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液膜流モデルを中心としたKdV方程式の摂動系に現れる波

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Abstract

Two different topics associated with perturbed KdV equations are studied. First one is the formal approach to the 2-D problem , that is how localized pulses behave, by considering pulse interactions. Second topic is the linearized stability of the periodic patterns in perturbed KdV equations in 1-D setting studied both theoretically and numerically. The stability depends upon the wavelength of the solution, namely, the periodic patterns with sufficiently short and long wavelength are unstable. This result coincides with the wavelength preference which can be observed in numerical solutions to the initial value problem of the PDE.

1 Introduction

In many physical problems, integrable systems such as the KdV or the NLS equations have been obtained by reductive perturbation method in its lowest order. These integrable systems are, however, sometimes insufficient to explain the original phenomena and one needs to take not only the lowest order but also higher order correction terms at the reductive perturbation step. These higher order terms usually contain dissipative effects. The Benney equation, which explains the wave motions on a liquid layer over an inclined plane, is one of the example of the nearly-integrable systems(see [KT]):

(1) $$u_t - uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \quad t \geq 0, \quad -\infty < x < \infty.$$

Here, ε is a small positive parameter, and the unperturbed equation is the KdV:

(2) $$u_t - uu_x + u_{xxx} = 0, \quad t \geq 0, \quad -\infty < x < \infty.$$

By considering the dispersion relation, intuitively speaking, two derivative terms in the perturbation, *i.e.* u_{xx} and u_{xxxx}, have instability and dissipative effects respectively.

The Benney equation is not only the model of the long surface wave on a thin liquid layer but also related to various other phenomena. And in many cases it is more realistic to study the Benney equation in the 2-D setting:

(3) $$u_t + uu_x + \Delta u_x + \varepsilon(u_{xx} + \Delta^2 u) = 0,$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and the parameter ε is assumed to be a small positive number. When the perturbation terms are absent it is equivalent to the Zakharov-Kuznetsov equation[ZK], one of the 2-D versions of the KdV equation:

(4) $$u_t + uu_x + \Delta u_x = 0.$$

The travelling wave solutions of the form $u = u(x - ct, y)$ satisfy the following equation:

$$\Delta u + \frac{1}{2}u^2 - cu = 0,$$

where c represents the wave velocity to be determined by solving this equation. We can scale out the velocity by $cu = U, \sqrt{c}x = X$ and $\sqrt{c}y = Y$ as:

$$\Delta^* U + \frac{1}{2}U^2 - U = 0, \tag{5}$$

where Δ^* denotes the Laplacian with respect to X and Y. It is easy to show that

$$U = 3\text{sech}^2[(X\cos\theta + Y\sin\theta)/2]$$

is an exact solution to (5). This solution is an oblique one-dimensional travelling wave which is naturally obtained from the 1-D KdV soliton. However, this 1-D travelling wave soloution is shown to be unstable by considering the eigenvalue problem.

It is known that the equation (5) admits the unique radially symmetric solution $F(r)$, that decays exponentially as $r \to \infty$. Therefore the Zakharov-Kuznetsov equation admits radially symmetric localized pulse solutions $u = cF(\sqrt{c(x^2 + y^2)})$ for an arbitrary positive velocity(amplitude).

Now let us go back to the perturbed equation (3). In [TIK] they have numerically found the quasi-stationary lattice patterns of pulses to (3). They reported that many localized pulses appear even when the initial data is random and these localized structures preserve their identities. These pulses travel as a whole changing their relative positions gradually and form mysterious lattice patterns. Also each of these pulses is well approximated by the radially symmetric localized pulse $u = cF(\sqrt{c(x^2 + y^2)})$ for a definite velocity(amplitude) c. It seems quite similar to the 1-D case, *i.e.*, the pulse solution to the 1-D Benney equation is well-approximated by the KdV soliton solution with definite amplitude. It is called amplitude selection. However, as far as we know there are no theoretical results on the behavior of solutions to the two-dimensional equation (3).

We shall discuss the amplitude selection of the pulse solution to the 2-D Benney equation and the specific regular patterns of pulses by considering the pulse interaction.

First, by applying Ei-Ohta's method to the 2-D Benney equation we shall obtain the selected velocity(amplitude) as a non-secularity condition, a kind of solvability condition. Second, the equations of motions of pulse positions can be obtained also from a non-secularity condition. And third, we shall study the ODE system which describes the motion of n-pulses under the periodic boundary conditions. Moreover, several elementary fixed points of the reduced ODE system is studied to compare the results with the numerical simulations of the full system. Though the ODE system has basically the simple repulsive character similar to the 1-D case, there are non trivial stable stationary patterns to the ODE system which would explain the lattice patterns in [TIK].

