2 + (1 - z_\lambda p)w^2\Big\} - z_\lambda\Big\{\frac{1}{p}(e^{pw} - 1) - w - \frac{1}{2}pw^2\Big\}\Big] z_2\, dz_1\, dz_2 = 2\gamma.$$

Lemma 5.3 *As $\varepsilon \downarrow 0$, the asymptotic equality*

$$\varepsilon^2\int_\Omega |\nabla\varphi_\varepsilon|^2\, dx = \varepsilon^2\Big\{\int_{\mathbf{R}^2_+}(w')^2\, dz - \psi''(0)\gamma\varepsilon + O(\varepsilon^2)\Big\}$$

holds. Moreover, in general, if $G : \mathbf{R} \mapsto \mathbf{R}$ is locally Hölder continuous and $G(0) = 0$, then

$$\int_\Omega G(\varphi_\varepsilon)\, dx = \varepsilon^2\Big\{\int_{\mathbf{R}^2_+} G(w)(1 - \psi''(0)\varepsilon z_2)\, dz_1\, dz_2 + O(\varepsilon^2)\Big\}$$

holds for $\varepsilon \downarrow 0$.

Lemma 5.4 *Let us define $h_\varepsilon(t)$ as*

$$h_\varepsilon(t) := \frac{t^2}{2}\Big(\int_\Omega(\varepsilon^2|\nabla\varphi_\varepsilon|^2 + c|\varphi_\varepsilon|^2)\, dx - z_\lambda\int_\Omega\Big\{\frac{1}{p}(e^{tp\varphi_\varepsilon} - 1) - t\varphi_\varepsilon - \frac{1}{2}p(t\varphi_\varepsilon)^2\Big\}\, dx.$$

Then for each $\varepsilon > 0$ sufficiently small, h_ε attains a unique positive maximum at $t = t_0(\varepsilon) > 0$ and

$$t_0(\varepsilon) = 1 + \beta\varepsilon + o(\varepsilon)$$

as $\varepsilon \downarrow 0$, where $\beta > 0$ is a constant.

Proof of Proposition 5.2. For the sake of simplicity, set

$$f(u) = z_\lambda(e^{pu} - 1 - pu) \quad \text{and} \quad F(u) = z_\lambda\{\frac{1}{p}(e^{pu} - 1) - u - \frac{1}{2}pu^2\}.$$

From the definition of $M[\varphi_\varepsilon]$, we have

$$M[\varphi_\varepsilon] = \frac{1}{2}t_0(\varepsilon)^2 \int_\Omega \{\varepsilon^2|\nabla\varphi_\varepsilon|^2 + (1 - z_\lambda p)\varphi_\varepsilon^2\}\, dx - \int_\Omega F(t_0\varphi_\varepsilon)\, dx.$$

By Lemmas 5.3 and 5.4, expanding $F(t(\varepsilon)\varphi_\varepsilon)$ in the Taylor series and the decay property of φ_ε, we have

$$\begin{aligned} M[\varphi_\varepsilon] &= \varepsilon^2\Big[\int_{\mathbf{R}^2_+} \Big\{\frac{1}{2}(w')^2 + (1 - z_\lambda p)w^2\Big\}\, dz \\ &+\varepsilon\Big[\beta\int_{\mathbf{R}^2_+} \Big\{(w')^2 + (1 - z_\lambda p)w^2 - wF'(w)\Big\}\, dz \\ &-\frac{\psi''(0)}{2}\{\gamma + (1 - z_\lambda p)\int_{\mathbf{R}^2_+} w^2 z_2\, dz - 2\int_{\mathbf{R}^2_+} F(w)z_2\, dz\}\Big\} + o(\varepsilon)\Big]. \end{aligned}$$

Since w is radial, we get

$$\begin{aligned} &2\int_{\mathbf{R}^2_+} \Big\{(w')^2 + (1 - z_\lambda p)w^2 - wF'(w)\Big\}\, dz \\ &\int_{\mathbf{R}^2} \Big\{|\nabla w|^2 + (1 - z_\lambda p)w^2 - wF'(w)\Big\}\, dz \\ &= \int_{\mathbf{R}^2} w(-\Delta w + (1 - z_\lambda p)w - F'(w))\, dz = 0. \end{aligned}$$

