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Ando-Bhatia-Kittaneh 不等式と等号条件

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1. Introduction.

Let $|||\cdot|||$ be a *unitarily invariant* norm for matrices, that is, it is a norm with

$$|||UAV||| = |||A|||$$

for all matrices A and unitary ones U and V. Since $|||A|||$ is a function of the singular values of A (see [5], where it is called a *symmetric norming function*), we have

$$(1) \qquad |||f(AA^*)||| = |||f(A^*A)|||$$

for all real functions f. For the order induced by positive-semidefinite matrices, it is known that

$$0 \leq A \leq B \quad \text{implies} \quad |||A||| \leq |||B|||.$$

The *operator norm* $\|A\| = \sup_{\|x\|=1} \|Ax\|$ is one of unitarily invariant ones and the following inequality holds for all matrices X and Y:

$$(2) \qquad |||XAY||| \leq \|X\| \; |||A||| \; \|Y\|.$$

In 1988, Ando [1] solved Bhatia's conjecture of estimations for unitarily invariant norms as the following general result:

Theorem A. *Let f be a nonnegative operator monotone function on $[0, \infty)$. If A and B are positive-definite matrices, then*

$$|||f(A) - f(B)||| \leq |||f(|A - B|)|||.$$

In [3], one of the authors gave an equality condition for Ando's inequality for the case of the operator norm: *For non-affine f, the equality holds if and only if $A = B$ and $f(0) = 0$.*

Recently, based on Ando's inequality, Bhatia and Kittaneh [2] generalized it, which is paraphrased in terms of the bounds of X^*X:

Theorem B. *Let f be a nonnegative operator monotone function on $[0, \infty)$. If A and B are positive-definite matrices and X satisfies $0 < m \leq X^*X \leq M$ for some positive numbers m and M, then*

$$(3) \qquad |||f(A)X - Xf(B)||| \leq \frac{1+M}{2} \left|\left|\left| f\left(\frac{2}{1+m}|AX - XB|\right)\right|\right|\right|.$$

One of the authors also gave an equality condition of the above inequality for the operator norm in [4]. But our approach does not suit for unitarily invariant norms. In this paper, we give equality conditions for the above inequalities making use of Ando's approach in [1]. Moreover, we generalize the above inequality considering the initial condition of f. Note that all the inequalities in this paper hold also for operators on a Hilbert space if the unitarily invariant norms for operators in discourse make sense.

To begin with, we discuss an extension of Ando's one in the next section since Theorem A played an essential role in the Bhatia-Kittaneh inequality.

2. Ando's inequality. As we pointed out in [4], we need not nonnegativity for f in the above inequalities, which is a sharp estimation even if f is nonnegative.

Theorem 1. *Let f be operator monotone on $[0, \infty)$. If A and B are positive-semidefinite matrices, then*

$$(4) \qquad |||f(A) - f(B)||| \leq \left|\left|\left| f(|A - B|) - f(0)\right|\right|\right|.$$

The equality holds for non-affine f and positive-definite A and B if and only if $A = B$.

The inequality (4) itself is obtained by applying Theorem A to a nonnegative operator monotone function $F(t) = f(t) - f(0)$. To see the equality condition, we have only to show the following lemma, which is essential also in Ando's inequality as in the proof of Theorem A in [1]. Ando's proof is based on the integral representation of a nonnegative operator monotone function F: There exists a positive Radon measure μ on $[0,\infty]$ with

$$F(A) = a + bA + \int_{(0,\infty)} (t : A)\ \frac{1+t}{t} d\mu(t).$$

In other words, if F is non-affine, then $F(A)$ is nothing but an variation of a parallel sum $1 : A = A(1+A)^{-1} = 1 - (1+A)^{-1}$.

Lemma 2. *Let F be a nonnegative non-affine operator monotone function on $[0,\infty)$. If A and B are positive-semidefinite, then*

$$|||F(A+B) - F(B)||| \leq |||F(A)|||.$$

Moreover, the equality does not hold if A is nonzero and B is positive-definite.

Proof. Considering the integral representation for F, we have only to show the above inequality for the case $F(x) = 1 : x$. Since

$$\begin{aligned}
0 &\leq 1 : (A+B) - 1 : B = (B+1)^{-1} - (A+B+1)^{-1} \\
&= (B+1)^{-1/2}\left(1 - ((B+1)^{-1/2}A(B+1)^{-1/2} + 1)^{-1}\right)(B+1)^{-1/2} \\
&= (B+1)^{-1/2} F\left((B+1)^{-1/2}A(B+1)^{-1/2}\right)(B+1)^{-1/2},
\end{aligned}$$

and $(B+1)^{-1} \leq 1/(k+1)$ for some $k \geq 0$ with $B \geq k$, we have

$$\begin{aligned}
|||1 : (A+B) - 1 : B||| &\leq \|(B+1)^{-1}\| \left|\left|\left| F((B+1)^{-1/2}A(B+1)^{-1/2}) \right|\right|\right| && \text{by (2)} \\
&\leq \frac{1}{k+1} \left|\left|\left| F((B+1)^{-1/2}A(B+1)^{-1/2}) \right|\right|\right| \\
&\leq \frac{1}{k+1} \left|\left|\left| F(A^{1/2}(B+1)^{-1}A^{1/2}) \right|\right|\right| && \text{by (1)} \\
&\leq \frac{1}{k+1} |||F(A)||| \leq |||F(A)|||.
\end{aligned}$$