There are several stages in the development of the 2-D Benney equation:generation of pulses, pulse interaction and fast transition process. We can conclude that the reduced ODE would explain the pulse interaction stages which are very slow.

We will not discuss the detail on the 2-D problem here, because this result is already published in [OL]. And we will reconsider the 1-D problem, say wavelength preference.

The following equation is a perturbation of the mKdV equation:

$$u_t + (-1)^k u^2 u_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \quad t \geq 0, \quad -\infty < x < \infty. \tag{6}$$

The equation (6), especially with $k = 1$, has a relation to a traffic congestion problem.(see [KS].) There are some other nearly-integrable systems which can be considered as perturbations of the NLS equation, e.g. the guiding-centre soliton in the optical communication theory by Hasegawa and Kodama[HK]. Here we don't mention it, however, we believe that there are many nearly-integrable systems which have similar properties to (1) and also our method can be applied to study their stability.

These nearly-integrable systems, especially (1) and (6), have similar properties: an amplitude selection and a wavelength preference. It is well-known that the KdV equation has pulse and periodic travelling wave solutions:

$$u^{(0)}(z) = 3c \operatorname{sech}^2(\sqrt{c}/2z), \quad \textit{and}$$

$$u^{(0)}(z) = a\mathrm{cn}^2(Bz, m).$$

Here, $B = \sqrt{a/12m^2}$, and moreover a is an arbitrary positive constant and $z = x - ct$ is a travelling coordinate with the velocity $c = (2 - m^{-2})a/3$. Also cn denotes Jacobi's elliptic cn-function with modulus $m \in (0, 1)$. This means the KdV admits travelling wave solutions with an arbitrary amplitude. We can show the existence of such solutions to the Benney equation (1) when ε is small enough. In this case, however, the equation (1) admits only one travelling wave solution up to phase shift and Galilei transformation for each wavelength. Here, Galilei transformation: $u = \tilde{u} + c, x = \tilde{x} - ct$ means to take a different travelling wave coordinate. In fact, we have the following:

Theorem 1. ([EMR],[Og]) There exists a positive number ε^* such that the following holds. For an arbitrary $l > 2\pi$ (1) has a unique periodic travelling wave solution with wavelength l, $\Phi(z; l)$, up to phase shift and Galilei transformation for all ε with $0 < \varepsilon < \varepsilon^*$. Φ can be approximated as:

$$\Phi(z; l) = \Phi^{(0)}(z; a(l)) + \varepsilon\Phi^{(1)}(z) + o(\varepsilon).$$

Here, $\Phi^{(0)}(z; a) = a\mathrm{cn}^2(Bz, m(l))$, where $m(l)$ is a well-defined smooth function with $Bl = 2K(m(l))$ so that $\Phi^{(0)}$ has wavelength l. $K(m)$ denotes the complete elliptic integral of the first kind. $a(l)$ is given by (the solvability condition):

$$\int_{-\infty}^{\infty} \left\{ \frac{\partial \Phi^{(0)}(z; a(l))}{\partial z} \right\}^2 dz - \int_{-\infty}^{\infty} \left\{ \frac{\partial^2 \Phi^{(0)}(z; a(l))}{\partial z^2} \right\}^2 dz = 0, \tag{7}$$

and is consequently smooth and monotone increasing function of l.(Figure 1(a)) Moreover, $\Phi(z;l)$ converges to the unique pulse solution $\Phi_\infty(z)$ compact uniformly as $l \to \infty$.

Therefore the amplitude selection means the property that only one travelling wave solution can persist to the perturbed equation (1) for each wavelength. It should be noted that other representation is also possible for Φ by an appropriate Galilei transformation. More precisely, we used the function $a\mathrm{cn}^2(Bz,m)$ to describe the periodic solution here for simplicity, however, $u^{(0)}(z) = u_0 + a\mathrm{cn}^2(Bz,m)$ is also possible by taking another speed $c = u_0 + (2-m^{-2})a/3$. In the next section we require the mean-zero constraint $\int \Phi dz = 0$, however, Theorem 1 still holds with the same amplitude function $a(l)$ because Galilei transformation does not affect the solvability condition (7).