Moreover, since

$$\frac{1}{2}(1 - z_\lambda p)\int_{\mathbf{R}^2_+} w^2 z_2\, dz - \int_{\mathbf{R}^2_+} F(w)z_2\, dz = 2\gamma - \frac{3}{2}\gamma = \frac{1}{2}\gamma,$$

we obtain

$$M[\varphi_\varepsilon] = \varepsilon^2\{\frac{1}{2}I(w) - \psi''(0)\gamma\varepsilon + o(\varepsilon)\}.$$

□

Remark 5.1 Proposition 5.2 is valid for any positive radial solution w to (5.2) which decays exponentially at infinity. In Theorem 1.2, we have seen that $||u_\varepsilon||_{L^\infty(\Omega)}$ is bounded from above. Here we will show that the L^∞-norm is also bounded away from 0.

Lemma 5.5 *Suppose that* u_ε *attains its maximum at* $x_0 \in \bar{\Omega}$. *Then*

$$u_\varepsilon(x_0) \geq \bar{u}$$

holds for any sufficiently small $\varepsilon > 0$. *Moreover there exists* $\eta_0 > 0$ *independent of* x_0 *and* ε *such that* $u_\varepsilon(x) \geq \eta_0$ *holds for any* $x \in B_\varepsilon(x_0) \cap \Omega$ *if* ε *is sufficiently small, where* $\bar{u}$ *is a positive constant solution of*

$$\varepsilon^2 \Delta u - (1 - z_\lambda p)u + z_\lambda(e^{pu} - 1 - pu) = 0 \quad \text{in } \Omega$$

with the homogeneous Neumann boundary condition, i.e., $\bar{u}$ *satisfies* $\bar{u} = z_\lambda(e^{p\bar{u}} - 1)$.

Proof. Suppose that $u_\varepsilon(x_0) < \bar{u}$. If $x_0 \in \Omega$, then

$$\varepsilon^2 \Delta u_\varepsilon = u_\varepsilon - z_\lambda(e^{pu} - 1) > 0$$

holds in a neighborhood of x_0. This contradicts the fact that $\Delta u(x_0) \leq 0$ because x_0 is a local maximum point. Hence $x_0 \in \partial\Omega$. Hence $u_\varepsilon(x) < u_\varepsilon(x_0)$ in a neighborhood of x_0. Then by the Hopf boundary point lemma, we conclude that $\partial u_\varepsilon/\partial \nu > 0$ at x_0, which contradicts the boundary condition $\partial u/\partial \nu = 0$ on $\partial\Omega$. Thus we obtain

$$u_\varepsilon(x_0) \geq \bar{u}.$$

As for the latter part, by using Lemma 4.2(the Harnack inequality), we can find a constant $\bar{C} > 0$ independent of $\varepsilon > 0$ such that

$$\sup_{B_\varepsilon(x_0)\cap\Omega} u_\varepsilon \leq \bar{C} \inf_{B_\varepsilon(x_0)\cap\Omega} u_\varepsilon.$$

Hence from the former part, we have

$$\inf_{B_\varepsilon(x_0)\cap\Omega} u_\varepsilon \geq \frac{1}{\bar{C}} u_\varepsilon(x_0) \geq \frac{1}{\bar{C}}\bar{u}.$$

□

6 Proof of Theorem 1.4

Finally, we have arrived at the position to prove Theorem 1.4.

Suppose that at $P_\varepsilon \in \bar{\Omega}$, u_ε attains its maximum. We will prove Theorem 1.4 in three steps. Although the way of proving Theorem 1.4 is almost identical to that of Theorem 1.2 of [10], we just give a sketch of a proof.

Step 1. We prove that there exists $C^* > 0$ independent of $\varepsilon > 0$ such that

$$\text{dist}\,(P_\varepsilon, \partial\Omega) \leq C^*\varepsilon$$

if $\varepsilon > 0$ is sufficiently small.

Suppose to the contrary that there exists a sequence $\{\varepsilon_j\}$ $(\varepsilon_j \downarrow 0)$ such that

$$\rho_j := \frac{\text{dist}\,(P_\varepsilon, \partial\Omega)}{\varepsilon_j} \to \infty$$

as $j \to \infty$. Let us define a scaled function $v_j(z) := u_{\varepsilon_j}(P_j + \varepsilon_j z)$, $z \in B_{\rho_j}$. Then by the elliptic regularity theory (see, e.g. [6]), we can extract a subsequence, still denoted by $\{v_j\}$, such that

$$v_j \to w \quad \text{in } C^2_{loc}(\mathbf{R}^2)$$

with $w(\not\equiv 0) \in C^2(\mathbf{R}^2) \cap W^{2,r}(\mathbf{R}^2)$.

