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NONEXISTENCE RESULTS OF HARMONIC MAPS BETWEEN HADAMARD MANIFOLDS

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1. NONEXISTENCE RESULTS

Harmonic maps have been studied by so many mathematicians since the famous paper by Eells-Sampson [10] was published. Especially, about the case that the source manifold is compact, existence problems have been studied very deeply and we know many results today. In contrast, about harmonic maps between noncompact manifolds, we do not know so many results yet. On existence, we know only results for some special cases. For example, in the case that both (source and target) manifolds are hyperbolic spaces, we know the results by [1], [22], [23], [24] and [2]. But, no general existence theorem is known. On the other hand, for noncompact case, we can expect not only existence results but also nonexistence results. For example, since the notion of " harmonic map" is a natural extension of one of " harmonic function", it is very reasonable to expect " Liouville-type theorem". In this article we introduce some nonexistence results.

Let M and N be complete Riemannian manifolds of dimension m and n $(m, n \geq 2)$ respectively. For a map $U \in C^1(M,N)$ we define the *energy density* $e(U)(p)$ of U at $p \in M$ by

$$e(U)(p) = \frac{1}{2}\|dU(p)\|^2,$$

where $\| \ \|$ denotes the norm induced from the tensor product norm on $T_p^*M \otimes T_{U(p)}N$. For a bounded domain $\Omega \subset M$, we define the energy of U on Ω by

$$E(U;\Omega) = \int_\Omega e(U)d\mu,$$

where $d\mu$ stands for the volume element on M. A map $U : M \to N$ is said to be *harmonic* if is of class C^2 and satisfies the Euler-Lagrange equation of the energy functional.

As mentioned above, it seems to be reasonable to expect that a Liouville-type theorem will hold about harmonic maps. In fact, a Liouville-type theorem has been proved by S.Hildebrandt-J.Jost-K.-O.Widman [17]. (See also [4], [11] and [28].)

Theorem 1.1 (Hildebrandt-Jost-Widman [17]**).** *Let U be a harmonic map of simple or compact Riemannian manifold M of class C^1 into a complete Riemannian manifold N of class C^3, the sectional curvature of which is bounded from above by a constant $\kappa^2 \geq 0$. Denote by $B_R(q)$ a geodesic ball in N with radius $R < \pi/(2\kappa)$ which does not meet the cut locus of its center q_0. Assume also that the range $U(M)$ of the map U is contained in $B_R(q_0)$. Then U is a constant map.*

Here, a Riemannian manifold is said to be *simple* if it is diffeomorphic to an Euclidean m-space $\mathbb{R}^m$ and furnished with a metric for which associated Laplace-Beltrami operator is uniformly elliptic.

For the case that the target manifold N has a *pole* q_0 (i.e. the exponential map at q_0 gives a diffeomorphism between N and an Euclidean space) and whose radial curvature is bounded by a sufficiently rapidly decreasing function of the distance from the pole, L.Karp [19] proved that nonconstant harmonic maps defined on a complete, noncompact manifold satisfy a certain growth order condition. This result implies nonexistence of nonconstant harmonic maps under some growth order condition.

Theorem 1.2 (Karp [19]). *Let $U : M \to N$ be a harmonic map and suppose N has a pole q_0 and all radial curvature at $q \in N$ are smaller than or equal to $K(r)$, $r = \mathrm{dist}(q, q_0)$, where $K : [0,\infty) \to \mathbb{R}$ satisfies $0 < 1 - \int_0^\infty rK(r)dr = \delta \le 1$. If U is not constant then*

$$\limsup_{r\to\infty} \frac{1}{r^2F(r)} \int_{B_r(q_0)} \{\mathrm{dist}(U(x), q_0)\}^p d\mu = +\infty$$

for every $F \in \mathcal{F}$ and every $p > 2 - \delta$, where

$$\mathcal{F} = \{F : (0,\infty) \to (0,\infty) | \int_1^\infty \frac{dr}{rF(r)} = +\infty\}.$$

These results show nonexistence of harmonic maps under the conditions on the growth of the maps. On the other hand, in [14] S.I.Goldberg-T.Ishihara-N.C.Petridis proved a nonexistence result of another type. (See also [13] and [27].)

Theorem 1.3 (Goldberg-Ishihara-Petridis [14]). *Let M be a complete connected locally flat Riemannian manifold and N a Riemannian manifold with negative sectional curvature bounded away from 0. Then a harmonic map of bounded dilatation $U : M \to N$ is a constant.*

Here, a map $\varphi : (M, h) \to (N, g)$ is said to have *bounded dilatation* if there exists a number K such that for each $x \in M$, either $d\varphi(x) = 0$ or $\lambda_1/\lambda_2 \le K$, where

$$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r > 0$$

are the positive eigenvalues of the pull-back metric $\varphi^* g$. W.S.Kendall [20] , [21] gave probabilistic extension of this result. (See also Theorem 1.2 of [3].)

