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Hysteresis, limit cycles and mode interactions of standing waves with Faraday excitation.

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Abstract
Complex interaction phenomena are known to arise among standing free-surface waves in fluids within containers subjected to small periodic vertical (Faraday) oscillations. Here, we review theoretical and experimental work concerning (i) hysteresis and limit-cycle behaviour of 'pure' standing waves; and (ii) instability of a 'pure' standing wave to a pair of neighbouring modes, and the subsequent modulations. We also discuss the special case of (iii) second-harmonic resonance.

1 Introduction

Study of wave motion excited by small periodic vertical vibrations of a cylindrical container began with the pioneering studies of Faraday[1] and Rayleigh.[2,3] Such vertical oscillation is now known as 'Faraday excitation'. The waves most prone to generation are those with frequencies close to one-half of the forcing frequency; a situation now commonly referred to as 'subharmonic', 'parametric', or 'Faraday' resonance. The linear theory developed by Benjamin & Ursell[4] established that, in the inviscid limit, each surface-wave mode is governed by an equation of Mathieu type, and so exhibits the many zones of instability characterised by this equation; but the strongest instability, and so that least likely to be suppressed by viscous damping, is the subharmonic one. The corresponding linear viscous problem is fully described by Kumar & Tuckerman[5].

During the past fifteen years, interest in Faraday excitation has greatly increased, due to important advances in the theory of nonlinear dynamical systems; and to influential experimental studies that revealed a rich variety of behaviour, not all yet fully understood. Most notably, Ciliberto & Gollub[6] studied standing waves in circular cylinders; Feng & Sethna[7], Simonelli & Gollub[8] those in square and almost square rectangular containers; Ezerskii *et al.*[9], Douady & Fauve[10] and Douady[11] studied short capillary waves in containers large compared with wavelength; Douady, Fauve & Thual[12] excited, in an annulus, waves which could be standing or travelling; Wu, Keolian & Rudnick[13] examined localised 'standing solitons' in a long rectangular tank; Craik & Armitage[14] and Decent & Craik[15] studied neighbouring plane standing-wave modes in a long narrow tank; Jiang, Ting, Perlin & Schultz[16] examined large-amplitude waves and their modulations due to slightly-perturbed tank vibrations. The earlier work, and related theory, are described in the review by Miles & Henderson[17]. Additionally, fine experimental studies and related theory of wave motion in tanks subject to horizontal, rather than vertical, vibration have been made by Funakoshi & Inoue[18], while Krasnopolskaya & van Heijst[19] have investigated wave-generation in an annular tank with radially-vibrating inner wall, finding both 'direct' generation of axisymmetric waves and parametric 'Faraday' excitation of non-axisymmetric waves at the subharmonic frequency.

The mutual stability and nonlinear interaction of different spatial modes with similar (or, in degenerate cases, identical) natural frequencies has also been a subject of much activity. In addition to the above-mentioned experimental studies, which also address theoretical issues, the theoretical papers of Meron & Procaccia[20], Nagata[21,22], Umeki & Kambe[23], Kambe & Umeki[24], Umeki[25] and Craik[26] may be mentioned. In all of these, the postulated interaction of two modes, each characterised by a (slowly-varying) time-dependent complex wave amplitude, leads to a pair of complex evolution equations. These have four real variables and the structure of temporal orbits can be remarkably complicated. There are typically several equilibrium states that correspond to fixed points of the governing equations, some stable and some unstable. The actual behaviour depends crucially on several constant parameters that appear in these equations; and these parameters in turn depend (sometimes very sensitively) on the precise experimental configuration. Theoretical determination of the parameter values for a given configuration is not a trivial task and, once accomplished, investigation of the nature of the solutions involves a mix of analysis - to determine fixed points and their local stability - and extensive computations of solution trajectories. Even after all this is accomplished, it is not easy to make meaningful connections between the different sets of results for different configurations. No one set of experimental or theoretical results is 'typical': the richness of possible behaviours and sensitivity to parameter values is too great.

A different scenario is more appropriate for interaction between modes with similar spatial structure in a long narrow tank like that of Craik & Armitage[14]. In their configuration, where neighbouring modes have very similar spatial structure but slightly different wavelengths, the interaction of three (and perhaps more) modes is important. Then, a single dominant finite-amplitude mode may be unstable to a pair of neighbouring modes. Such instability has similarities with the well-known Eckhaus[27] and Benjamin-Feir[28] instability, but with additional forcing effects. This topic has recently been comprehensively treated by Decent & Craik[29], extending an earlier exploratory analysis of Craik[30], and it is discussed in section 3 below.