Now we estimate the minimax value c_{ε_j} from below. Using $I(w)$, we have

$$c_{\varepsilon_j} \geq \varepsilon_j^2(I(w) - C_3 \exp(-\mu_2 R))$$

for any $j \geq j_m$ with C_3 and $\mu_2 > 0$ inidepedent of j and m.

On the other hand, we have from Proposition 5.2 and Remark 5.1

$$c_{\varepsilon_j} < \varepsilon_j^2 \frac{1}{2} I(w)$$

if ε_j is sufficiently small. This is a contradiction. Thus we have proved

$$\text{dist}\,(P_\varepsilon, \partial\Omega) \leq C^*\varepsilon.$$

Step 2. We will prove $P_\varepsilon \in \partial\Omega$ if $\varepsilon > 0$ is sufficiently small. Suppose to the contrary that there exists a decreasing sequence $\{\varepsilon_k\}$ $(\varepsilon_k \downarrow 0$ as $k \to \infty)$ such that $P_{\varepsilon_k} \in \Omega$. From Step 1, we may suppose that $P_k =: P_{\varepsilon_k} \to P \in \partial\Omega$

as $k \to \infty$ choosing a subsequence if necessary. We may regard $P = (0,0)$. We use the diffeomorphism Φ introduced in (5.1). The inverse image Ψ of Φ straightens a boundary portion of P. We may suppose that Φ is defined in an open set containing the closed ball $\bar{B}_{2\kappa}$ and that $Q_k := \Psi(P_k) \in B_\kappa^+$ for all k. Put

$$v_k(y) := u_{\varepsilon_k}(\Phi(y)) \quad \text{for } y \in \bar{B}_{2\kappa}^+, \tag{6.1}$$

and extend it to $\bar{B}_{2\kappa}$ by reflection:

$$\tilde{v}_k(y) := \begin{cases} v_k(y) & \text{if } y \in \bar{B}_{2\kappa}^+, \\ v_k(y_1, -y_2) & \text{if } y \in \bar{B}_{2\kappa}^-, \end{cases} \tag{6.2}$$

where $B_{2\kappa}^- = \{y \in B_{2\kappa} \,|\, y_2 < 0\}$. Moreover, we define a scaled function $w_k(z)$ by

$$w_k(z) := \tilde{v}_k(Q_k + \varepsilon_k z) \quad \text{for } z \in \bar{B}_{\kappa/\varepsilon_k}. \tag{6.3}$$

Let $Q_k = (q_k, \alpha_k \varepsilon_k)$ with $q_k \in \mathbf{R}$ and $\alpha_k > 0$. Then by the Step 1, $\{\alpha_k\}$ is bounded. It is easily seen that

$$w_k \in C^2(\bar{B}_{\kappa/\varepsilon_k} \backslash \{z_2 = -\alpha_k\}) \cap C^1(\bar{B}_{\kappa/\varepsilon_k})$$

since $\partial v_k / \partial y_2 = 0$ on $\{y_2 = 0\}$. Similar to Step 1, we obtain a convergent subsequence (still denoted by $\{w_k\}$) such that

$$w_k \to w \quad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}^2)$$

and $w \in C^2(\mathbf{R}^2) \cap W^{2,r}(\mathbf{R}^2)$. The limit w satisfies

$$a_{11}(0)\frac{\partial^2 w}{\partial z_1^2} + 2a_{12}(0)\frac{\partial^2 w}{\partial z_1 \partial z_2} + a_{22}(0)\frac{\partial w}{\partial z_2^2} - (1 - z_\lambda)w + f(w) = 0 \quad \text{in } \mathbf{R}^2.$$

However, in fact, since $Q_k \to (0,0)$ as $k \to \infty$ and $D\Psi(0) = [D\Phi(0)]^{-1} = I$, we have

$$\Delta w - (1 - z_\lambda p)w + f(w) = 0 \quad \text{in } \mathbf{R}^2.$$

Moreover, w is raidalyy symmetric with respect to the origin and decays exponentially at infinity. Fix $R > 0$ sufficietly large. Then we can find an integer k_R such that, for $k \geq k_R$,

$$||w_k - w||_{C^2(\bar{B}_{4R})} \leq \eta_R. \tag{6.4}$$

This shows that w_k has only one maximum point in B_R. If $\alpha_k > 0$, then by the definition of $\tilde{v}_k$, $Q_R^* = (q_k, -\alpha_k\varepsilon_k)$ is also a local maximum point of $\tilde{v}_k$, i.e., $(0, -\alpha_k)$ is another local maximum point of w_k in B_R, which is a contradiction. Step 2 is complete.