If B is positive-definite, then k is positive, or $1/(k+1) < 1$. So the last inequality is exchanged for

$$\frac{1}{k+1}|||F(A)||| < |||F(A)|||$$

for nonzero A. $\square$

Remark. If $F(t) = a + bt$ for $a, b \geq 0$, a nonnegative affine function, then we have $F(A+B) - F(B) = bA \leq F(A)$ and hence

$$|||F(A+B) - F(B)||| \leq |||F(A)|||,$$

in which the equality holds if $F(0) = a = 0$.

For completeness, we sketch a proof of the above theorem:

Proof of Theorem 1. For a nonnegative operator monotone functon $F(t) = f(t) - f(0)$, we have

$$|F(A) - F(B)| \leq F(|A-B| + B) - F(B)$$

and Lemma 2 shows

$$|||F(|A-B| + B) - F(B)||| \leq |||F(|A-B|)|||,$$

so the required inequality yields. Moreover Lemma 2 shows $|A - B| = 0$, that is $A = B$.

3. The Bhatia-Kittaneh inequality. Now we extend the Bhatia-Kittaneh inequality and discuss an equality condition. The following extension is the same formula as in [4] except norms:

Theorem 3. *Let f be operator monotone on $[0, \infty)$ and matrices A and B are positive-semidefinite. If a matrix X satisfies $0 \leq m \leq X^*X \leq M$ for some real numbers m and M, then*

$$|||f(A)X - Xf(B)||| \leq \frac{1+M}{2}\left|\left|\left|f\Big(\frac{2}{1+m}|AX - XB|\Big) - f(0)\right|\right|\right|. \tag{5}$$

The equality holds for non-affine f and positive-definite A and B if and only if $AX = XB$.

Though (5) itself follows from (3) via $F(t) = f(t) - f(0)$, we will prove (5) to observe the equality condition considering their proof in [2]. The basic fact is the following lemma which is easily obtained by Theorem 1:

Lemma 4. *Let f be operator monotone on $[0, \infty)$. If A is positive-semidefinite and U is unitary, then*

$$|||f(A)U - Uf(A)||| \le |||f(|UA - AU|) - f(0)|||.$$

If f is non-affine and A is positive-definite, then the equality holds if and only if A commutes with U.

Proof. It follows from Theorem 1 that

$$\begin{aligned} |||f(A)U - Uf(A)||| &= |||U^*f(A)U - f(A)||| = |||f(U^*AU) - f(A)||| \\ &\le |||f(|U^*AU - A|) - f(0)||| = |||f(|AU - UA|) - f(0)|||. \end{aligned}$$

Moreover the equality condition is $U^*AU = A$, or $AU = UA$. □

Now we show Theorem 3 by using their excellent idea in [2]:

Proof of Theorem 3. First we show the case $X = X^*$ and $A = B$. Take a unitary operator $U = (X - i)(X + i)^{-1}$ by the Cayley transform. Then

$$\begin{aligned} f(A)X - Xf(A) &= 2i\left(f(A)(1 - U)^{-1} - (1 - U)^{-1}f(A)\right) \\ &= 2i(1 - U)^{-1}\left(f(A)U - Uf(A)\right)(1 - U)^{-1} \end{aligned}$$

and $AU - UA = 2i(X + i)^{-1}(AX - XA)(X + i)^{-1}$. Since

$$\begin{aligned} ||(1 - U)^{-1}||^2 &= \frac{||X^2 + 1||}{4} \le \frac{M + 1}{4}, \\ ||(X + i)^{-1}||^2 &= ||(X^2 + 1)^{-1}|| \le \frac{1}{1 + m}, \end{aligned}$$

it follows from Lemma 4 that

$$\begin{aligned}|||f(A)X - Xf(A)||| &\le \frac{M+1}{2}|||f(A)U - Uf(A)||| \\ &\le \frac{M+1}{2}|||f(|AU - UA|) - f(0)||| \\ &\le \frac{M+1}{2}\left|\left|\left|f\left(\frac{2}{1+m}|AX - XA|\right) - f(0)\right|\right|\right|.\end{aligned}$$