On the other hand, numerical simulations suggest that the time evolutions of the solution to the equation (1) is much more interesting.(See [TK].) The equation (1) is solved numerically in a finite interval $[0,L]$ with the periodic boundary conditions. Even if the initial data is small, many pulses appear at the first stage by the instability effect. Then at the second stage amplitudes of these pulses change to become close. Moreover, at the third stage, pulse positions of these pulses are modulated gradually and the solution seems to converge to a periodic solution finally. For a fixed interval $[0,L]$ we have n different periodic travelling wave solutions from the Theorem 1, where $n < \frac{L}{2\pi} \leq n+1$. However, numerical solutions converge to only a few of them if we start from many different initial data. In Figure 1(b) we show the numerically "stable" region of each different periodic solution. The most interesting point in the numerical simulation is that solutions to (1) exhibit the wavelength preference(See Figure 2). That is only the periodic traveling wave solutions with wavelength $l \in (l_*, l^*)$ are stable. Here, l_* and l^* are independent of L : the domain length.

The motivation of this research is to understand why the specific wavelength is preferred. We study the linearized eigenvalue problem $(EP)_l$ around each periodic travelling wave solution $\Phi(z;l)$ for all $l > 2\pi$. Let us define the stable wavelength region S by

$$S := \{l \in (2\pi, +\infty) | \mathrm{spec}((EP)_l) \subset \{\lambda | \mathrm{Im}\lambda < 0\} \cup \{0\}\}.$$

We can intuitively understand that S is bounded. Because of the dispersion relation a long wave instability appears. It is, however, still non-trivial. In this report we introduce the theoretical-numerical approach to this problem. By this approach we can "show" that $S \subset (l_*, l^*)$. The result is still not rigorous, however, suggests that there are two types of instability mechanisms which corresponds the upper and lower bounds l_*, l^*. It should be noted that if we restrict the problem for the finite interval as the numerical simulations, we can calculate all the critical eigenvalues accurately.

2 Eigenvalue problems

Let us first linearize (1) around the periodic solution $\Phi(z;l)$ to obtain the eigenvalue problem:

$(EP)_{l,N}$ $$Lv = \sigma v, \quad z \in [0, Nl],$$

where L is the linearized operator with l -periodic coefficients:

$$Lv = cv_z - (\Phi v)_z - v_{zzz} - \varepsilon(v_{zz} + v_{zzzz}).$$

The eigenvalue problem can be written as the first order system by using the notation $y = (u, u_z, u_{zz}, u_{zzz})^t$:

$$\frac{d}{dz}y = A(z;\sigma)y.$$

Here A is a matrix with l -periodic entry, *i.e.*, $A(z+l;\sigma) = A(z;\sigma)$. Let $Y(z;\sigma)$ be a fundamental solution matrix. The floque theory tells us that $Y(z;\sigma) = \Gamma(z)e^{\Lambda z}$, where Γ is l-periodic and Λ is a constant matrix. This means that σ is an eigenvalue of $(EP)_{l,N}$ if and only if $F(\sigma, \frac{n}{N}; l) = \det(Y(l;\sigma) - e^{2\pi i \frac{n}{N}} I) = 0$. In this case the corresponding eigenfunction is Nl-periodic and oscillating n-times. By restricting the problem to the finite interval the spectrum becomes discrete and they can be treated as the perturbation of the eigenvalues of the KdV equation. The spectrum of L, $spec((EP)_l)$, in the space such as $BC(\mathbf{R})$ is expected to be obtained by the suitable limit of the set of all the eigenvalues of $(EP)_{l,N}$, however, we don't mention this problem here.(See for example [Mi] for the related topic.) Therefore the question is to determine the set $C_l = \overline{\{\sigma | F(\sigma, \frac{n}{N}; l) = 0, n, N \in \mathbf{Z}\}}$, trivially, a subset of $spec((EP)_l)$, for each $l > 2\pi$.

Our approach owes [EMR] very much. They first considered the eigenvalue problem of (1) as the perturbation of that of the KdV:

(8) $$\sigma^{(0)}v = L^{(0)}v = cv_z - (\Phi^{(0)}v)_z - v_{zzz}.$$

They also calculated the first order solvability condition of the perturbation formally. We can justify this point by the geometric singular perturbation technique as in [OS]. More crucial point is that the eigenfunctions of the KdV can be written by using that of Hill's equation.[MT] The eigenvalues of the KdV lie densely on the imaginary axis. That means the travelling wave solutions of the KdV is neutrally stable. Therefore we need to study ε's first order correction terms from the perturbation effect to determine the stability.

