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R^N 上の楕円型問題の解について

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1. Introduction . この講演では, 次の楕円型問題の正値解の存在を考える。

$$(P) \qquad \begin{cases} -\Delta u+u=g(x,u), & u>0, \quad \text{in } R^N \\ u\in H^1(R^N), & N\geq 2 \end{cases}$$

where $f:R^N\to R$ and $g:\Omega\times R\to R$ is continuous with $g(x,0)=0$ for $x\in\Omega$. 楕円型問題 (P) については, 過去１０年間の間に, その解の存在と性質について多くの研究がなされている。最近、 半線形楕円型問題

$$(P_Q) \qquad \begin{cases} -\Delta u+u=Q(x)\mid u\mid^{p-1}u, & x\in R^N \\ u\in H^1(R^N), & N\geq 2 \end{cases}$$

の正値解については, $1<p$ for $N=2$, $1<p<(N+2)/(N-2)$ for $N\geq 3$, および,$Q(x)$ が positive bounded continuous function という条件下で, 何人かの研究者によって, 結果が得られている。 $Q(x)$ が radial function で ある場合には、問題 (P_Q) は無限個の解を持つことが得られる。これは, 解を radial functions の中から探すことにより、常微分方程式に帰着させることによって得ることができる。(cf. [1]). $Q(x)$ が必ずしも radial でない場合には, 領域が無限であることにより、comact 性の欠如という問題に突き当たる。 すなわち、Sobolev type の compact embedding が成り立たない為に、解の存在が容易に示せないということになる。この問題は, P.L. Lions(cf.[6,7]) によって、いわゆる concentrate compactness method という方法によって部分的に解決された. この方法によれば、P_Q のような問題は, 適当な条件下で解くことができる。

P.L.Lions は彼の方法を用いて次のような結果を得た: Assume that

$$lim_{|x|\to\infty}Q(x)=\overline{Q}(>0) \ \text{ and } \ Q(x)\geq\overline{Q} \text{ on } R^N,$$

then problem (P_Q) has a positive solution.

この結果は次のような観察に基づいている。 すなわち、問題 P_Q の ground state level c_Q, すなわち

$$I_Q(u)=\frac{1}{2}\int_{R^N}(\mid\nabla u\mid^2+\mid u\mid^2)dx-\frac{1}{p+1}\int_{R^N}Q(x)u^{p+1}dx$$

の lowest critical level が ground state level $c_{\overline{Q}}$, すなわち $I_{\overline{Q}}$の lowest critical level よりも小さい. このようなときには、我々は concentrate compactness method を用いること

ができる。問題 (P) についても同様の議論ができる。すなわち、$g: R^N \times R \to R$ が条件 $\lim_{|x|\to\infty} g(x,t) = t^p$ を満たし,

$$I(u) = \frac{1}{2}\int_{R^N}(|\nabla u|^2 + |u|^2)dx - \int_{R^N}\int_0^{u(x)} g(x,t)dtdx,$$

$u \in H^1(R^N)$, の least critical level c_1が

$$I^\infty(u) = \frac{1}{2}\int_{R^N}(|\nabla u|^2 + |u|^2)dx - \frac{1}{p+1}\int u^{p+1}dx.$$

のそれよりも小さいとする。このとき、適当な条件下で,(P) の正値解の存在は Ding & Ni[4] や Stuart[10] 等によって示されている。また, 最近 Cao[2] は (P_Q) の正値解について, $c_Q \le c_{\overline{Q}}$ を満たす場合について, $\lim_{\|x\|\to\infty} Q(x) = \overline{Q}$ かつ $Q(x) \ge 2^{(1-p)/2}\overline{Q}$ on R^N という条件下で証明している。

