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Random Strained-Vortices の統計法則

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Statistical Laws of Random Strained-Vortices in Turbulence

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Abstract

Statistical properties of random distribution of strained vortices (Burgers vortices) in turbulence are studied, and the scaling behaviors of structure functions are investigated. It is found within the scale-range of interest (corresponding to the inertial range) that the third-order structure function is negative and the scaling exponent is nearly unity in accordance with the Kolmogorov's four-fifths law. The inertial-range scaling exponents are estimated up to the 25th-order which are in good agreement with those obtained from experiments and direct numerical simulations once the probability distribution of the vortex strength is taken into account.

In recent computer simulations and experiments of homogeneous isotropic turbulence at high Reynolds numbers, a number of elongated intense vortex structures are observed to distribute randomly in space, which are often called *worms* [1–5]. Each worm-structure is found approximately to be a Burgers' vortex under local straining and is responsible for the signals usually referred to as the *intermittency* [5]. Bearing these in mind, we investigate the statistical properties of a model field associated with random distribution of Burgers vortices.

High Reynolds number flows are characterized by the statistical properties of the velocity field $\boldsymbol{v}(\boldsymbol{x})$ and the difference at two points $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{s}$: $\Delta\boldsymbol{v}(\boldsymbol{x},\boldsymbol{s})=\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{s})-\boldsymbol{v}(\boldsymbol{x})$. Defining the *longitudinal* difference in the direction $\boldsymbol{s}$ by

$$\Delta v_\ell(\boldsymbol{x},\boldsymbol{s})=\Delta\boldsymbol{v}(\boldsymbol{x},\boldsymbol{s})\cdot\frac{\boldsymbol{s}}{s} \tag{1}$$

where $s=|\boldsymbol{s}|$, the pth-order longitudinal structure function S_p is given by $S_p=\langle(\Delta v_\ell)^p\rangle$, where $\langle\cdot\rangle$ is an ensemble average for a fixed s. In the homogeneous isotropic turbulence, the structure function S_p follows a power-law in the inertial range of s:

$$S_p(s)\equiv\langle(\Delta v_\ell(\boldsymbol{x},\boldsymbol{s}))^p\rangle\sim s^{\zeta_p} \tag{2}$$

where ζ_p is the scaling exponent of the p-th order structure function.

The skewness $S_3/(S_2)^{3/2}$ in turbulence is always found to be negative for small s. The *negative* skewness is related to the enstrophy production [6] and the non-Gaussian statistics of the velocity derivatives [7]. An example of a vortex under external straining (considered below) has such negative skewness. In particular, the third-order structure function is described by the Kolmogorov's four-fifths law [8,9],

$$\left\langle (\Delta v_\ell)^3 \right\rangle = -\frac{4}{5}\varepsilon s = -\frac{4}{5}\nu\bar{\omega}^2 s \tag{3}$$

for the values of s in the inertial-range, where the rate of energy dissipation ε is replaced by an equivalent form $\varepsilon = \nu\bar{\omega}^2$, $\bar{\omega}$ being the *rms* vorticity. The parameter ε may be termed more appropriately as the energy transfer across a wave number in the inertial range. In the Kolmogorov 1941 theory [10], the average $\langle|\Delta v_\ell|\rangle$ at the scale s in the inertial range is given by dimensional arguments as $\langle|\Delta v_\ell|\rangle \sim (\varepsilon s)^{1/3}$, and in general the exponent ζ_p is represented as $\zeta_p = p/3$ (referred to as K41 below).

According to the scenario of Kambe and Hosokawa [11], the present analysis aims at clarifying statistical properties of a mathematical model endowed with a characteristic of the isotropic homogeneous turbulence, namely a random system of strained vortices. This approach is consistent with the idea of the multifractal model of turbulence field. It is assumed that, in the limit of large Reynolds number, there is an invariant measure of the Navier-Stokes turbulence, for which a probability distribution function $P(s, \Delta v_\ell)$ is defined [9]. The pth-order structure function S_p is expressed as an integral $S_p(s) = \int (\Delta v_\ell)^p P(s, \Delta v_\ell)\, d\Delta v_\ell$, which leads to a power-law in a certain interval of s corresponding to the inertial range, as actually obtained for the present model below.