Let us briefly review the known facts about the eigenvalues of Hill's equation:

$$-y'' + \frac{u(x)}{6}y = \frac{\lambda}{6}y,$$

where $u(x)$ is a periodic function of $[0,1]$. Then, there exists an infinite sequence of eigenvalues

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots \to \infty$$

with eigenfunctions of period 1 or 2. These eigenvalues are well characterized by the so-called Floquet discriminant. Let $Y(x;\lambda)$ be the fundamental solution matrix of Hill's equation with $Y(0;\lambda) = I$. Then it follows that $\det Y(x;\lambda) \equiv 1$ and consequently the eigenvalues of $Y(1,\lambda)$ are $(\Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 4})/2$ where $\Delta(\lambda) = \text{trace}Y(1;\lambda)$, the Floquet discriminant. Thus Hill's equation has a bounded solution only when $\Delta(\lambda) \leq 2$. Moreover, it has 1-periodic solution if and only if $\Delta(\lambda) = 2$ and has 2-periodic solution if and only if $\Delta(\lambda) = -2$. Now

$$\Delta(\lambda_k) = 2 \quad \text{when} \quad k = 0 \quad \text{or} \quad 2n-1, 2n, \quad \text{where} \quad n = 2, 4, 6, \cdots \quad \text{and}$$

$$\Delta(\lambda_k) = -2 \quad \text{when} \quad k = 2n-1, 2n, \quad \text{where} \quad n = 1, 3, 5, \cdots.$$

The surprising fact is that there is only one instability gap, *i.e.* the open interval $(\lambda_{2N-1}, \lambda_{2N})$, in the sequence of $\{\lambda_k\}$ if and only if the potential function $u(z)$ is a periodic travelling wave solution of the KdV equation with period 1/N.

Let us recall here our original equation:

$$u_t - uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \quad x \in [0, l], \tag{9}$$

with periodic boundary condition. For the sake of convenience we rewrite it by

$$\tilde{x} = \beta x, \quad \tilde{u} = u/\beta^2, \quad \tilde{t} = \beta^3 t, \quad \text{and} \quad \tilde{\varepsilon} = \varepsilon/\beta \tag{10}$$

as

$$\tilde{u}_{\tilde{t}} - \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{\varepsilon}(\tilde{u}_{\tilde{x}\tilde{x}} + \beta^2 \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}) = 0, \quad \tilde{x} \in [0, 1] \tag{11}$$

so that we can consider all the different periodic travelling wave solutions have period 1. We will omit the notation ˜ as far as it is clear. Every periodic travelling wave solution of the KdV, can be written by the elliptic function as follows

$$u_{(0)} = \alpha_2 - (\alpha_2 - \alpha_1)\text{cn}^2(\sqrt{\frac{\alpha_3 - \alpha_1}{12}}(z - z_0)),$$

where $\alpha_1 < \alpha_2 < \alpha_3$ are arbitrary real numbers and the modulus of Jacobi's cn-function should be $m = (\alpha_2 - \alpha_1)/(\alpha_3 - \alpha_1)$. The speed of its travelling wave solution is $c^{(0)} = -(\alpha_1 + \alpha_2 + \alpha_3)/3$. It is called the cnoidal wave solution. Let us chose these α's to satisfy the following three condition:

- $u^{(0)}$ has period 1,
- $\displaystyle\int_0^1 \left\{\frac{\partial u^{(0)}}{\partial z}\right\}^2 dz = \beta^2 \int_0^1 \left\{\frac{\partial^2 u^{(0)}}{\partial z^2}\right\}^2 dz,$
- $\displaystyle\int_0^1 u^{(0)} dz = 0.$

Here, the second condition corresponds the solvability condition in Theorem 1, therefore these three conditions determine a unique triple $(\alpha_1, \alpha_2, \alpha_3)$ for $\beta < 1/(2\pi)$. By the above fact, simple eigenvalues of Hill's equation are $\lambda_0 = (\alpha_1 + \alpha_2)/2$, $\lambda_1 = (\alpha_1 + \alpha_3)/2$ and $\lambda_2 = (\alpha_2 + \alpha_3)/2$ with the corresponding eigenfunctions $y_0 = \mathrm{dn}(sz)$, $y_1 = \mathrm{cn}(sz)$ and $y_2 = \mathrm{sn}(sz)$, where $s = \sqrt{(\alpha_3 - \alpha_1)/12}$. The other eigenvalues are all double and the corresponding eigenfunctions are

$$
\begin{aligned}
y_{2n-1}, y_{2n} = & \sqrt{\lambda_n^d + u^{(0)}(z)/2 + 3c^{(0)}/2} \\
& \cdot \exp(\pm\sqrt{(\lambda_n^d - \lambda_0)(\lambda_n^d - \lambda_1)(\lambda_n^d - \lambda_2)/6} \int_0^z \frac{d\tau}{\lambda_n^d + u^{(0)}(\tau)/2 + 3c^{(0)}/2})
\end{aligned} \tag{12}
$$

for $n > 1$. Here, $\lambda_n^d = \lambda_{2n-1} = \lambda_{2n}$ and these values are determined by three simple λ's and n. All of the above facts are in [MT] and more detailed review can be found in [EMR].