Step 3. We shall show that u_ε has at most one local maximum point. Suppose to the contrary that there exists a decreasing sequence $\{\varepsilon_k\}$such that u_{ε_k} has two local maxima at P_k and P_k'. From Step 2, P_k and P_k' are on the boundary. Moreover, we may assume

$$\frac{|P_k - P_k'|}{\varepsilon_k} \to \infty \quad (k \to \infty)$$

since otherwise, the scaled function w_k has two local maxima in B_R, which contradicts Step 2.

We introduce the diffeomorphism $y = \Psi(x)$ which straightens a boundary portion around P_k as in Section 5 and define $v_k, \tilde{v}_k$ and w_k by (6.1), (6.2) and (6.3), respectively. Then by the compactness argument as in Step 2, we see that $\{w_k\}$ has a convergent subsequence, still denoted by $\{w_k\}$, converging to $w \in C^2(\mathbf{R}^2) \cap W^{2,2}(\mathbf{R}^2)$ in the $C^2_{\text{loc}}(\mathbf{R}^2)$ topology and w is a positive radial solution to (5.2).

Now we estimate c_{ε_k} from below. Similar to Step 1, we have

$$c_{\varepsilon_k} \geq \varepsilon_k^2 \Big\{ \frac{1}{2} I(w) + C_4 - C_5 e^{-\mu R} - C_6 \varepsilon_k \Big\} \tag{6.5}$$

with positive constants C_4, C_5 and C_6 by making use of the exponential decay of w. Now choosing $P \in \partial\Omega$ such that $\psi''(0) > 0$, we see from Proposition 5.2 that

$$c_{\varepsilon_k} < \varepsilon_k^2 \frac{1}{2} I(w)$$

if ε_k is sufficiently small, which is inconsistent with (6.5). Therefore u_ε has at most one maximum. □

References

[1] Adams, R., *Sobolev Spaces*, Academic Press, New York (1975).

[2] Ambrosetti, A . and Rabinowitz, P., Dual variational methods in critical point theory and applications, J. Funct. Anal. **14** (1973), 349–381.

[3] Berestycki, H., Gallouët, T. and Kavian, O., Equations de champs scalaires euclidiens nonlinéaire dans le pla, C.R. Acad. Sci. Paris, Série I Math. **297** (1983), 307–310.

[4] Donaldson, T.K. and Trudinger, N. S., Orlicz-Sobolev spaces and imbedding theorems, J. Funct. Anal. **8** (1971), 52–75.

[5] Gidas, B., Ni, W.-M. and Nirenberg. L., Symmetry of positive solutions of semilinear elliptic equations in $\mathbf{R}^n$, Advances in MAth., Suppl. Studies **7A** (1981), 369–402.

[6] Gilbarg, D. and Trudinger, N. S., *elliptic Partial Differential Equations of Second Order,* Second edition, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1983).

[7] Keller, E. F. and Segel, L. A., Initiation of slime mold aggregation viewed as instability, J. Theoret. Biol. **26** (1970), 399–415.

[8] Lin, C.-S. and Ni, W.-M., On the diffusion coefficient of a semilinear Neumann problem, in *Calculus of Variations and Partial Differential Equations*, Lect. Note Math. **1340**, 160–174, Springer-Verlag, Berlin Heidelberg (1988)

[9] Lin C.-S., Ni, W.-M. and Takagi, I., Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations **72** (1988), 1–27.

[10] Ni, W.-M. and Takagi, I., On the shape of least-energy solutions to a semilinear Neumann problem, Comm. Pure Appl. Math., **44** (1991), 819–851.

[11] Ni, W.-M. and Takagi, I., Location of the peaks of least-energy solutions to a semilinear Neumann problem, Duke Math. J. **72** (1993), 247–281.

[12] Ni, W.-M. and Takagi, I., Point condensation generated by a reaction-diffusion system in axially symmetric domains, Japan J. Indust. Appl. Math. **12** (1995), 327–365.