At, and near to, a precise frequency, second-harmonic resonance occurs among capillary-gravity waves: see e.g. McGoldrick[31]. Then, two waves have respective frequencies and wavenumbers in the ratio 1 : 2. When Faraday forcing has frequency close to twice that of one of the resonant pair of waves, an interesting mutual interaction occurs that is not described by theories that exclude such resonance. Such situations have been considered by Henderson & Miles[32], and similar equations arise for a forced resonant double pendulum (Becker & Miles[33]). Recent work of Forster & Craik[34] draws attention to the fact that the simplest model equations for this situation admit solutions that display unbounded growth : see section 4.

2 Hysteresis and single-mode limit cycles

Even for a single dominant standing-wave mode, theoretical description is far from straightforward, for the simplest approximation does not yield results in agreement with observation. A full account, and a particular examination of hysteresis of such waves as the imposed frequency and amplitude of vibrations are altered, is given by Decent & Craik[15,35]. Earlier, Miles[36] and Craik & Armitage[14] had shown that nonlinear forcing and damping can significantly affect single-mode hysteresis boundaries. Decent & Craik[15,35] retained also higher-order conservative terms, obtaining results that agree fairly well with their experiments and with those of Craik & Armitage[14] for three separate liquid depths. A novel feature of their results is the prediction of a periodically-modulating standing wave, corresponding to a single-mode limit-cycle solution, in a limited region of parameter space: but clear experimental confirmation of this does not yet exist. Their results for one spatial mode with liquid depth of 2cm are reproduced in Figure 1, together with experimental results of Craik & Armitage[14] on the linear instability boundary and nonlinear lower hysteresis boundary for that mode. The horizontal axis represents a scaled frequency parameter Ω which measures the small difference between half the forcing frequency and the natural linear frequency of the mode; the vertical axis denotes the scaled parameter F which measures the amplitude of the tank vibrations. In contrast, results for 1cm and 1.3cm depths display no limit cycle behaviour.

The analysis of Decent & Craik[15,35] incorporates an assumed value for their nonlinear damping coefficient N; and the range of approximate validity of their composite evolution equation (obtained by combining two rationally-derived evolution equations at successive orders of a governing small parameter ε) is unknown. Nevertheless, their results show quite good agreement with experiment; and a later attempt by Decent[37] to estimate theoretically the parameter N gives a value consistent with that previously assumed. Certainly, their results at all three liquid depths are in reasonable agreement with observation.

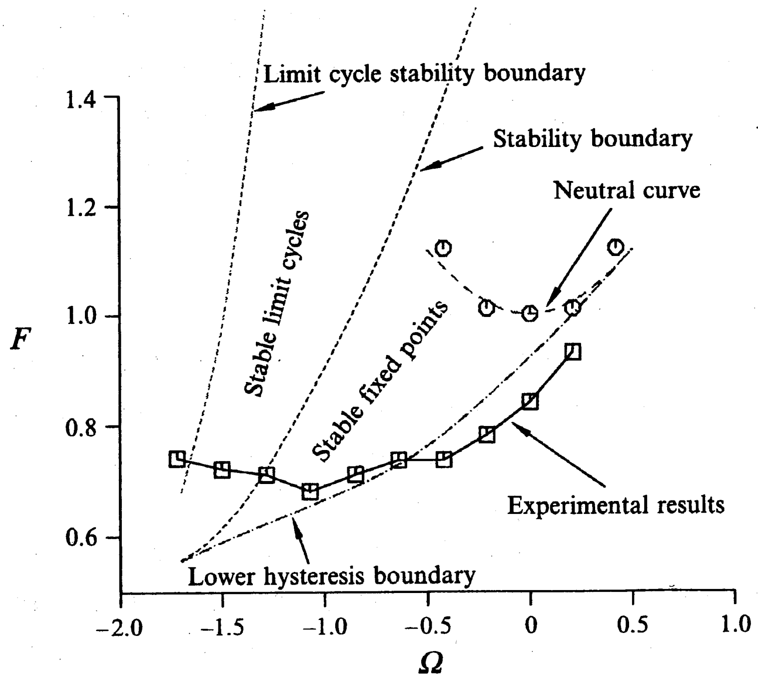

Figure 1. Location of neutral curve, hysteresis boundary and limit-cycle boundaries for one mode at 2cm depth, from Decent & Craik[15]. The experimental points for neutral curve and hysteresis boundary are from Craik & Armitage[14]. The axes are frequency-detuning and forcing parameters Ω and F.