Now the eigenvalue problem should also be scaled as

$$
\sigma^\varepsilon v^\varepsilon = L^\varepsilon v^\varepsilon = D[(u^\varepsilon + c^\varepsilon)v^\varepsilon] - D^3 v^\varepsilon - \varepsilon(D^2 + \beta^2 D^4)v^\varepsilon, \tag{13}
$$

where $u^\varepsilon = u^{(0)} + \varepsilon u^{(1)} + \cdots$ is a periodic travelling wave solution and $c^\varepsilon = c^{(0)} + \varepsilon c^{(1)} + \cdots$. The lowest order of (13) is equivalent to (8) and solved by squared eigenfunctions of Hill's equation.

Fact 1.([EMR]) If $u(z)$ is the KdV cnoidal wave solution with period 1, then the derivatives:

$$
v_k^{(0)} = D(y_k^{(0)})^2, \quad k = 3, 4, 5, \cdots
$$

of the squared eigenfunctions y_k of Hill's equation together with $v_1 = \partial u^{(0)}/\partial c$ and $v_2 = \partial u^{(0)}/\partial z$ are a set of eigenfunctions of (8) with the eigenvalues

$$
\sigma_{2n-1}^{(0)}, \sigma_{2n}^{(0)} = \pm\frac{4}{3} i\sqrt{(\lambda_n^d - \lambda_0)(\lambda_n^d - \lambda_1)(\lambda_n^d - \lambda_2)/6},
$$

for $n = 2, 3, \cdots$. Moreover, the functions:

$$
w_k^{(0)} = (y_k^{(0)})^2, \quad k = 3, 4, 5, \cdots
$$

together with $w_1 = u^{(0)}$ and $w_2 = \int_0^z [u^{(0)}/\partial c] dz$ are a set of solutions to the adjoint eigenvalue problem $L^{(0)*} w = \overline{\sigma^{(0)}} w$. Also, the following biorthogonality holds:

$$
< v_j^{(0)}, w_k^{(0)} >= 0 \quad if \quad j \neq k, \quad k, l \geq 1,
$$

where $< f, g >:= \int_0^1 f\bar{g} dz$.

Fact 1 says that ε's zero'th order of the eigenvalue equation can be solved exactly and eigenvalues $\sigma_k^{(0)}$ lie on the imaginary axes. By using the formal expansions

$$
\begin{cases} \sigma_k^\varepsilon = \sigma_k^{(0)} + \varepsilon\sigma_k^{(1)} + \cdots \quad , \\ v_k^\varepsilon(z) = v_k^{(0)}(z) + \varepsilon v_k^{(1)}(z) + \cdots , \end{cases} \tag{14}
$$

for each $k \geq 3$ they ([EMR]) obtained the formal solvability condition of $v_k^{(1)}$ as,

$$\sigma_k^{(1)} = \frac{-< u^{(1)}v_k^{(0)}, v_k^{(0)} > + < \partial v_k^{(0)}, v_k^{(0)} > -\beta^2 < \partial^2 v_k^{(0)}, \partial v_k^{(0)} >}{< v_k^{(0)}, w_k^{(0)} >}. \tag{15}$$

Here, for the eigenvalue equation (13) is a singular perturbation of (8), the existence of eigenfunctions is not automatically trivial from the above formal solvability condition. Therefore we apply the similar reduction to [OS] by using a geometric singular perturbation technique so that we obtain the equivalent regular perturbation problem which solvability condition is the same as (15) up to $O(\varepsilon^2)$.

Theorem 2. All the eigenvalues of (13) can be expressed as (14) with $\sigma_k^{(1)}$ determined as (15).

At this stage we still need the detailed information of the periodic solution, *i.e.* $u^{(1)}$, to determine $\sigma_k^{(1)}$. However, by the following lemma we can obtain the required information for stability without using $u^{(1)}$.

Lemma 3. $\mathrm{Re} < v_k^{(0)}, w_k^{(0)} >= \mathrm{Re} < \partial v_k^{(0)}, v_k^{(0)} >= \mathrm{Re} < \partial^2 v_k^{(0)}, \partial v_k^{(0)} >= 0$. Therefore,

$$\mathrm{Re}\sigma_k^{(1)} = \frac{< \partial v_k^{(0)}, v_k^{(0)} > -\beta^2 < \partial^2 v_k^{(0)}, \partial v_k^{(0)} >}{< v_k^{(0)}, w_k^{(0)} >}. \tag{16}$$

This lemma can be proved by direct calculation by the exact representation of y_k.