$c_Q = c_{\overline{Q}}$が成り立つ場合には,concentrate compactness method を用いることができないので、証明が難しい. 一方、gが $Q(x)t^p$という形で与えられていない場合は, Lagrange's method が 使えない為に, 新たな困難が生じる。(P_Q) に関しては, 解を得る為には, minimizing problem

$$\inf\{I_Q(u) : u \in V_\lambda\},$$
$$V_\lambda = \{u \in H^1(R^N), u > 0, \int_{R^N} Q(x)u^{p+1}dx = 1\}$$

の解を求めればよかった。すなわち, 得られた解,u に対して c を適当に選べば, cu が (P_Q) の解となる。Lagrange's method は残念ながら一般の g にたいしては有効に働かない。我々の方法は, こうした一般の場合にも有効となるものである。すなわち、g が $g(0) = 0$, $g(t) \to t^p$ as $t \to \infty$ を満たす場合に問題 (P) の解を考えることができる。また、nonhomoginous な場合:

$$(P_f) \qquad \begin{cases} -\Delta u + u = |u|^{p-1}u + f, & x \in R^N \\ u \in H^1(R^N), & N \ge 3 \end{cases}$$

ここで $p > 1$ for $N = 1$, $1 < p < (N+2)/(N-2)$ for $N \ge 3$, も同様の文脈で考えることができる。nonhomogeneous な場合については, Zhu[12] が解の存在を考えている。[12] において少なくとも 2 つの正値な（P）の解が次の条件下で存在することが示されている。すなわち、$f \in L^2(R^N)$ は L^2−norm が十分に小さく、exponential decay

$$f(x) \le Cexp\{-(1+\epsilon)|x|\}, \qquad \text{for } x \in R^N.$$

を持つ。我々の結果は, $f \in L^q(R^N)$($q = (p+1)/p$) が正値であれば, その decay の速度については条件がいらない。

この講演では, 次のような結果に対するアプローチを示す。

以下では,y $|\cdot|_q$ は $L^q(R^N)$ のノルムをあらわす。. $g: R^N \times R \to R$にたいしては次のような条件を仮定する。

(g1) There exists a positive number $d < 1$ such that
$$-dt + (1-d)t^p \le g(x,t) \le dt + (1+d)t^p$$
for all $(x,t) \in R^N \times [0,\infty)$;

(g2) there exists a positive number C such that
$$| g_t(x,0) | < 1 \quad \text{and} \quad 0 < t^2 g_{tt}(x,t) < C(1+t^p)$$
for all $(x,t) \in R^N \times [0,\infty)$;

(g3)
$$\lim_{|x|\to\infty} g(x,t) = | t |^{p-1} t$$
uniformly on bounded intervals in $[0,\infty)$,

where $1 < p$ for $N = 2$ and $1 < p < (N+2)/(N-2)$ for $N \ge 3$, and $g_t(\cdot,\cdot)$ stands for the derivative of g with respect to the second variable.

この講演では次のような結果について述べる.

Theorem 1. *(g2)* および *(g3)* を仮定する. このとき、$d_0 > 0$ が 存在して, もし *(g1)* が $d < d_0$なる値に対して成立するならば *(P)* は正値解を持つ.

(P_f), に関しては次の結果が成り立つ.

Theorem 2. $C > 0$ について 各 $f \in L^q(R^N)$ が $f \ge 0$ かつ $| f |_q < C$ を満たすならば, (P_f) は少なくとも二つの解を持つ.

以下では, 上記の定理の証明の概略を与える。

2. Preliminaries. We just give a sketch of a proof of Theorem 1 to show that how the singular homology theory works for the proof of existence of positive solutions. We put $H = H^1(R^N)$. Then H is a Hilbert space with norm
$$\| u \| = \left(\int_{R^N} (| \nabla u |^2 + | u |^2)dx\right)^{1/2}.$$