Recently a phenomenological step is advanced [12–14]. This is a statistical model taking account of a hierarchy of fluctuating vortex-filament structures which is found to have properties of the *log-Poisson* statistics. The resulting exponent of the p-th structure function is given as $\zeta_p = p/9 + 2 - 2(2/3)^{p/3}$, which is found to be not far from the direct numerical simulation(DNS) [1] and the experimental observation [15,16].

Turbulence is regarded as a field of rate-of-strains. At each point, three principal rates of strain α, β and γ are defined, and they satisfy the relation $\alpha+\beta+\gamma = 0$ by the solenoidality of the velocity field. Assuming the property $\alpha \geq \beta \geq \gamma$, we have always $\alpha \geq 0$ and $\gamma \leq 0$. The intermediate eigenvalue β takes either a positive or a negative value.

We consider a velocity field of a strained vortex. The vorticity distribution is assumed to have only the axial component $\omega(r)$ in the cylindrical coordinate system (r, θ, z). Hence the vorticity vector is $\boldsymbol{\omega} = (0, 0, \omega(r))$ with the axial component $\omega(r)$ specified later. The velocity associated with $\boldsymbol{\omega}$ is $\boldsymbol{v}_\omega = (0, v_\theta(r), 0)$, having only the azimuthal component $v_\theta(r)$. This vortex is exposed to an irrotational straining field given by $\boldsymbol{v}_{\mathrm{e}} = (-ar, 0, 2az)$ satisfying the solenoidal property. The total flow field $\boldsymbol{v}$ is the superposition of $\boldsymbol{v}_\omega$ and $\boldsymbol{v}_{\mathrm{e}}$:

$$\boldsymbol{v}(\boldsymbol{x}) = (-ar, v_\theta(r), 2az) \tag{4}$$

Local principal rates of strain e_1, e_2 and e_3 of the velocity field $\boldsymbol{v}(\boldsymbol{x})$ are readily calculated as $e_1 = -a + |e_{\mathrm{r}\theta}|$, $e_2 = 2a$ and $e_3 = -a - |e_{\mathrm{r}\theta}|$, where $e_{\mathrm{r}\theta} = (v_\theta'(r) - r^{-1}v_\theta(r))/2$. If $|a|$ is sufficiently small compared with $|e_{\mathrm{r}\theta}|$, then $\alpha = e_1$, $\beta = e_2$ and $\gamma = e_3$. In the following, the parameter a is assumed to be positive.

In this circumstance, it can be shown [17] that, with an arbitrary initial axisymmetric distribution, the axial vorticity $\omega(r)$ (only non-zero-component) tends to the final steady distribution $\omega_B(r)$ asymptotically as $t \to \infty$:

$$\omega_B(r) = \frac{\Gamma}{\pi r_\mathrm{b}^2} \exp\left(-\hat{r}^2\right), \quad v_\theta(r) = \frac{\Gamma}{2\pi r_\mathrm{b}} \frac{1-\exp\left(-\hat{r}^2\right)}{\hat{r}}, \tag{5}$$

where $\hat{r} = r/r_\mathrm{b}$, $r_\mathrm{b} = (2\nu/a)^{1/2}$ and Γ is the strength. This is the *Burgers* vortex of radius r_b [18] (Fig. 1).

The vortex axes are randomly oriented spatially in isotropic turbulence. In the present single-worm case, the average is taken over a sphere centered at a chosen reference point $\boldsymbol{x}$. For example, local third-order moment $\hat{s}_3 = \langle(\partial v_\ell/\partial s)^3\rangle_\mathrm{sp}|_{s=0}$ (skewness without normalization) of the longitudinal derivative at $\boldsymbol{x}$ is calculated [19,20] as $\hat{s}_3 = (8/35)\, e_1 e_2 e_3 = -(16/35)\, a\, (e_{\mathrm{r}\theta}^2 - a^2)$, where the spherical average $\langle\cdot\rangle_\mathrm{sp}$ is an integral over the solid angle with respect to the direction $\boldsymbol{s}$ divided by 4π. It is found that for a pure vortex $\boldsymbol{v}_\omega$ without any external strain (hence $a = 0$), $\hat{s}_3$ is zero, while the converse case of a pure straining $\boldsymbol{v}_\mathrm{e}$ without the vortex (thus $e_{\mathrm{r}\theta} = 0$), $\hat{s}_3 = (16/35)\, a^3$ is positive. However, the composite flow field considered above gives a negative $\hat{s}_3$ as far as $|e_{\mathrm{r}\theta}| > a$. Therefore the space surrounding the intense vortex under the straining of $\boldsymbol{v}_\mathrm{e}$ is characterized as a field of negative skewness. Local rate of energy dissipation is given as $\varepsilon_\mathrm{loc}(r) = \nu\,\{12a^2 + (2e_{\mathrm{r}\theta})^2\}$, where $2e_{\mathrm{r}\theta} \equiv v_\theta'(r) - r^{-1}v_\theta(r) = (\Gamma/\pi r_\mathrm{b}^2)\,[\exp\left(-\hat{r}^2\right) - \hat{r}^{-2}\left(1-\exp(-\hat{r}^2)\right)]$. If $\Gamma/\left(\pi r_\mathrm{b}^2\right)$ is sufficiently large compared with a, the energy is strongly dissipated at around r_b, while at the center of vortex scarcely dissipated. Taking an average of the local third-order moment over a spherical surface of radius $s = |\boldsymbol{s}|$, we have $\langle(\Delta v_\ell)^3\rangle_\mathrm{sp} \approx \hat{s}_3 s^3$ when s is sufficiently small. Owing to the solenoidal property of the velocity, the average $\langle\Delta v_\ell\rangle_\mathrm{sp}$ vanishes identically.