Let $\mu(z,\lambda) = \lambda + u^{(0)}(z)/2 + 3c^{(0)}/2$ and $\gamma = \sqrt{(\lambda-\lambda_0)(\lambda-\lambda_1)(\lambda-\lambda_2)/6}$. Then moreover, we have

Lemma 4.

$$< v, w >= 2\gamma i \int_0^1 \mu(z,\lambda) dz,$$

$$< \partial v, v >= \gamma i \int_0^1 \frac{(\partial u/\partial z)^2 + 16\gamma^2}{2\mu(z,\lambda)} dz,$$

$$< \partial^2 v, \partial v >= \gamma i \int_0^1 \{\frac{3(\partial^2 u/\partial z^2)^2}{2\mu} + \frac{1}{\mu^3}(32\gamma^4 - 10\gamma^2(\partial u/\partial z)^2 - \frac{(\partial u/\partial z)^4}{6})\} dz.$$

Here, λ satisfies $\lambda_0 < \lambda < \lambda_1$ or $\lambda_2 < \lambda$.

Remark 1. Only when $\lambda = \lambda_k^d$ the integrals in the above lemma give the information for $\sigma_k^{(1)}$. These eigenvalues correspond 1-periodic and k-times oscillating eigenfunctions. Since we are interested in the eigenvalues corresponding to a rational rotation number we will obtain the information for C_l by taking λ as in Lemma 4.

Remark 2. The function $m(z;\lambda)$ has zeros only when $\Delta(\lambda) \leq -2$. Therefore, integrals appearing in the second and third equations in Lemma 4 become singular when λ approaches λ_1 or λ_2.

3 Numerical estimates and wavelength preference

We numerically calculate $\mathrm{Re}\sigma^{(1)}$ by using the formula in Lemma 4. As we remarked there, we can integrate them as much accuracy as we want except the neighborhood of λ_1 and λ_2. By Fact 1, imaginary part of σ^ε is controlled by $\sigma^{(0)}$ and $\sigma^{(0)} \to 0$ when $\lambda \to \lambda_i$.

The numerical results in Figure 3 describe eigenvalues by plotting $(\beta^3\sigma^{(0)}, \beta^2\mathrm{Re}\sigma^{(1)})$. The factors β^3 and β^2 come from a rescaling procedure to the original $(EP)_l$. The spectrum curve C_l is expected to be obtained after the real coordinate scaling by the factor of ε. Three pictures(different scale) in one row are associated with the same periodic travelling wave solution. Figure 3 shows the results for six different wavelengths. There are basically two curves, bounded and unbounded ones. Unbounded curve comes from the spectrum branch of Hill's equation for $\lambda > \lambda_2$. And bounded curve comes from that for $\lambda_0 < \lambda < \lambda_1$. These results are summarized as

- $\Phi(z;l)$ is linearly unstable when $2\pi < l < 8.43\ldots$ by the perturbation with wavelength less than l.
- $\Phi(z;l)$ is linearly unstable when $l > 26.3\ldots$ by the perturbation with wavelength larger than l.

Therefore we can obtain the wavelength preference by the linearized eigenvalue approach. Still many problems remain open. First, as we noticed above, these numerically determined curves are reliable except the neighborhood of the real axes. Therefore we need a local theory to determine the connectivity to those numerical curves. To do that we know only 0 eigenfunction given by the derivative of the solution. And we can say that the spectrum curve passing through 0 stays locally in the left half plane for all $l > 2\pi$ by the local calculus around 0. (We haven't mention the detail here.) However, numerical results suggest that spectrum curve crosses the real axes three times, while we don't know other real eigenvalues. Second, this numerically obtained lower bound for the stable wavelength coincides quite well with Figure 2, while the upper bound we obtained is much larger than Figure 2.

We believe that these theoretical-numerical approach to detect the spectrum about the periodic solutions can be applied to many other nearly-integrable systems. In fact, we can do it for the equation (6) to obtain the similar wavelength preference. Also, we have an example which does not have wavelength preference. Consider the following perturbation of the KdV:

$$u_t - uu_x + u_{xxx} - \varepsilon(u + u_{xx}) = 0, t \geq 0, -\infty < x < \infty. \tag{17}$$

We can similarly construct the family of periodic solutions as in Theorem 1, while, in this case, spectrum curve intersects with right half plane for all $l > 2\pi$. This also coincides with the numerical simulations of the initial value problem of (17).