For the case that N is a Hadamard manifold whose sectional curvature is bounded above by a nonpositive constant, in [30] the author has shown nonexistence of a harmonic map U from an Euclidean m-space $\mathbb{R}^m$ to N under certain nondegeneracy condition (1.2) below. Moreover, the above result was extended in [31].

Theorem 1.4 (Tachikawa [31]). *Let M be a Riemannian m-manifold with a pole $p_0 \in M$, $(x^1, \dots, x^m)$ a normal coordinate system centered at p_0 and $k_M(x)$ the minimum of the sectional curvature of M at x. Assume that*

$$(1.1) \qquad -\min\{k_M(x), 0\} \le O(r^{-2}) \text{ as } r = |x| \to \infty.$$

Let N be an Hadamard n-manifold whose sectional curvature are bounded above by a negative constant $-\kappa^2$. Then there exists no harmonic map $U : M \to N$ which satisfies the following condition.

$$(1.2)\qquad |x|^2 \left\{\frac{\kappa}{\sinh(\kappa\rho(x))}\right\}^2 \{e(U)(x) - e(\rho)(x)\} \geq \varepsilon_0 > 0$$

where $\rho(x) = \mathrm{dist}_N(U(0), U(x))$ and $e(\rho)(x) = h^{\alpha\beta}(x)D_\alpha\rho(x)D_\beta\rho(x)$.

Let $u(x)$ be an expression of $U(x)$ with respect to a normal coordinate system $(u^1, ..., u^n)$ on N centered at $U(p_0)$. We can see that

$$e(u)(x) - e(\rho)(x) = |u|^2 g_{ij}(u)h^{\alpha\beta}(x)D_\alpha\frac{u^i}{|u|}D_\beta\frac{u^j}{|u|},$$

where $|u(x)| = \sqrt{\sum_{i=1}^n (u(x)^i)^2}$. Moreover, the assumption on the curvature of N implies that

$$|u|^2 g_{ij}(u)X^iX^j \geq \left\{\frac{\sinh(\kappa|u|)}{\kappa}\right\}^2 |X|^2, \text{ for } X \in \mathbb{R}^n \text{ with } g_{ij}(x)X^iu^j = X^iu^j = 0.$$

(See Lemma 2.1 of [31].) Thus, we get

$$\left\{\frac{\kappa}{\sinh(\kappa\rho(x))}\right\}^2 \{e(u)(x) - e(\rho)(x)\}$$

$$\geq \sum_{i=1}^n h^{\alpha\beta}(x)D_\alpha\frac{u^i}{|u|}D_\beta\frac{u^i}{|u|}.$$

Therefore, writing $\xi = u/|u|$ we can employ the condition

$$(1.3)\qquad e(\xi)(x) = \sum_{i=1}^n h^{\alpha\beta}(x)D_\alpha\xi^iD_\beta\xi^i \geq \varepsilon_0/|x|^2,$$

instead of (1.2). This is the reason to call (1.2) a rotational nondegeneracy condition.

On the other hand, A.Ratto-M.Rigoli[26] showed a nonexistence result of similar type for harmonic maps U from a *model* $M^m(f)$ to N as above. Here a *model* $M^m(f)$ is a warped product manifold $[0,\infty) \times_f S^{m-1}$ i.e.

$$M^m(g) = ([0,\infty) \times S^{m-1}, dr^2 + f^2(r)d\theta^2).$$

Theorem 1.5 (Ratto-Rigoli[26]). *Let N be a Hadamard manifold with sectional curvature bounded above by a negative constant and $M^m(f)$ a model such that $[f]^{-1} \notin L^1(+\infty)$ and f' is bounded above by some positive constant. Then, there are no nonconstant harmonic maps $U : M^m(f) \to N$ such that, on $\{x \in M^m(f) : U(x) \neq U(0)\}$*

$$(1.4)\qquad e(\xi) \geq \frac{c}{f^2(r)} \text{ for some constant } c > 0,$$

where ξ is as in (1.3).

Anyway, in these results, global conditions (1.2) or (1.4) are assumed. In the following theorem, the global condition (1.2) are replaced by a condition on the asymptotic behavior of U.