3 Standing-wave instability and modulations

When the experimental configuration admits modes of similar spatial structure at similar frequencies, as does the long narrow channel of Craik & Armitage[14], then a single finite-amplitude standing wave is prone to instability due to growth of its two 'nearest neighbours': for, the latter, though linearly stable when the liquid surface is flat, may be unstable when the standing wave is present. The stability of a pure standing wave to its neighbours, and the nature of the resultant three-mode interactions, are subjects of a recent paper of Decent & Craik[29].

Their analysis incorporates all cubic conservative interaction terms involving the three modes, and estimates parametrically the effect of nonlinear damping and quintic conservative terms. When their equations are linearised with respect to infinitesimal 'sideband' modes, with complex amplitudes A and C say, and the standing wave amplitude B corresponds to the known finite-amplitude equilibrium solution, a 4-dimensional eigenvalue problem results. Its numerical solution determines the instability threshold for the growth of the modes A and C.

Though their results for 1cm depth do not agree particularly well with the observed threshold of Craik & Armitage's experiment, those for 2cm depth show much better agreement. The latter are shown in Figure 2. The observed onset of wave modulations associated with sideband growth agrees rather well with the theoretical results. Note that the limit-cycle region shown in Figure 1 is much reduced by the availability of the sideband instability; but this effect will be absent in experimental configurations that prohibit such 'close neighbours'.

When the neighbouring modes grow, mutual interactions occur and three-mode nonlinear solutions display rich structure, often with fast and slow timescales. One feature, however, displays no modulations at all. This is the region labelled 'six-dimensional stationary point'. Within this, the pure standing wave B is unstable to the sideband modes A and C but the resultant state, in which all three spatial modes are present, displays no temporal modulation: this, therefore, is a three-mode standing wave, with each component locked in phase. Decent & Craik point out that this standing wave never passes through a flat surface during its oscillation. Experimental confirmation of such standing-wave motion remains to be found.

Temporal modulations can be of various sorts. Decent & Craik found that, for a water depth of 1cm, intervals of strong wave activity are separated by recurrent nearly calm periods; but this recurrent calming does not occur with the larger depth of 2cm, for which modulations are typically periodic or chaotic. These findings are in broad agreement with some observations of Craik & Armitage. Two of Decent & Craik's figures are reproduced in Figures 3 and 4 below. The former shows a case for 1cm depth, in which quite long calm periods are seen, between bouts of wave activity. The central B-mode appears to grow, and equilibrate, before the sideband modes A and C are driven unstable; but the growth of the sidebands causes modulations that lead to the decay of all three modes to an almost calm state. Qualitatively similar behaviour was observed by Craik & Armitage at this