The norm of the dual space $H^{-1}(R^N)$ of H is also denoted by $\|\cdot\|$. B_r stands for the open ball centered at 0 with radius r. We denote by $\langle\cdot,\cdot\rangle$ the pairing between $H^1(R^N)$ and $H^{-1}(R^N)$. For each $r > 1$, the norm of $L^r(R^N)$ is denoted by $|\cdot|_r$. For simplicity, we write $|\cdot|_*$ instead of $|\cdot|_{p+1}$. For $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$. We denote by C_p the minimal constant satisfying
$$| u |_* \le C_p \| u \| \qquad \text{for } u \in H. \tag{2.1}$$

It is easy to check that critical points of I are solutions of (P). It is also obvious that nonzero critical points of I^∞ are solutions of (P) with $g(t) = t^p$ for $t \geq 0$. For each functional F on H and $a \in R$, we set $F_a = \{u \in H : F(u) \leq a\}$. We put

$$M = \{u \in H \backslash \{0\} : \| u \|^2 = \int_{R^N} ug(x,u)dx\}$$
$$M^\infty = \{u \in H \backslash \{0\} : \| u \|^2 = \int_{R^N} u^{p+1}dx\}$$

For the proof of the following two propositions are crucial:

Proposition 2.1. *There exists positive number $d_0 < \widetilde{d_0}$ and ϵ_0 satisfying that if (g1) holds with $d \leq d_0$, then for each $0 < \epsilon < \epsilon_0$,*

$$H_*(I^\infty_{c+\epsilon}, I^\infty_\epsilon) = H_*(I_{c+\epsilon}, I_\epsilon)$$

where $H_*(A,B)$ denotes the singular homology group for a pair (A,B) of topological spaces(cf. Spanier[8]).

Proposition 2.2. *For each positive number $\epsilon < \epsilon_0$,*

$$H_q(I^\infty_{c+\epsilon}, I^\infty_\epsilon) = \begin{cases} 2 & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Here we give a proof for Proposition 2.2.

We set

$$T_{u_\infty}(M^\infty) = \{\lim_{t\to 0}(c(t) - u_\infty)/t : c \in C^1((-1,1); M^\infty) \text{ with } c(0) = u_\infty\},$$

$$\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+ = \{-\tau_x u_\infty : x \in R^N\} \cup \{\tau_x u_\infty : x \in R^N\}$$

and

$$T_{u_\infty}(\mathcal{C}) = \{\lim_{t\to 0}(u_\infty(\cdot + tx) - u_\infty(\cdot))/t : x \in R^N\}.$$

It follows from the definition of M^∞ that the codimension of $T_{u_\infty}(M^\infty)$ in H is one. It is also obvious that $\dim T_{u_\infty}(\mathcal{C}) = N$. We denote by $\widetilde{H}$ the subspace such that $H = \widetilde{H} \oplus T_{u_\infty}(\mathcal{C})$. For each $r > 0$, we set $B^0_r = B_r \cap \widetilde{H}$. Here we consider the linealized equation

$$(L) \qquad -\Delta u + u - h(x)u = \mu u, \qquad u \in H, \mu \in R,$$

where $h(x) = p \mid u_\infty(x) \mid^{p-1}$ for $x \in R^N$. Since $-\Delta$ is positive definite and $h(x)I$ is compact, we find by Freidrich's theory that the negative spectrums of $A = -\Delta - h(x)I$ are finite and each eigenspace corresponding to a negative eigenvalue is finite dimensional. Then each eigenspace corresponding to a nonpositive eigenvalue of $L = -\Delta + I - h(x)I$ is finite dimensional. Then there exists $c_0 > 0$ and a decomposition $H = H_- \oplus H_0 \oplus H_+$ such that $H_0 = ker(L)$ and L is positive(negative) definite on $H_+(H_-)$ with

$$\langle Lv, v\rangle \geq c_0 \parallel v \parallel^2 (\leq -c_0 \parallel v \parallel^2) \qquad \text{for } v \in H_+(H_-).$$

Since each $u \in \mathcal{C}$ is a solution of problem (P_∞), we can see that $T_{u_\infty}(\mathcal{C}) \subset H_0$.

Lemma 2.3. $dim H_- = 1$.