Here we notice that the equality condition is $AU = UA$, hence $AX = XA$. Now we have the required inequality by taking $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ instead of A and X respectively. In this case, the equality condition

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

implies $AX = XB$. $\square$

Example. The logarithm is a typical operator monotone function on $(0, \infty)$, so we put an operator monotone function on $[0, \infty]$ by $f_\varepsilon(t) = \log(t + \varepsilon)$ for all $\varepsilon > 0$. Then we have the following inequality by Theorem 3:

$$|||\log(A+\varepsilon)X - X\log(B+\varepsilon)||| \le \frac{1+M}{2}\left|\left|\left|\log\left(\frac{2}{1+m}|AX - XB| + \varepsilon\right) - \log\varepsilon\right|\right|\right|.$$

But (3) implies the following inequality for $a \ge 1$:

$$|||\log(A+a)X - X\log(B+a)||| \le \frac{1+M}{2}\left|\left|\left|\log\left(\frac{2}{1+m}|AX - XB| + a\right)\right|\right|\right|.$$

Thus, compared with the latter inequality, the former one gives a new inequality for $0 < \varepsilon < 1$ and a sharp one for $\varepsilon > 1$.

4. Variations. In this section, we discuss variations for Theorem 3. Bhatia and Kittaneh [2] showed that

$$|||f(A)X - Xf(B)||| \le \frac{5}{4}|||f(|AX - XB|)||| \qquad (6)$$

for all contractions X. By virtue of Bhatia-Kittaneh's idea, we have the following corollary by substituting tX for X in Theorem 3:

Corollary. *Let f be operator monotone on $[0,\infty)$ and matrices A and B positive semi-definite. If an operator X satisfies $0 \le m \le X^*X \le M$ for some real numbers m and M, then*

$$(7)\qquad |||f(A)X - Xf(B)||| \le \frac{1+Mt^2}{2t}\left|\left|\left|f\left(\frac{2t}{1+mt^2}|AX-XB|\right) - f(0)\right|\right|\right|$$

for any positive number t. The equality holds for non-affine f and positive-definite A and B if and only if $AX = XB$.

Hereafter in this section, we leave out the common equality condition for the following inequality since they are all based on the above corollary. Now, as a generalization for (6), we have an estimation of $|||f(A)X - Xf(B)|||$ in terms of $f(|AX-XB|)$.

Theorem 5. *Let f be operator monotone on $[0,\infty)$, matrices A and B positive semi-definite and $0 \le m \le X^*X \le M$ for some real numbers m and M. If $0 < m \le 1$, then*

$$(8)\quad |||f(A)X - Xf(B)||| \le \frac{m^2 + M(2-m-2\sqrt{1-m})}{2m(1-\sqrt{1-m})}|||f(|AX-XB|) - f(0)|||.$$

If $m = 0$, then

$$(8')\qquad |||f(A)X - Xf(B)||| \le \left(1 + \frac{M}{4}\right)|||f(|AX-XB|) - f(0)|||.$$

Proof. Solving an equation $2t/(1+mt^2) = 1$ for $0 < m \le 1$, we have $t = (1 \pm \sqrt{1-m})/m$. Then, Corollary implies (8) since the value $(1+Mt^2)/(2t)$ at $t = t_1 = (1-\sqrt{1-m})/m$ is not greater than that at $t = t_2 = (1+\sqrt{1-m})/m$. In fact, $t_1t_2 = 1/m > 0$ shows

$$\begin{aligned}\frac{1+Mt_2^2}{2t_2} - \frac{1+Mt_1^2}{2t_1} &= \frac{M(t_1t_2-1)(t_2-t_1)}{2t_1t_2}\\ &= \frac{(M-m)(t_2-t_1)}{2} \ge 0\end{aligned}$$

For $m = 0$, we have (8') by putting $t = 1/2$. □

Remark. We may say that (8') is the extreme case for (8) since

$$\lim_{m\downarrow 0}\frac{1-\sqrt{1-m}}{m}=\frac{1}{2}$$

and

$$\lim_{m\downarrow 0}\frac{m^2+M(2-m-2\sqrt{1-m})}{2m(1-\sqrt{1-m})}=1+\frac{M}{4}.$$

Putting $M=1$ in (8'), we have the following (6') for any contraction X, which is an extension of (6):

$$(6')\qquad |||f(A)X-Xf(B)|||\le\frac{5}{4}|||f(|AX-XB|)-f(0)|||$$

Though Bhatia and Kittaneh did not mention the other type of inequality, we show the following one similarly:

Theorem 6. *Let f be operator monotone on $[0,\infty)$, matrices A and B positive semi-definite and $0\le m\le X^*X\le M$ for some real numbers m and M. If $0\le M\le 1$, then*

$$|||f(A)X-Xf(B)|||\le\left|\left|\left|f\left(\frac{2M(1-\sqrt{1-M})}{M^2+m(2-M-2\sqrt{1-M})}|AX-XB|\right)-f(0)\right|\right|\right|.$$