Theorem 1.6 (Akutagawa-Tachikawa [3]). *Let M be a simple Riemannian m-manifold with a pole $p_0 \in M$, $(x^1, ..., x^m)$ a normal coordinate system centered at p_0 and $k_M(x)$ the minimum of the sectional curvature of M at x. Assume that $k_M(x)$ satisfies (1.1). Let N be an Hadamard n-manifold whose sectional curvature are bounded above by a negative constant $-\kappa^2$. Then there exists no harmonic map $U : M \to N$ which satisfies the following condition.*

$$\text{(1.5)} \qquad \liminf_{|x|\to\infty} (\log |x|) \left\{ |x|^2 \left(\frac{\kappa}{\sinh(\kappa\rho)} \right)^2 (e(U)(x) - e(\rho)(x)) \right\} > 0,$$

where $\rho(x) = dist_N(U(x), q_0)$ for an arbitrarily fixed point $q_0 \in N$.

Now, to state the next result, let us introduce some notations.

For a Riemannian manifold $P = (P^p, \gamma)$, $(\ ,\ )_{\gamma(q)}$ denotes the inner product on the tangent space T_qP with respect to the metric γ and $\|X\|_{\gamma(q)} = \sqrt{(X, X)_{\gamma(q)}}$. If P has a pole q_0, let $\sigma(q_0, q)(t)$ be the geodesic curve such that $\sigma(q_0, q)(0) = q_0$ and $\sigma(q_0, q)(1) = q$, $K_P(q; \pi)$ the sectional curvature of P at q with respect to the plane section π and $k_{P,\text{rad}}(q\ ; q_0)$ the maximum of the radial curvature of P at q, i.e.

$$\text{(1.6)} \qquad k_{P,\text{rad}}(q\ ; q_0) := \max\{K_P(q\ ; \pi) : \pi \ni \sigma'(q_0, q)(1)\}.$$

Moreover, let us define the minimum eigen value of $\gamma(q)$ with respect to the tangent vectors which are orthogonal to the $\sigma'(q_0, q)(1)$, $\lambda_P(q; q_0)$ by

$$\text{(1.7)} \qquad \lambda_P(q; q_0) := \inf \left\{ \|\xi\|^2_{\gamma(q)} / |\xi|^2\ ;\ (\xi, \sigma'(q_0, q)(1))_{\gamma(p)} = 0 \right\}.$$

Here and in the sequel, $|\ |$ denotes the standard Euclidean norm.

For the case that the source manifold M is an Euclidean space, the assumption on the curvature of the target manifold N can be weaken as follows.

Theorem 1.7 ([32]). *Let $N = (N^n, g)$ be a Hadamard n-manifold. Assume that*

$$\text{(1.8)} \qquad \{\text{dist}(p_0, p)\}^2 |k_{N,\text{rad}}(p\ ; p_0)| \geq \kappa > 0 \text{ as } \text{dist}(p_0, p) \to \infty.$$

Then there exists no harmonic map $U : \mathbb{R}^m \to N$ which satisfies the following condition.

$$\text{(1.9)} \qquad \liminf_{|x|\to\infty} \left\{ |x|^2 \left(\frac{1}{\rho^2 \lambda_N(U(x); q_0)} \right) (e(U)(x) - e(\rho)(x)) \right\} > 0,$$

where $\rho(x) = \text{dist}_N(U(x), q_0)$ for some $q_0 \in N$.

Remark. The case that N satisfies the curvature condition in Theorem 1.6, we can take

$$\lambda_N(U(x); q_0) = \left(\frac{1}{\kappa} \sinh \kappa |u(x)| \right)^2.$$

Therefore, the difference between the conditions (1.5) and (1.9) is only $\log |x|$.

In the next section, we show the outline of the proof of Theorem 1.7.

2. Outline of the Proof of Theorem 1.7

First of all, we need some differential geometric estimates which are based on Rauch's comparison theorem (cf. Lemma 6 of [17]).

Lemma 2.1. *Let N be a Riemannian n-manifold with a pole q_0, $(y^1, \cdots, y^n)$ a normal coordinate system centered at q_0 and $(g_{ij}(y))$ the metric tensor with respect to the normal coordinate system. Let f be a function of class $C^2(\mathbb{R}_+, \mathbb{R}_+)$ which satisfy*

$$\lim_{t\to 0}\frac{f(t)}{t} = 1, \ \ f(t) > 0 \ \forall t \in (0,\infty). \tag{2.1}$$

Assume that

$$k_{N,\mathrm{rad}}(0;y) \le -\frac{f''(t)}{f(t)}, \tag{2.2}$$

where $t = |y| = \sqrt{\sum_{j=1}^n (y^j)^2}$. Then we have the following estimates

$$g_{ij}(y)(X^iX^j + y^k\Gamma^i_{kl}(y)X^jX^l) \ge |\zeta|^2 + t\frac{f'(t)}{f(t)}g_{ij}(y)\xi^i\xi^j, \tag{2.3}$$

$$g_{ij}(y)X^iX^j \ge |\zeta|^2 + \frac{f^2(t)}{t^2}|\xi|^2, \tag{2.4}$$

for all $y, X \in \mathbb{R}^n$, where $t = |y|$, $\zeta = (X,y)y/t^2$ and $\xi = X - \zeta$.