Next, we investigate the behaviors of the longitudinal velocity difference $\Delta v_\ell(\boldsymbol{s})$ at large distances, in particular, general pth-order structure functions. Fixing a reference point $\boldsymbol{x}$ at $(r_0, 0, z_0)$ in the cylindrical system K_1: (r, θ, z), we define a spherical polar coordinates K_2: (s, ζ, ϕ) centered at $\boldsymbol{x}$ to represent the relative position of the point $\boldsymbol{x} + \boldsymbol{s}$, where ζ is the polar angle and ϕ the azimuthal angle. For the velocity field (4) and (5), the longitudinal velocity difference is represented as

$$\Delta v_\ell(\boldsymbol{x}, s, \zeta, \phi) = as(3\cos^2\zeta - 1) + r_0 W(r, r_0)\sin\zeta\sin\phi \tag{6}$$

where $W(r, r_0) = r^{-1}v_\theta(r) - r_0^{-1}v_\theta(r_0)$. The spherical average is calculated by

$$\langle(\Delta v_\ell)^p\rangle_\mathrm{sp}(\boldsymbol{x}, s) \equiv \frac{1}{4\pi}\int_{-\pi}^{\pi} d\phi \int_0^{\pi} (\Delta v_\ell)^p \sin\zeta\, d\zeta. \tag{7}$$

This average will depend on the point $\boldsymbol{x}$ as well as the separation vector $\boldsymbol{s}$ and have different scaling behaviors with respect to s at different $\boldsymbol{x}$'s in accordance with the multifractal aspect.

The statistical average $\langle\cdot\rangle$ is taken firstly by the spherical average $\langle\cdot\rangle_\mathrm{sp}$ with respect to the running point $\boldsymbol{x} + \boldsymbol{s}$, and secondly by volume average with respect to the reference point $\boldsymbol{x}$:

$$\langle\cdot\rangle(s) = \frac{1}{\pi R_0^2\,\Delta z}\int_0^{\Delta z} dz_0 \int_0^{R_0} \langle\cdot\rangle_\mathrm{sp}\, 2\pi r_0\, dr_0 \tag{8}$$

(the average with respect to z_0 is trivial). Thus we obtain the statistical properties of isotropy and homogeneity from the velocity field (4).

The structure functions are estimated for three different strengths of the Burgers vortex with $R_\Gamma \equiv \Gamma/\nu = 600$, 2000 and 10000. In Fig. 2, the third-order structure functions

$S_3(s)$ are shown. At small distances $s/r_b < 1$, the function $S_3(s)$ is proportional to s^3 as anticipated for the continuous smooth field. However for $s/r_b > 1$ the function $S_3(s)$ shifts to another scaling law of a different slope. It is found that the third-order scaling exponent ζ_3 in the second scaling range is about unity and almost independent of the magnitude of R_Γ. Straight lines with unit slope are obtained from Kolmogorov's four-fifths law (3), where mean energy dissipation rate is defined as $\varepsilon = (\pi R_0^2)^{-1}\int_0^{R_0} \varepsilon_{\rm loc}(r_0) 2\pi r_0\, dr_0$. The limit of r_0-integral is given by $R_0 = 2.5\, r_b$ so as to be consistent with the four-fifths law for the second scaling range. The first scaling range of the exponent 3.0 is identified as the viscous range, and the second range of the exponent 1.0 as the inertial range which is wider for larger R_Γ. In Fig. 3, the scaling exponents ζ_p up to $p = 25$ are shown for the three values of R_Γ, and compared with those of K41 and log-Poisson model. Increasing the magnitude R_Γ, the exponents ζ_p decrease more below the K41. The even-p exponents fall lower than the line of the odd-p exponents, which is in agreement with the general behavior of the experimental data [15].