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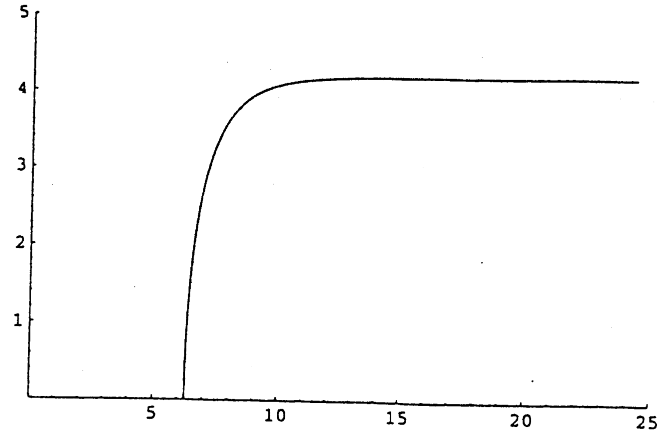


WAVELENGTH vs AMPLITUDE (Benney equation)

Figure 1 (a)

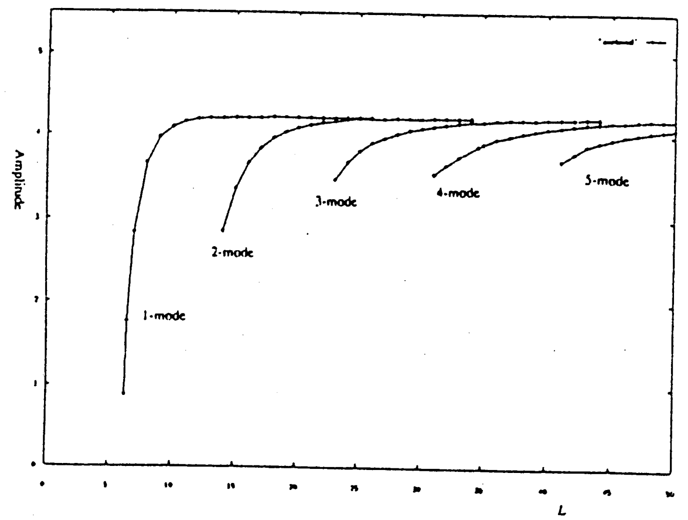


Figure 1 (b)

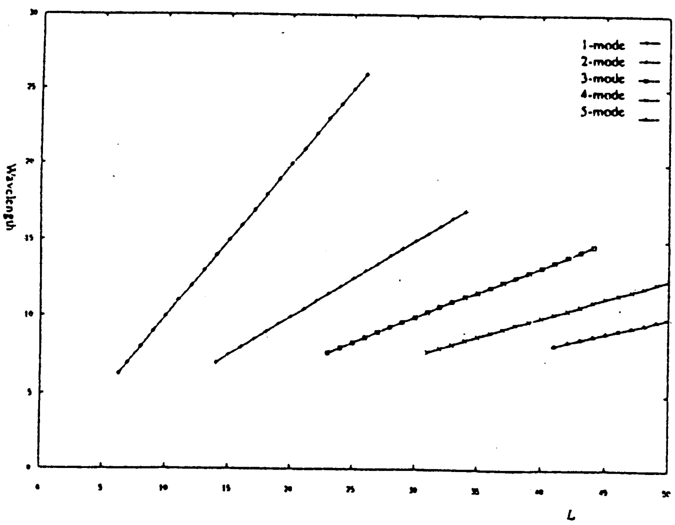
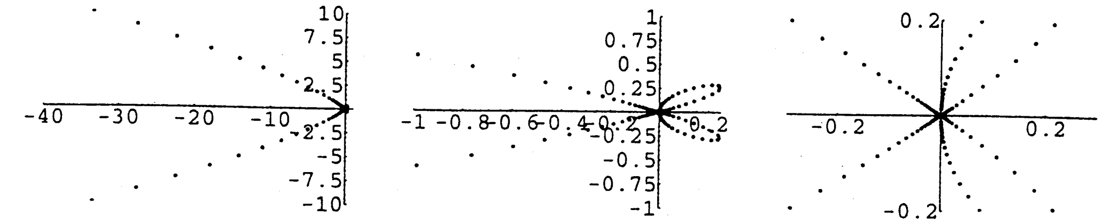
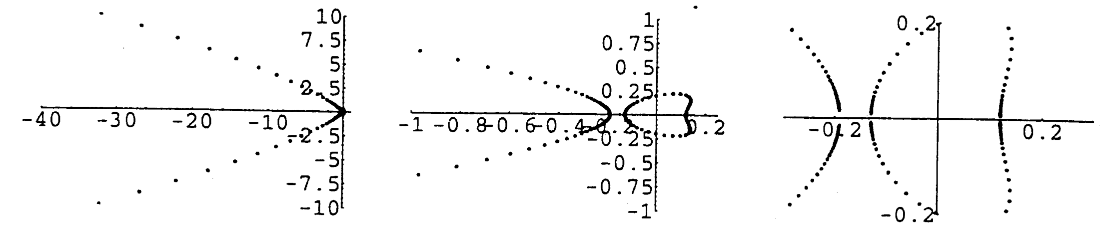