The estimates as (2.3) are used very often to estimates nonlinear terms of the equations of harmonic maps.

Let $u = (u^1(x), \cdots, u^n(x))$ be the expression of a harmonic map $U : \mathbb{R}^m \to N$ in terms of a normal coordinate system centered at any fixed point q_0 in N. Then u satisfies the following equation of weak form.

$$\int_{\mathbb{R}^m}\sum_{\alpha=1}^m g_{ij}\{D_\alpha u^i D_\alpha\varphi^j + \varphi^k\Gamma^j_{kl}D_\alpha u^l D_\alpha u^j\}dx = 0, \quad \forall\varphi \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^n). \tag{2.5}$$

Proposition 2.2. *Let N, f be as in Lemma 2.1 and u the expression of a harmonic map $U : \mathbb{R}^m \to N$ with respect to a normal coordinate system on N centered at an arbitrary fixed point $q_0 \in N$. Then we have the following differential inequality for $|u|$.*

$$\Delta|u|(x) - \frac{f'(|u|)}{f(|u|)}\{e(u)(x) - e(|u|)(x)\} \ge 0, \tag{2.6}$$

where $|u| = \sqrt{\sum_{i=1}^n (u^i)^2}$, $e(u) = \sum_{\alpha=1}^m g^{ij}(u)D_\alpha u^i D_\alpha u^j$, $e(|u|) = \sum_{\alpha=1}^m D_\alpha|u|D_\alpha|u|$.

Moreover, if u satisfies (1.9), then we get

$$\Delta|u| - \frac{\varepsilon_0}{|x|^2}ff'(|u|) \ge 0 \text{ on } \mathbb{R}^m\backslash B_{R_0}(0) \tag{2.7}$$

for some $\varepsilon_0 > 0$ and $R_0 > 0$.

Proof. Taking $\varphi = u\eta$, $\eta \in C_0^\infty(\mathbb{R}^m, \mathbb{R})$ in (2.5), we get

$$\int_{\mathbb{R}^m} \sum_{\alpha=1}^m \{\frac{1}{2} D_\alpha |u|^2 D_\alpha \eta + \eta g_{ij}(D_\alpha u^i D_\alpha u^j + u^k \Gamma^j_{kl} D_\alpha u^l D_\alpha u^j)\} dx \tag{2.8}$$
$$= 0.$$

In (2.3) take $X^i = D_\alpha u^i$, and sum up with respect to α, then we get the following inequality

$$\begin{aligned} &\sum_{\alpha=1}^m g_{ij}(u)(D_\alpha u^i D_\alpha u^j + u^k \Gamma^j_{kl} D_\alpha u^l D_\alpha u^j) \\ &\geq |\zeta|^2 + \sum_{\alpha=1}^m |u| \frac{f_k'(|u|)}{f_k(|u|)} g_{ij}(u) \xi^i_\alpha \xi^j_\alpha, \end{aligned} \tag{2.9}$$

where

$$\zeta = (\zeta^i_\alpha),\ \zeta^i_\alpha = \frac{\sum_{j=1}^n u^j D_\alpha u^j}{|u|^2} u^i \text{ and } \xi = (\xi^i_\alpha) = (D_\alpha u^i - \zeta^i_\gamma).$$

Moreover, we can see that

$$\begin{aligned} |\zeta|^2 &= \sum_{\alpha=1}^m \sum_{i=1}^n (\zeta^i_\alpha)^2 = \frac{1}{4|u|^2} \sum_\alpha^m D_\alpha |u|^2 D_\alpha |u|^2 = \frac{\|D|u|^2\|^2}{4|u|^2}, \\ \sum_{\alpha=1}^m & g_{ij}(u) \xi^i_\alpha \xi^j_\alpha = e(u)(x) - \|(D|u|)\|^2(x) = e(u)(x) - e(|u|)(x). \end{aligned} \tag{2.10}$$

From (2.8), (2.9) and (2.10), we can deduce that $|u|$ satisfies the differential inequality (2.6).