Proof. Since I^∞ attains its minimal on M^∞ at u_∞, we have that $T_{u_\infty}(M^\infty) \subset H_+ \oplus H_0$. Then since the codimension of M^∞ is one, we find that $\dim H_- \leq 1$. On the other hand, we have

$$\begin{aligned} \langle Lu_\infty, u_\infty\rangle &= \int_{R^N} (\mid \nabla u_\infty \mid^2 + \mid u_\infty \mid^2 - p \mid u_\infty \mid^{p+1}) dx \\ &< \int_{R^N} (\mid \nabla u_\infty \mid^2 + \mid u_\infty \mid^2 - \mid u_\infty \mid^{p+1}) dx = 0. \end{aligned} \tag{2.2}$$

Then we have that $\dim H_- \geq 1$. This completes the proof. ∎

In the following we denote by φ an element of H_- with $\parallel \varphi \parallel = 1$. Here we note that since $h \in C^\infty(R^N)$, each solution u of (L) is in $C^1(R^N)$. It then follows that if u has the form

$$u(r, \theta) = \psi(r)\xi(\theta_1, \cdots, \theta_{n-1}), \qquad \text{with } \xi \not\equiv \text{const.},$$

in spherical coordinate, ψ satisfies that $\psi(0) = 0$.

We denote by H_r the set of all radial functions in H and by (L_r) the problem (L) restricted to H_r. Then, in spherical coordinates, the problem (L_r) with $\mu > 0$ is reduced to

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + (h-1)\psi = -\mu\psi(r), \qquad r > 0, \psi \in C_r, \tag{2.3}$$

$$\frac{d\psi(r)}{dr}(0) = 0, \tag{2.4}$$

where $C_r = \{\psi \in C[0, \infty) : \lim_{r\to\infty} \psi(r) = 0\}$.

We next consider nonradial solutions of (L). In case of nonradial functions, the problem (L) is deduced to

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h-1) - \frac{\alpha_k}{r^2})\psi(r) = -\mu\psi(r), \qquad r > 0, \tag{2.5}$$

$$\psi(0) = 0 \tag{2.6}$$

where $\psi \in H_r$, $\alpha_k = k(k+n-1)$, $k = 1, 2, \cdots$. Note that α_k are the eigenvalues of Laplacian $-\Delta$ on S^{n-1}, the unit sphere, and the dimension of the eigenspace S_k associate with α_k is

$$\rho_k = \binom{k+n-2}{k} \frac{n+2k-2}{n+k-2}.$$

That is there exists smooth functions $\{\varphi_{k,i} : i = 1, \cdots, \rho_k\}$ defined on S^{n-1} such that $S_k = span\{\varphi_{k,1}, \cdots, \varphi_{k,\rho_k}\}$, and the functions $u = \psi(r)\varphi_{k,i}(\theta)$ are the solutions of (L).

By using (2.5) and (2.6) , we can see

Lemma 2.4. *$dimH_0 \leq N+1$.*

Here we recall that H has a decomposition $H = \widetilde{H} \oplus T_{u_\infty}(\mathcal{C})$ and then $H = \tau_x\widetilde{H} \oplus \tau_x T_{u_\infty}(\mathcal{C})$ for each $x \in R^N$. Then since $\mathcal{C}_\pm$ are smooth $N-$manifolds, we have that there exists $r_0 > 0$ such that

$$\tau_x((-1)^i u_\infty + B^0_{r_0}) \cap \tau_y(u_\infty + B^0_{r_0}) = \phi \tag{2.7}$$