In particular, if $M=1$, then

$$|||f(A)X-Xf(B)|||\le\left|\left|\left|f\left(\frac{2}{1+m}|AX-XB|\right)-f(0)\right|\right|\right|.$$

5. Inverse inequalities. Ando [1] gave the inverse inequality for Theorem A making use of the fact that if hermitian matrices A and B satify $|||A|||\le|||B|||$ for all unitarili invariant norms, then $|||h(A)|||\le|||h(B)|||$ for all monotone increasing convex functions h: *Let g be nonnegative monotone increasing convex function on $[0,\infty)$ such that g^{-1} is operator monotone on $[0,\infty)$ and matrices A and B positive-definite. Then*

$$(9.)\qquad |||g(|A-B|)|||\le|||g(A)-g(B)|||$$

According to this idea, Bhatia and Kittaneh [2] also gave inverse inequalities for Theorem B:

$$(10) \qquad \left|\left|\left| g\left(\frac{2}{1+M}|AX - XB|\right)\right|\right|\right| \le \frac{2}{1+m}|||g(A)X - Xg(B)|||.$$

In the above results, the nonnegativity of g implies that $g(A)$ and $g(B)$ are positive-(semi)definite. Also the condition $\min g(t) = 0$ assures that A and B are arbitrary positive-(semi)definite matrices. But now the domain interval of g does not have to be nonnegative any longer:

Theorem 7. *Let g and h be nonnegative monotone increasing convex functions on $[c, \infty)$ for $c \le 0$ such that g^{-1} is operator monotone and $g(c) = 0$. If A and B be positive-semidefinite and $0 \le m \le X^*X \le M$, then*

$$\left|\left|\left| h\left(\frac{2}{1+M}|AX - XB|\right)\right|\right|\right| \le \frac{2}{1+m}\left|\left|\left| h\left(g^{-1}(g(A)X - Xg(B)) - g^{-1}(0)\right)\right|\right|\right|.$$

Proof. Since $f = g^{-1}$ is operator monotone on $[0, \infty)$ and there exist positive-semidefinite operators $C = g(A)$ and $D = g(B)$, we can apply Theorem 3 for C and D:

$$\left|\left|\left| \left(\frac{2}{1+M}|f(C)X - Xf(D)|\right)\right|\right|\right| \le \frac{2}{1+m}\left|\left|\left| f(CX - XD) - f(0)\right|\right|\right|.$$

So the convexity of h assures the required inequality. $\square$

Remark. Of course h can be g itself. But the independence h of g sometimes convenient. In our example for Theorem 3, $f_\varepsilon(t) = \log(t + \varepsilon)$ means $g_\varepsilon(t) = e^t - \varepsilon$, in which ε should not be greater than 1 by the assumption $g^{-1}(0) \le 0$ in Theorem 6. In fact $\varepsilon > 1$ implies $A = \log C + \varepsilon \ge \log \varepsilon > 0$, which restricts A. So the above example shows

$$\begin{aligned}\left|\left|\left| \frac{2}{1+M}|AX - XB|\right|\right|\right| &\le \left|\left|\left| \log\left(\frac{2}{1+m}|(e^A - \varepsilon)X - X(e^B - \varepsilon)| + \varepsilon\right) - \log \varepsilon\right|\right|\right| \\ &\le \left|\left|\left| \log\left(\frac{2}{\varepsilon(1+m)}|e^A X - Xe^B| + 1\right)\right|\right|\right|.\end{aligned}$$

If we apply Theorem 6 for $h = g_\varepsilon$, then

$$\left|\left|\left|\exp\left(\frac{2}{1+M}|AX - XB|\right) - \varepsilon\right|\right|\right| \le \left|\left|\left|\left(\frac{2}{\varepsilon(1+m)}|e^A X - X e^B| + 1\right) - \varepsilon\right|\right|\right|.$$

For the case $h(t) = e^t$, we have

$$\left|\left|\left|\exp\left(\frac{2}{1+M}|AX - XB|\right)\right|\right|\right| \le \left|\left|\left|\left(\frac{2}{\varepsilon(1+m)}|e^A X - X e^B| + 1\right)\right|\right|\right|$$

for all $0 < \varepsilon \le 1$. If $M \ge 1$ in addition, then the convexity of the function $t \mapsto t^{(1+M)/2}$ implies

$$\left|\left|\left|\exp(|AX - XB|)\right|\right|\right| \le \left|\left|\left|\left(\frac{2}{\varepsilon(1+m)}|e^A X - X e^B| + 1\right)^{(1+M)/2}\right|\right|\right|.$$

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