Now, assume that u satisfies (1.9). Then there exist $\varepsilon_0 > 0$ and $R_0 > 0$ such that

$$|x|^2 \left(\frac{1}{|u|^2 \lambda_{N,\mathrm{rad}}(0; u(x))} \right) \{e(u)(x) - e(|u|)(x)\} \geq \varepsilon_0 \text{ for } x \in \mathbb{R}^m \backslash B_{R_0}. \tag{2.11}$$

On the other hand, (2.4) implies that

$$\lambda_{N,\mathrm{rad}}(0, u(x)) \geq \frac{f^2(|u(x)|)}{|u(x)|^2}. \tag{2.12}$$

Thus, combining (2.11) and (2.12), we get

$$|x|^2 \frac{1}{f^2(|u|)} \{e(u)(x) - e(|u|)(x)\} \geq \varepsilon_0 \text{ for } x \in \mathbb{R}^m \backslash B_{R_0}. \tag{2.13}$$

Now, from (2.6) and (2.13), we get the differential inequality (2.7). □

Now, we can prove Theorem 1.1 by comparing $|u|$ with a blow-up supersolution of (2.7). The following theorem due to T.Nagasawa [25] gives us blow up solutions of (2.7).

Theorem 2.3 (Nagasawa [25]). *For $m \geq 2$ and $\mu > 0$, let us consider the initial value problem*

$$r''(t) + \frac{m-1}{t} r'(t) - \frac{\mu^2}{t^2} f(r) f'(r) = 0 \tag{2.14}$$

$$r(0) = 0. \tag{2.15}$$

Assume that

$$(ff')'(r) \geq 0 \text{ for } r \geq 0, \tag{2.16}$$

$$f(r) = br + O(r^3) \text{ as } r \downarrow 0 \text{ for some } b > 0, \tag{2.17}$$

$$\int^{\infty} \frac{dr}{f(r)} < \infty. \tag{2.18}$$

Then the following facts hold.

(1) *There exists a solution $r(t)$ to (2.14)-(2.15) which blows up in finite time.*
(2) *The set of all solutions is a one-parameter family $\{r_\lambda(t) = r(\lambda t))\}_{\lambda \geq 0}$. Here $r(t)$ is the solution in the first assertion. In particular there exists no global solution except zero solution.*
(3) *For any $T \in (0, \infty)$ there exists a unique solution to (2.14)-(2.15) which blows up at $t = T$.*

Moreover it is known that the solutions of (2.14) - (2.15) are nondecreasing.

Proof of Theorem 1.7.

Let $u(x)$ be the expression of a harmonic map $U : \mathbb{R}^m \to N$ with respect to a normal coordinate system $y = (y^1, \cdots, y^n)$ on N centered at arbitrary fixed point $q_0 \in N$. Take R_0 as in Proposition 2.1 and put $\xi = \sup_{B_{R_0}} |u|$. Assume that U is not a constant map. Then $|u|$ can not remain bounded because of a Liouville-type theorem due to [17]. Thus, there exists a compact set $D_0 \subset \mathbb{R}^m \backslash B_{R_0}$ on which $|u| \geq \xi + 1$.

Under the assumption on N in Theorem 1.7, one can find a function f which satisfies the assumptions in Lemma 2.1 and Theorem 2.3. Thus there exist a one-parameter family of solutions $r_\lambda(t)$ to (2.14), or equivalently to the equation

$$\Delta r_\lambda(|x|) - \frac{\varepsilon_0}{|x|^2} ff'(r_\lambda(|x|)) = 0, \tag{2.19}$$

which blow up at $|x| = T/\lambda$ for some $T > 0$ as in Theorem 2.1. Since $r(0) = 0$, we can take $\lambda_0 > 0$ sufficiently small so that $D_0 \subset B_{T/\lambda_0}$ and

$$r_{\lambda_0}(|x|) < 1 \text{ on } D_0. \tag{2.20}$$

Let

$$\psi(x) = r_{\lambda_0}(|x|) + \xi,$$

then $\psi(x)$ satisfies

(2.21) $$\Delta\psi(x) - \frac{\varepsilon_0}{t^2} f f'(\psi(x)) \le 0 \text{ in } \mathbb{R}^m.$$

(2.22) $$\psi(x) \ge \xi \text{ on } \partial B_{R_0} \text{ and } \lim_{|x|\to T/\lambda} \psi(x) = +\infty.$$

Now, using comparison theorem for elliptic equations, we can see that

(2.23) $$|u(x)| \le \psi(x) \text{ on } B_{T/\lambda} \backslash B_{R_0}.$$

On the other hand (2.20) implies that $u(x) > \psi(x)$ on D_0. This is a contradiction. □

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