The probability distribution functions of the vortex Reynolds number R_Γ and the Burgers' radius r_b in turbulence are estimated by Jimenéz *et al.* [2] in DNS and by Belin *et al.* [5] experimentally. In particular, distributions of the normalized values $R_\Gamma/R_\lambda^{1/2}$ are independent of the value of the Reynolds number R_λ based on the Taylor microscale λ. Taking into account of the probability distribution, the structure functions are estimated [21].

In Fig. 4 and Fig. 5, behaviors of such structure functions are illustrated. It is observed that there exist two scaling ranges in each structure function, in which the second one corresponds to the inertial range. Here the inertial range is defined as the range within which the variance of the third-order structure function with respect to the four-fifth law is least. In Fig. 6, the scaling exponents in the inertial range are plotted and compared with those obtained from other models, DNS and experiments. It is found that the present analysis can predict the scaling exponents which are remarkably coincident with those of DNS [1] and the experiments [15,16].

If the vortex is absent (therefore $v_\theta = 0$), we have $S_p(s) = C_p\, a^p s^p \propto s^p$ from Eq. (6) and Eq. (7), where C_p is a constant. On the other hand, if the external strain is absent (therefore $a = 0$), we find that the structure functions of the odd-order are identically zero by the antisymmetric property of Eq. (6). Hence the present scaling exponents consistent with the homogeneous isotropic turbulence have resulted from the combined field of the vortex and the turbulence straining.

The present study is summarized as follows.

1. It is found from the velocity field of random distribution of Burgers vortices that the third-order structure function is negative in the inertial range and the scaling exponent is nearly unity and independent of the vortex Reynolds number R_Γ, and that the second-order structure function has the scaling exponent of about two-thirds, in accordance with the general turbulence properties.

2. The scaling exponents of the high-order structure functions deviate increasingly below K41 as R_Γ becomes larger. A Burgers vortex in turbulence causes more and more the degree of intermittency in the field as its strength gets larger.

3. The scaling exponents ζ_p are in good agreement with the experiments and DNS data once the distribution function of R_Γ (taken from the experiments and DNS) is taken into account.

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[19] A. A. Townsend, Proc. Roy. Soc. London A **208**, 534 (1951).

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[21] In order to estimate the mean vortex Reynolds number, it is assumed for isotropic turbulence that $\sigma = v_{\mathrm{rms}}/\lambda$ and $\Gamma = 2\pi r_{\mathrm{b}} v_{\mathrm{rms}}$, where σ is the axial stretching rate of worm ($\sigma = 2a$ in case of Burgers' vortex) and v_{rms} the root-mean-square velocity. The consequence is $R_\Gamma / R_\lambda^{1/2} = 4\pi$, in good agreement with the value obtained by Jimenéz *et al.* in DNS [2]. Thus the PDF of R_Γ is defined as $P(R_\Gamma) = (C^3/2)\, R_\Gamma^2 \exp(-C R_\Gamma)$ with $C = (3/4\pi) R_\lambda^{1/2}$, so that the mean value of R_Γ is $4\pi R_\lambda^{1/2}$ and the PDF has the similar form as DNS [2] and experiment [16] especially in the limit of $R_\Gamma \to 0, \infty$.