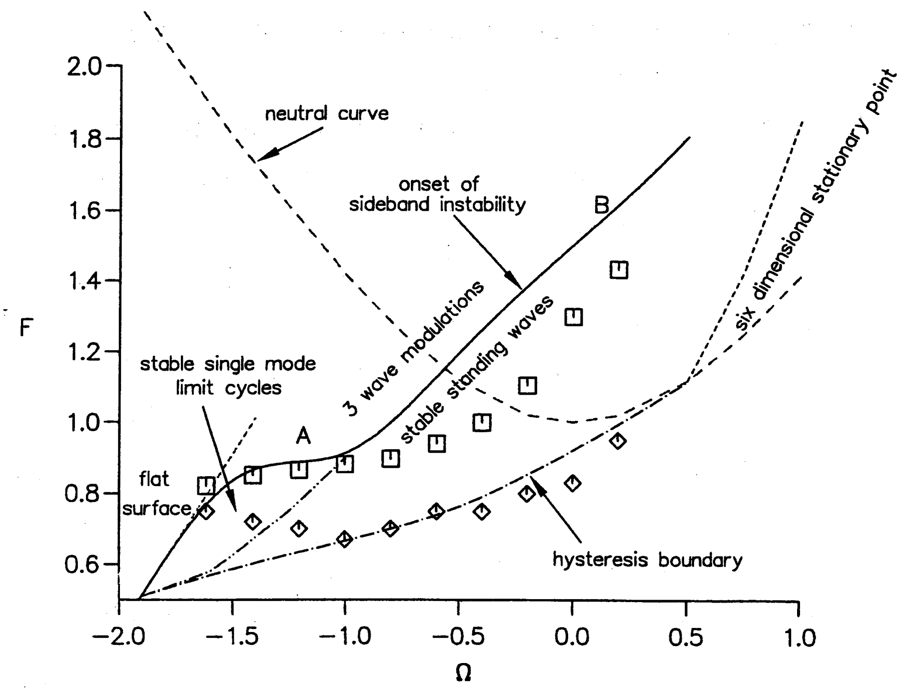

Figure 2. Three-mode stability diagram from Decent & Craik[29] for water depth of 2cm. The axes are frequency-detuning and forcing parameters Ω and F. Experimental points are from Craik & Armitage[14]. Diamonds and dotted-dashed curve show the measured and theoretical lower hysteresis boundary (cf Figure 1 above); squares and solid curve denote observed and theoretical onset of temporal modulations as F is increased. The dashed curve is the linear stability boundary for a flat surface. Regions of stable single-mode limit cycles and 6-dimensional (3-mode) stationary points are also indicated.

depth, and is recorded on videotape. In contrast, no such calming was observed by them with water of 2cm depth; and none is found theoretically either. Figure 4 shows a typical theoretical example at this larger depth. Somewhat similar behaviour, observed experimentally by Armitage & Sterratt (unpublished), is reported, with permission, by Decent & Craik[29].

In recent experiments, Jiang, Ting, Perlin & Schultz[16] reported spontaneous temporal modulation of a single gravity-wave mode, of the sort expected of the limit cycle described above. However, these authors express doubt over the origins of this modulation, which was not a consistently reproducible feature of their observations. Their subsequent investigations, employing deliberately-introduced sideband perturbations to the tank vibrations, showed that weak perturbations produced strong wave modulations, with a pronounced resonance peak. Certainly, inadvertent or deliberate signal noise is a possible source of modulations; but their dismissal of the possibility of a limit cycle seems premature without closer study.

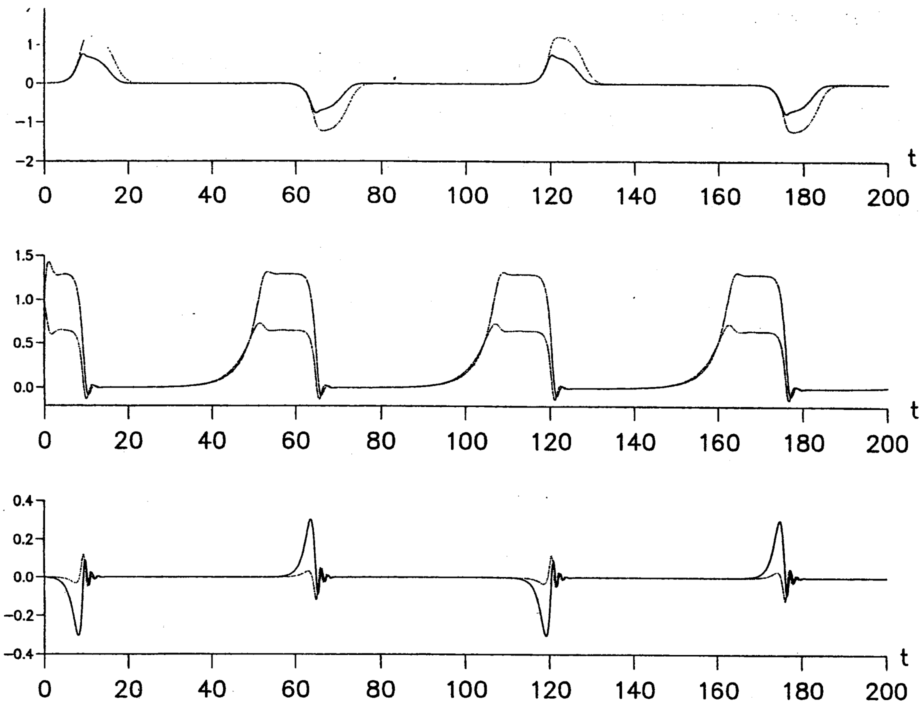

Figure 3. An example of three-mode modulations, for 1cm depth, from Decent & Craik[29].

