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Keller-Segel System and the Concentration Lemma

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1 Introduction

We consider time-global existence and blow-up of solutions of the following system related to chemotaxis

$$\text{(P)} \qquad \begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \gamma v + \alpha u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, χ, γ and α are postive constants and u_0 is a non-negative smooth function on $\overline{\Omega}$.

There exists a unique solution (u, v) to (P) defined on a maximal interval of existence $[0, T_{max})$, which is smooth in $x \in \overline{\Omega}$ and $0 < t < T_{max}$. If $u_0 \not\equiv 0$ in Ω, the solution satisfies that $u(x,t) > 0$, $v(x,t) > 0$ for $(x,t) \in \Omega \times (0, T_{max})$. If $T_{max} < \infty$, we can observe the following.

Proposition 1 *If $T_{max} < \infty$, then the following relations hold.*

(i) $\lim_{t \to T_{max}} \|u \log u\|_{L^1(\Omega)} = \infty$

(ii) $\lim_{t \to T_{max}} \|\nabla v\|_{L^2(\Omega)} = \infty$.

(iii) *For $a > \chi/2$, then $\lim_{t \to T_{max}} \int_\Omega e^{av(x,t)} dx = \infty$.*

Then, if $T_{max} < \infty$, we have that

$$\lim_{t \to T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \lim_{t \to T_{max}} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty,$$

which we mean that the solution blows up in finite time.

Let L be an arbitrary positive constant and let $D_L = \{x \in \mathbf{R}^2 | |x| < L\}$. We have the following results.

Theorem 1 *Suppose*

$$\Omega = D_L \quad \text{and} \quad \|u_0\|_{L^1(D_L)} < 8\pi/(\alpha\chi). \tag{1}$$

Let $u_0(x) = u_0(-x)$ *on* D_L. *Then (P) admits a unique classical solution* (u, v) *on* $\overline{D_L} \times (0, \infty)$ *satisfying*

$$\sup_{t \geq 0} \{\|u(\cdot, t)\|_{L^\infty(D_L)} + \|v(\cdot, t)\|_{L^\infty(D_L)}\} < \infty.$$

Definition 1 *We say that* q *is a blow-up point of* u *if there exists* $\{t_k\}_{k=1}^{\infty} \subset [0, T_{max})$ *and* $\{x_k\}_{k=1}^{\infty} \subset \overline{\Omega}$ *satisfying* $u(x_k, t_k) \to \infty$, $t_k \to T_{max} < \infty$ *and* $x_k \to q \in \overline{\Omega}$ *as* $k \to \infty$. *We denote the set of all blow-up points of* u *by* $\mathcal{B}$.

Theorem 2 *Let (1) hold. Let* a_* *be a root of* $a_* - \chi/2 - \|u_0\|_{L^1(\Omega)}\alpha a_*^2/16\pi = 0$ *such that* $a_* < \chi$. *If* $T_{max} < \infty$, *then there exists a point* $q \in \mathcal{B} \bigcap \partial\Omega$ *satisfying*

$$\limsup_{t \to T_{max}} \int_{\Omega \bigcap B(q, \varepsilon)} u(x, t)dx \geq \frac{2\pi}{a_* \alpha} \qquad \textit{for any } \varepsilon > 0.$$

Definition 2 *For* $q \in \mathcal{B}$, *we say that* q *is an isolated blow-up point if there exists* $\delta > 0$ *such that*

$$\sup\{u(x, t)|\ 0 \leq t < T_{max} \textit{ and } x \in \overline{\{B(q, \delta) \backslash B(q, \varepsilon)\} \bigcap \Omega}\} < \infty \qquad \textit{for any } \varepsilon \in (0, \delta).$$

We denote the set of all isolated blow-up points of u *by* $\mathcal{B}_I$.