FIGURES

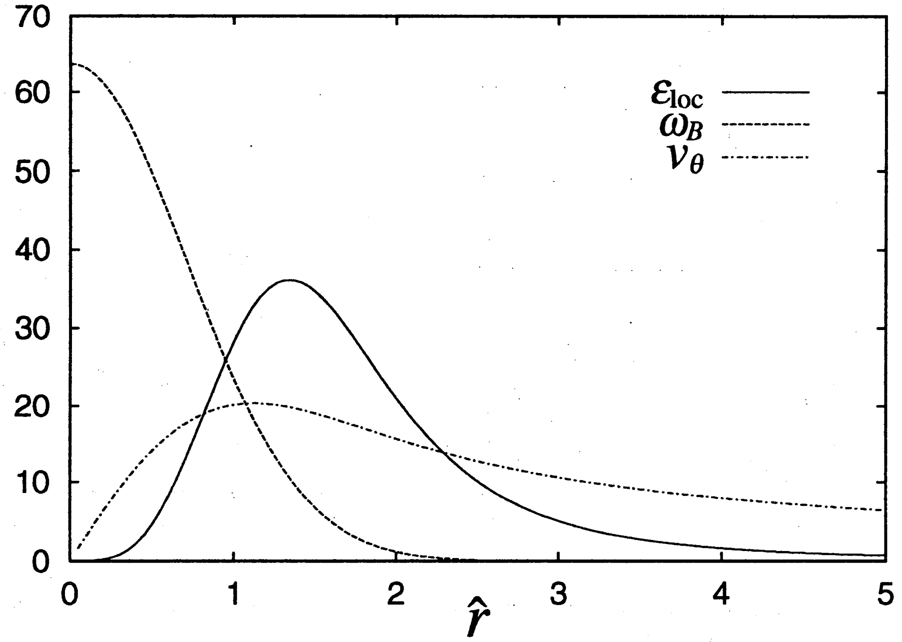

FIG. 1. The local energy dissipation rate $\varepsilon_{\mathrm{loc}}$, the axial vorticity ω_B and the azimuthal velocity v_θ of the Burgers' vortex for $R_\Gamma \equiv \Gamma/\nu = 2000$ normalized by $\nu = 0.1$.

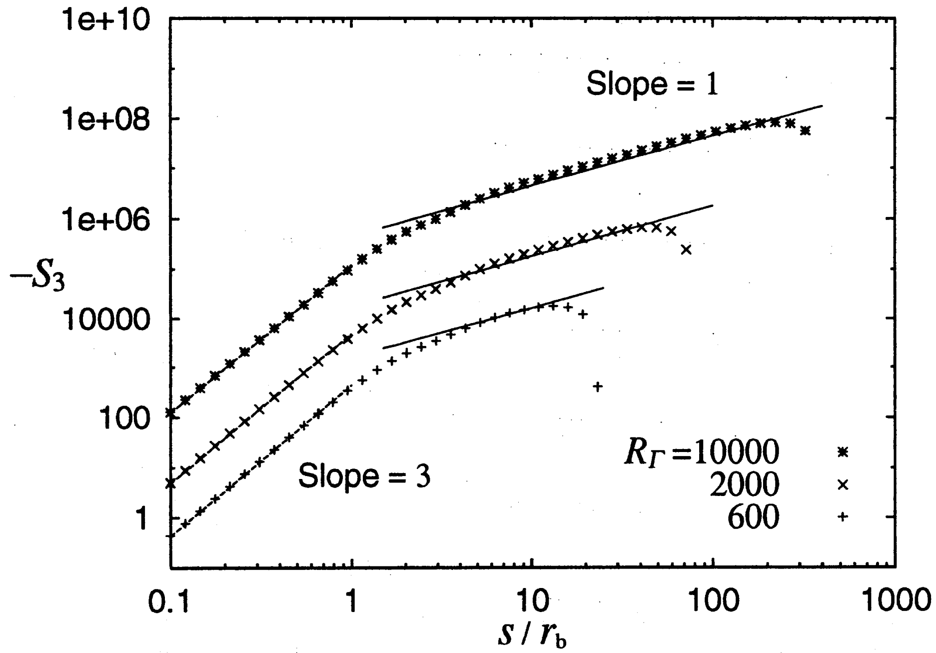

FIG. 2. The third-order structure functions times -1 for $R_\Gamma = 600, 2000, 10000$ with $\nu = 1$. Straight lines with unit slope are obtained from the Kolmogorov's four-fifths law (3).