for all $x, y \in R^N$ with $x \neq y$, and $i = 0, 1$. Here we consider a restriction $I^\infty \mid_{u_\infty + \widetilde{H}}$ of I^∞ on $u_\infty + \widetilde{H}$. Then from Lemma 3.2 and Lemma 3.3, we have by Gromoll-Meyer theory[3] that there exists subspaces H_1 $H_{2,1}$, $H_{2,2}$ of $\widetilde{H}$, a positive number $r_1 < r_0$, a mapping $\beta \in C^1((H_{2,2} \cap B^0_{r_1}), R)$ and a homeomorphism $\psi : u_\infty + B^0_{r_1} \to u_\infty + \widetilde{H}$ such that $\widetilde{H} = H_1 \oplus H_{2,1} \oplus H_{2,2}$ and

$$I^\infty \mid_{u_\infty + \widetilde{H}} (\psi(u)) = c - \| u_1 \|^2 + \| u_{2,1} \|^2 + \beta(u_{2,2}) \tag{2.8}$$

for each $u \in u_\infty + B^0_{r_1}$ with $u = u_\infty + u_1 + u_{2,1} + u_{2,2}$, $u_1 \in H_1$, $u_{2,i} \in H_{2,i}$, $i = 1, 2$. It follows from Lemma 2.3 that $H_{2,2}$ is one dimensional. Noting that $T_{u_\infty}(M) \subset H_0 \oplus H_+$ and u_∞ is the minimal point of I^∞ on M, we have by choosing r_1 sufficiently small that $\beta(t\varphi_2)$ is strictly increasing as $\mid t \mid$ increases in $[-r_1, r_1]$, where $\varphi_2 \in H_{2,2}$ with $\| \varphi_2 \| = 1$.

Since I^∞ is even, it is obvious that I^∞ has the form (2.8) on $-(u_\infty + B^0_{r_1})$. We also note that for each $x \in R^N$, (2.8) holds for each $u \in \tau_x(u_\infty + B^0_{r_0})$ with ψ replaced by $\tau_{-x} \circ \psi$.

Proof of Proposition 2.2. By the deformation property(cf. theorem 1.2 of Chang[3]) and the homotopy invariance of the homology groups, we have

$$H_q(I^\infty_{c+\epsilon}, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_c, I^\infty_{c-\epsilon}), \text{ and}$$

$$H_q(I^\infty_c \backslash \mathcal{C}, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_{c-\epsilon}, I^\infty_{c-\epsilon}) \cong 0.$$

From the exactness of the singular homology groups ,

$$H_q(I^\infty_c \backslash \mathcal{C}, I_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_c \backslash \mathcal{C}) \to H_{q-1}(I^\infty_c \backslash \mathcal{C}, I^\infty_{c-\epsilon}) \to \cdots$$

we find

$$0 \to H_q(I_c^\infty, I_{c-\epsilon}^\infty) \to H_q(I_c^\infty, I_c^\infty \backslash \mathcal{C}) \to 0.$$

That is

$$H_q(I_c^\infty, I_{c-\epsilon}^\infty) \cong H_q(I_c^\infty, I_c^\infty \backslash \mathcal{C}).$$

Noting that $\cup\{\tau_x(\pm u_\infty + B_{r_1}^0) : x \in R^N\}$ are disjoint open neighborhoods of $\mathcal{C}_\pm$ respectively, and that I^∞ is invariant under the translations τ_x, we find from the excision property and (2.8) that

$$\begin{aligned}
&H_*(I_{c+\epsilon}^\infty, I_\epsilon^\infty) \\
&\cong H_*(I_c^\infty, I_c^\infty \backslash \mathcal{C}) \\
&\cong H_*(I_c^\infty \cap (\cup_{i=\pm 1} \cup_x \tau_x(iu_\infty + B_{r_1}^0)), \\
&\qquad\qquad I_c^\infty \cap (\cup_{i=\pm 1} \cup_x \tau_x(iu_\infty + B_{r_1}^0) \backslash \mathcal{C})) \\
&\cong H_*(u_\infty + B_{r_1}^1, (u_\infty + B_{r_1}^1) \backslash \{u_\infty\}) \\
&\qquad \oplus H_*(-u_\infty + B_{r_1}^1, (-u_\infty + B_{r_1}^1) \backslash \{u_\infty\}) \\
&\cong H_*([0,1], \{0,1\}) \oplus H_*([0,1], \{0,1\}).
\end{aligned}$$