Theorem 3 *Suppose* $T_{max} < \infty$. *Then the following properties hold.*

For $q \in \mathcal{B}_I$, *there exist two positive constants* δ, $m \geq m_*$ *and a non-negative function* $f \in L^1(E(q, \delta)) \bigcap C(E(q, \delta) \backslash \{q\})$ *such that*

$$w^*\text{-}\lim_{t \to T_{max}} u(\cdot, t) = m\delta_q + f \qquad \textit{in } \mathcal{M}(E(q, \delta)),$$

where $E(q, \delta) = \overline{B(q, \delta) \bigcap \Omega}$,

$$m_* = \begin{cases} 4\pi/(\alpha\chi) & \textit{if } q \in \partial\Omega, \\ 8\pi/(\alpha\chi) & \textit{if } q \in \Omega \end{cases}$$

and $\mathcal{M}(S)$ *is the Banach space consisting of all Radon measures on a compact Hausdorff space* S *with the usual norm.*

For a set K, we denote the number of elements of K by $^\sharp E$. The following corollary is an immediate consequence of Theorem 3 .

Corollary 1 *Suppose* $T_{max} < \infty$. *Then* $\mathcal{B}_I$ *satisfies that*

$$^\sharp\{\mathcal{B}_I \bigcap \Omega\} + \frac{1}{2}{}^\sharp\{\mathcal{B}_I \bigcap \partial\Omega\} \leq \frac{\alpha\chi}{8\pi}\|u_0\|_{L^1(\Omega)}.$$

The following collorary is an immediate consequence of Theorem 3 and [6].

Corollary 2 *Suppose* $\Omega = D_L$ *and that* u_0 *be radially symmetric. If* $\int_{D_L} u_0(x)dx > 8\pi/(\alpha\chi)$ *and* $\int_{D_L} u_0(x)|x|^2 dx$ *is sufficiently small, then* $T_{max} < \infty$ *and there exist a positive constant* $m \geq 8\pi/(\alpha\chi)$ *and a non-negative function* $f \in L^1(D_L) \bigcap C(\overline{D_L} \backslash \{0\})$ *such that*

$$w^*\text{-}\lim_{t \nearrow T_{max}} u(\cdot, t) = m\delta_0 + f \qquad \textit{in } \mathcal{M}(\overline{D_L}).$$

2 Proof of Theorem 1

Lemma 1 *Let (u,v) be a solution to (P). Put*

$$W(t)=\int_\Omega\left\{u\log u-\frac{\chi}{2\alpha}\left(|\nabla v|^2+\gamma v^2\right)\right\}dx.$$

Then, it follows that

$$\frac{d}{dt}W(t)+\int_\Omega u|\nabla(\log u-\chi v)|^2dx=0 \qquad \textit{for } t\in(0,T_{max}).$$

Proof of Lemma 1: Multiplying $\log u-\chi v$ by the first equation of (P) and using the second equation of (P), then we have this lemma.

Lemma 2 *Suppose that (u,v) is a solution to (P). Let a be an arbitrary positive constant and let $M=\|u_0\|_{L^1}$. Then, the inequality*

$$a\int_\Omega uvdx\le\int_\Omega u\log udx+M\log\left(\int_\Omega e^{av}dx\right)-M\log M$$

holds for $0\le t<T_{max}$.

Proof of Lemma 2. Let

$$\mu=\int_\Omega e^{av}dx \text{ and } \psi=\frac{M}{\mu}e^{av}.$$

Then, we have that

$$\begin{aligned}0&=-\log\left(\int_\Omega\frac{\psi u}{u\mu}dx\right)\\&\le\int_\Omega\left(-\log\frac{\psi}{u}\right)\frac{u}{M}dx,\end{aligned}$$

by which together with Jensen's inequality we get this lemma.

Proposition 2 *If w is a function on $\overline{D_L}$ satisfies that $w\in C^1(D_L)$, $w(x)=w(-x)$ on ∂D_L and*

$$\frac{\partial w}{\partial n}=0 \qquad \textit{on } \partial D_L,$$

then there exists absolute constants C and K such that

$$\log\left(\int_\Omega e^{w}dx\right)\le\frac{1}{16\pi}\int_\Omega|\nabla w|^2dx+C\|w\|_{L^1}+K.$$

Proof of Theorem 1. By Lemmas 1 , 2 and the second equation of (P), we have that

$$\left(a-\frac{\chi}{2}\right)\frac{1}{\alpha}\int_{\Omega}\left(|\nabla v(x,t)|^2+\gamma|v(x,t)|^2\right)dx \le W(0)+M\log\left(\int_{\Omega}e^{av(x,t)}dx\right)-M\log M,$$

by which togehter with Proposition 2 it follows that

$$\left\{\left(a-\frac{\chi}{2}\right)\frac{1}{\alpha}-\frac{Ma^2}{16\pi}\right\}\int_{\Omega}\left(|\nabla v(x,t)|^2+\gamma|v(x,t)|^2\right)dx \le W(0)+M\left(\frac{CaM}{|D_L|}+K-\log M\right).$$