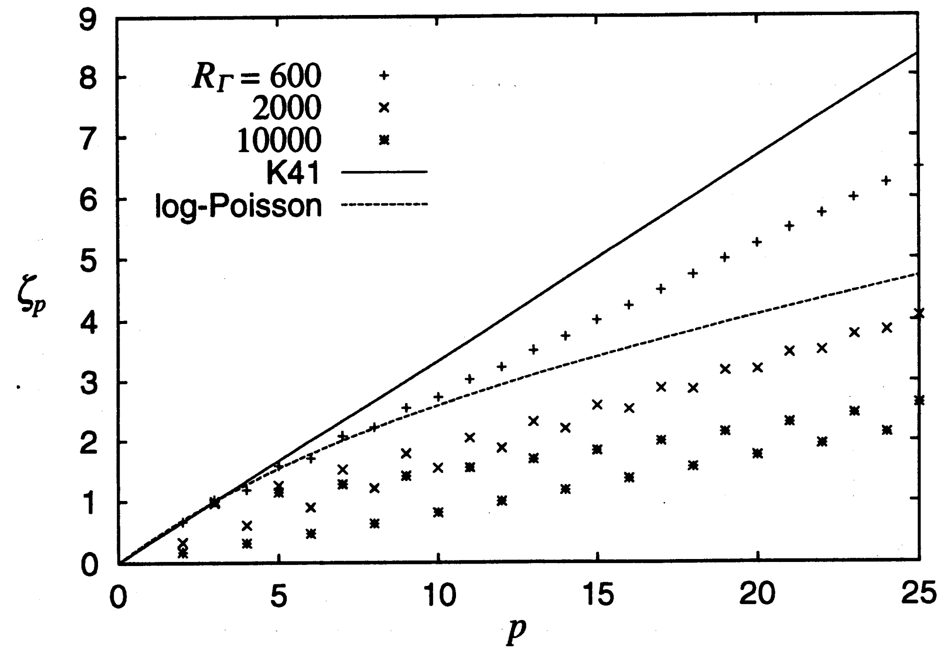

FIG. 3. The exponents ζ_p of the structure functions for $R_\Gamma = 600, 2000, 10000$ with K41 [10] and log-Poisson model [12–14].

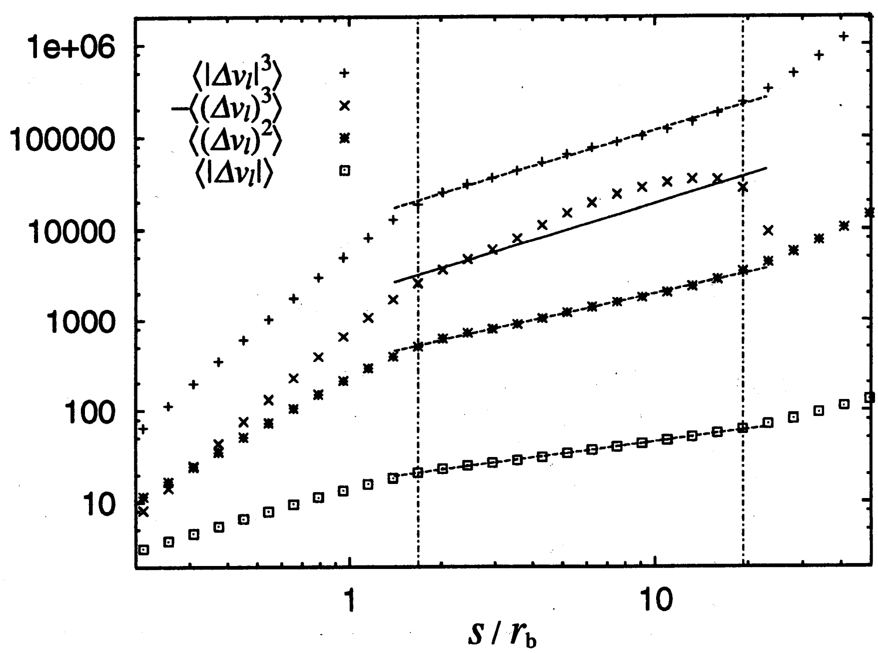

FIG. 4. The first-, second- and third-order structure functions for $R_\lambda = 2000$. The region between the dotdased lines is regarded as inertial range. Solid line is given by the Kolmogorov's four-fifths law (3) with $\varepsilon = (\pi R_0^2)^{-1} \int_0^\infty dR_\Gamma \int_0^{R_0} \varepsilon_{\rm loc}(R_\Gamma, r) P(R_\Gamma)\, 2\pi r\, dr$, and dashed lines are the least-log-squares fits within the inertial range.