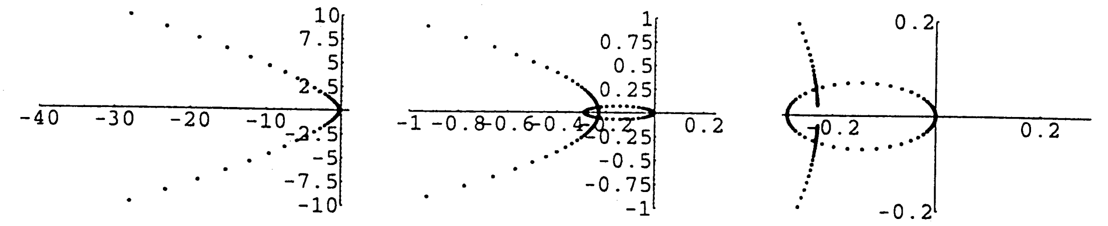
Figure 2



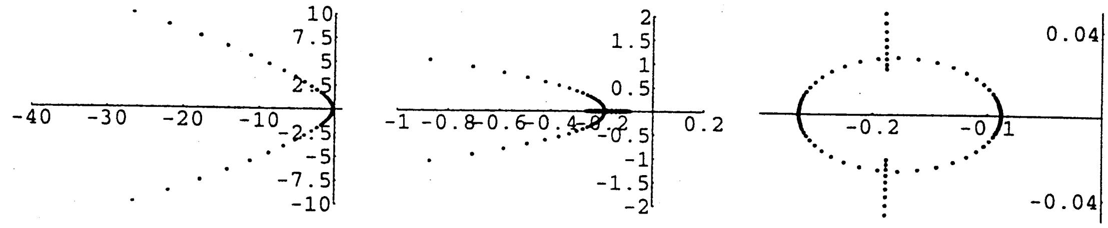
WaveLength = 6.28322



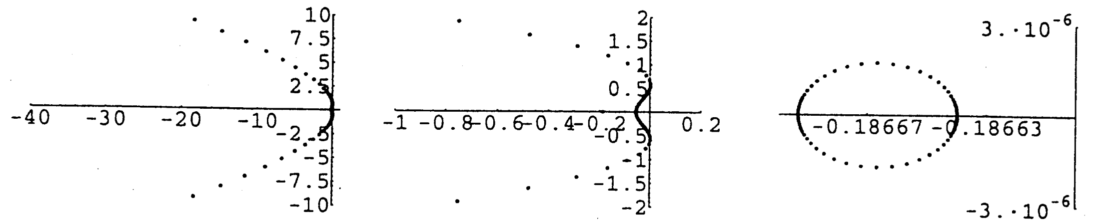
WaveLength = 6.66667



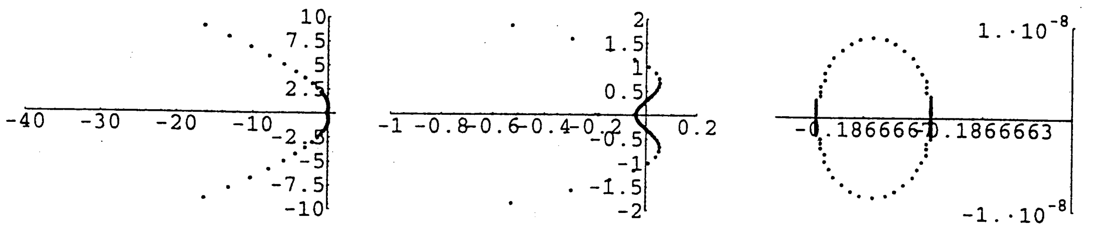
WaveLength = 8.43882



WaveLength = 10.0000



WaveLength = 26.3158



WaveLength = 35.2584

Figure 3