Because of $\|u_0\|_{L^1(\Omega)} = M < 8\pi/\alpha\chi$, we can take a constant a satisfying

$$\left(a-\frac{\chi}{2}\right)\frac{1}{\alpha}-\frac{Ma^2}{16\pi} > 0.$$

Then gives

$$\sup_{0\le t<T_{max}}\int_{D_L}(|\nabla v|^2+\gamma|v|^2)dx < \infty.$$

and hence $T_{max} = \infty$ by the case (ii) of Proposition 1 . □

3 Proof of Theorem 2

Proposition 3 *Let $h > 0$ and $\mathcal{E} = \{w \in C^1(\overline{D_L})|\frac{\partial w}{\partial n} = 0$ on ∂D_L and $\|w\|_{L^1} \le h\}$. Then, for any $\mathcal{F} \subset \mathcal{E}$ satisfy 1 or 2:*

(i) *For any $\varepsilon > 0$, there exists a positive constant C_ε s.t.*

$$\log\left(\int_{\Omega}e^w dx\right) \le \frac{1+\varepsilon}{16\pi}\int_{\Omega}|\nabla w|^2dx + C_\varepsilon \text{ for } w \in \mathcal{F}.$$

(ii) *There exist a sequence $\{w_k\} \subset \mathcal{F}$, a point $q \in \partial D_L$, a constant $m \in [1/2, 1)$ and a regular measure ψ s.t.*

$$w^*\text{-}\lim_{k\to\infty}\frac{exp(w_k)p_*}{\int_{D_L}exp(w_k)p_*dx} = m\delta_q + \psi \qquad \text{in } \mathcal{M}(\overline{D}),$$

where $p_* = \frac{8}{L^2(1+(x/L)^2)^2}$.

Next we consider the following elliptic problem related to the second equation of (P).

$$\text{(EE)} \qquad \begin{cases} 0 = \Delta w - \gamma w + f, & x \in \Omega, \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega \end{cases}$$

with $f \in L^1(\Omega), \ge 0$.

Proposition 4 *Let $q \in \Omega$. Then there exists a positive constant η_0 such that for $\eta \in (0, \eta_0)$ there exists a positive constant C_{η_0} satisfying*

$$\int_{\Omega \bigcap B(q,\eta)} e^w dx \le exp\left(\frac{C_{\eta_0}}{\eta_0 - \eta}\|f\|_{L^1(\Omega)}\right) \int_{|x|<2\eta_0} \frac{dx}{|x|^\theta},$$

where

$$\theta = \begin{cases} \frac{1}{2\pi}\int_{B(q,\eta_0)} |f| dx & \text{if } q \in \Omega, \\ \frac{1+O(\eta_0)}{\pi}\int_{\Omega \bigcap D(q,\eta_0)} |f| dx & \text{if } q \in \partial\Omega. \end{cases}$$

Proof of Theorem 2: Let a be a positive constant satisfying

$$\left(a - \frac{\chi}{2}\right)\frac{1}{\alpha} - \frac{Ma^2}{16\pi} > 0.$$

We assume that $\{av\}$ satisfies 1 of Proposition 3, by which together with the arguments of proof of Theorem 1 it follows that $T_{max} = \infty$. It is the contradiction. Then, for any positive constant a with

$$\left(a - \frac{\chi}{2}\right)\frac{1}{\alpha} - \frac{Ma^2}{16\pi} > 0,$$

we observe that

$$\begin{aligned} \infty &= \limsup_{t \to T_{max}} \int_\Omega e^{av} dx \\ &\le \frac{1}{m} \limsup_{t \to T_{max}} \int_{\Omega \bigcap B(q,\varepsilon)} e^{av} dx \qquad \text{for any } \varepsilon > 0, \end{aligned} \tag{2}$$

by which together with Proposition 4 we have this theorem. □

4 Proof of Theorem 3

By using Proposition 1 and Lemmas 1 and 2, we have (2) for any $q \in \mathcal{B}_I$ and any $a > \chi/2$, by which together with Proposition 4 we have this theorem.

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