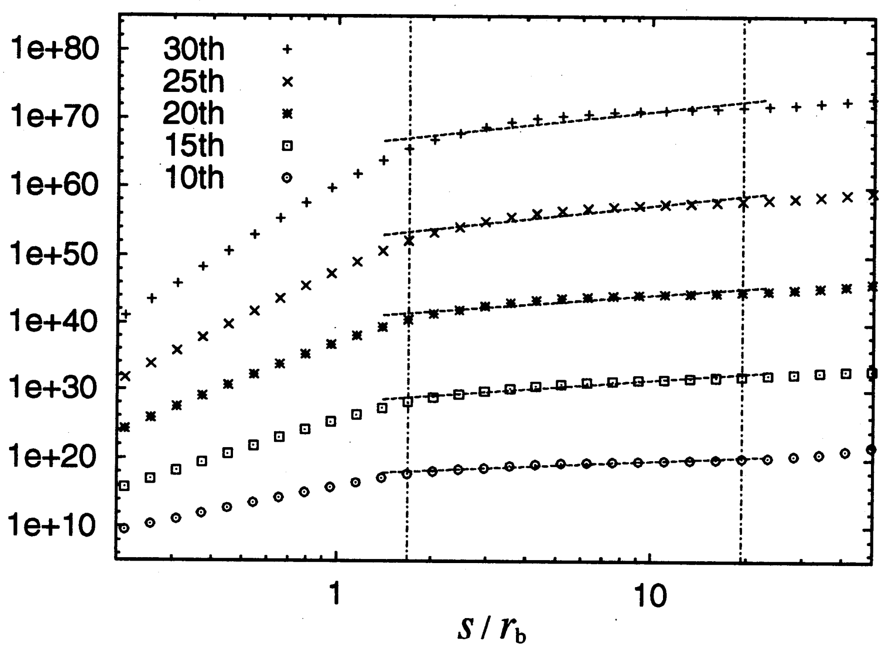

FIG. 5. High-order structure functions with fitting lines in the inertial range for $R_\lambda = 2000$.

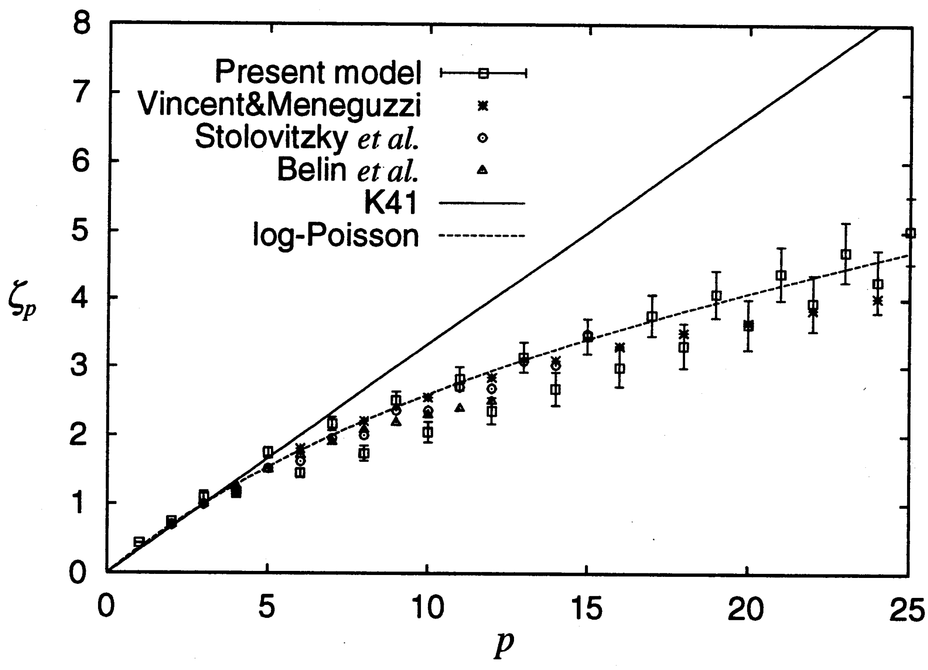

FIG. 6. The exponent ζ_p of the structure function for $R_\lambda = 2000$ with K41 [10], log-Poisson model [12–14], DNS for $R_\lambda = 200$ by Vincent and Meneguzzi [1], a wind tunnel experiment for $R_\lambda = 200$ by Stolovitzky *et al.*, obtained from taking the pollution of viscous range into account [15], and a helium gas experiment for $R_\lambda = 2000$ by Belin *et al.*, obtained by use of extended self-similarity [16].