This completes the proof. ∎

3. Proof of Theorem 1. We next consider a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying the following conditions:

(1) $U \cap (-U) = \phi$;

(2) $\{\tau_x u_\infty :|\, x \,|\geq r\} \subset intK$ for some $r > 0$;

(3) $cl(I_{c+\epsilon} \cap K) \subset int(I_{c+\epsilon} \cap U)$;

(4) $H_{N-1}(I_{c+\epsilon} \cap U) = 1, \quad H_1(I_{c+\epsilon} \cap U) = 0$;

(5) I_ϵ is a strong deformation retract of $I_{c+\epsilon} \backslash (K \cup (-K))$;

(6) $H_{N-1}((I_{c+\epsilon} \cap U) \backslash K) = 2$ or $H_0((I_{c+\epsilon} \cap U) \backslash K) \geq 2$ *holds.*

Proposition 3.1. *There exists a triple $(U, K, \epsilon) \subset H \times H \times R^+$ which satisfies (1) - (6).*

We omit the proof of Proposition 3.1.

Lemma 3.2. *Suppose that there exist a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying (1)-(6). Suppose in addition that $H_{N-1}((I_{c+\epsilon} \cap U) \backslash K) \geq 2$. Then $H_N(I_{c+\epsilon}, I_\epsilon) \geq 2$.*

Proof. We put $\widetilde{K} = K \cup (-K)$. Since I_ϵ is a strong deformation retract of $I_{c+\epsilon} \backslash \widetilde{K}$, we find that

$$H_q(I_{c+\epsilon} \backslash \widetilde{K}, I_\epsilon) \cong H_q(I_\epsilon, I_\epsilon) \cong 0.$$

Then from the exactness of the singular homology groups of the triple $(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}, I_\epsilon)$ we have

$$0 \to H_q(I_{c+\epsilon}, I_\epsilon) \to H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}) \to 0.$$

That is

$$H_q(I_{c+\epsilon}, I_\epsilon) \cong H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}).$$

From (1), we find

$$H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}) \cong H_q(W, W\backslash K) \oplus H_q(-W, (-W)\backslash(-K))$$

where $W = I_{c+\epsilon} \cap U$. Then since $H_{N-1}(W\backslash K) \geq 2$, we have from (4) and the exactness of the sequence

$$\to H_q(W, W\backslash K) \to H_{q-1}(W\backslash K) \to H_{q-1}(W) \to H_{q-1}(W, W\backslash K) \to \tag{3.1}$$

with $q = N$ that $H_N(I_{c+\epsilon}, I_\epsilon) \cong H_N(W, W\backslash K) \oplus H_N(W, W\backslash K) \geq 2$. ∎

Lemma 3.3. *Suppose that* $(U, K, \epsilon) \subset H \times H \times R^+$ *satisfies (1) - (6). Suppose in addition that* $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U)\backslash K) = 1$. *Then* $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ *or* $H_0(I_{c+\epsilon}, I_\epsilon) = 2$ *holds.*

We can now prove Theorem 1.

Proof of Theorem. Let (U, K, ϵ) be the triple constructed above. We have by Proposition 2.1 and Proposition 2.2 that $H_1(I_{c+\epsilon}, I_\epsilon) = 2$ and $H_q(I_{c+\epsilon}, I_\epsilon) = 0$ for $q \neq 1$. Now suppose that $(I_{c+\epsilon} \cap U)\backslash K$ is disconnected. Then since $H_0((I_{c+\epsilon} \cap U)\backslash K) \geq 2$, we find by Lemma 3.2 that $H_N(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. On the other hand, if $U\backslash K$ is connected, then $H_0(U\backslash K) = 1$. Then by Lemma 3.3, we have $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ or $H_0(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. Thus we obtain that there exists a positive solution of (P). ∎

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