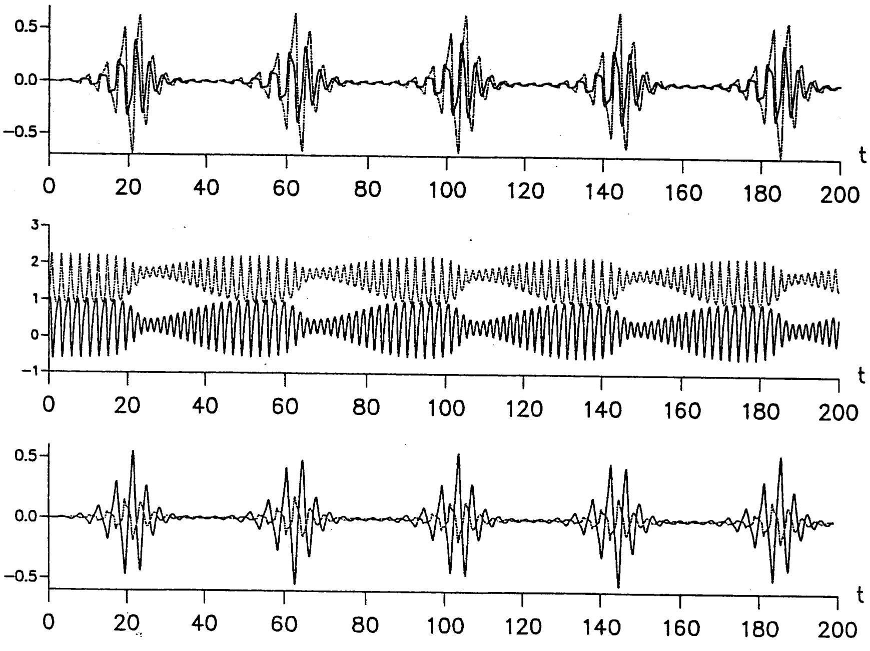

Figure 4. An example of three-mode modulations, for 2cm depth, from Decent & Craik[29].

4 Second-harmonic resonance

Second-harmonic wave resonance with Faraday excitation was considered by Henderson & Miles[32], who derived coupled evolution equations identical to those governing a forced resonant double pendulum (Becker & Miles[33]). Then, two standing waves have wavenumbers in the ratio 1 : 2 and natural frequencies also equal to, or very close to, that ratio. One or other of these waves is supposed excited by Faraday excitation close to twice it's natural frequency. Various aspects of the structure of solutions are examined by these authors; but they do not mention that their model equations permit unbounded wave growth under suitable circumstances. Recent work of Forster & Craik[34] draws attention to such unbounded solutions. Though unlimited growth is certainly 'unphysical', the presence of such solutions must indicate a transition to larger amplitudes that cannot adequately be described by the truncation, at quadratic order, implicit in the model equations.

The equations studied by Forster & Craik are a subset of those of Henderson & Miles, restriced to exactly resonant tuning with no viscous damping. These are either

$$\dot{a}_1 = \lambda_1 a_1^* a_2 + \mu a_1^*, \quad \dot{a}_2 = \lambda_2 a_1^2,$$

or

$$\dot{a}_1 = \lambda_1 a_1^* a_2, \quad \dot{a}_2 = \lambda_2 a_1^2 + \mu a_2^*,$$

for the respective complex wave amplitudes a_1 and a_2, depending on whether the forcing drives the first (a_1)or second (a_2) harmonic. Here, the overdot denotes time-derivative, the star denotes complex conjugate, λ_1, λ_2 are known real constants with opposite signs, and μ is a known imaginary constant. The former set is particularly simple, for the forcing term in μ may be eliminated by a simple change of variables, yielding the unforced equations which are solved in terms of elliptic functions.

The second set, with forcing at the second harmonic, is more challenging. These may be rescaled to

$$\dot{B}_1 = -B_1^* B_2, \qquad \dot{B}_2 = B_1^2 + B_2^*$$

where the overdot is now the rescaled time-derivative. Expressed in real and imaginary parts $B_1 \equiv x_1 + iy_1$, $B_2 \equiv x_2 + iy_2$, the corresponding real four-dimensional autonomous system is

$$\dot{x}_1 = -x_1 x_2 - y_1 y_2, \qquad \dot{y}_1 = x_2 y_1 - x_1 y_2,$$
$$\dot{x}_2 = x_1^2 - y_1^2 + x_2, \qquad \dot{y}_2 = 2x_1 y_1 - y_2.$$

Various computed solutions, both bounded and showing unbounded growth, are given by Forster & Craik. Transformed equations yield further insight and better enable delineation of the sets of initial data that lead to bounded evolution and unbounded growth respectively. In particular, a Hamiltonian constant of motion may be employed as a parameter; and this eventually leads to a two-dimensional set of coupled first-order non-autonomous equations, with a phase angle as independent variable. Poincaré sections then graphically reveal the domain of bounded initial data corresponding to the chosen value of the constant of motion.

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