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Integral Functional and Euler-Lagrange Inclusion

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Abstract

Let X and Y be separable Banach spaces. A control problem

$$\begin{array}{ll} \text{minimize} & J(x,u) \doteq \int_a^b g(t,x(t),\dot{x}(t),u(t))\,dt \\ \text{subject to} & x(t) \in F(t,x(t),u(t)), \\ & u(t) \in U(t) \subset Y \end{array}$$

is reduced to the problem

$$(P) \qquad \begin{cases} \text{minimize} & J(x,u) = \int_a^b L_t(x,\dot{x}))\,dt \\ \text{subject to} & x(t) \in A_X^p, \quad 1 \le p \le \infty \ \text{ with} \\ & L_t(x,\dot{x}) = L(t,x(t),\dot{x}) = \inf_{u(t)\in U(t)} g(t,x(t),\dot{x}(t),u(t)). \end{cases}$$

In this note, we establish the generalized Euler-Lagrange inclusion for a non-convex, non-locally Lipschitz Lagrangian $L_t(\cdot,\cdot)$ in (P) to be that for any solution x of (P), there exists an absolutely continuous function $\alpha : [a,b] \to X^*$, the separable dual of X, such that

$$(\dot{\alpha}(t), \alpha(t)) \in \partial L_t(x,\dot{x}) \quad \text{for a.a. } t \in [a,b].$$

If $L_t(\cdot,\cdot) \in C^1$, then the above differential inclusion reduces to the usual Euler-Lagrange equation

$$\frac{d}{dt} L_{t,\dot{x}}(x,\dot{x}) = L_{t,x}(x,\dot{x}).$$

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1. Introduction

In control theory, we often consider an integral functional including a state variable x and a control function u where x and u are obeying some differential equations or inclusion. While a control u is given in a practical system, we assume that one can solve the state x from the system of differential equations.

The integral functional may be regarded as the cost when the system is functioned. As u varies in a control space, then one will minimize the cost functional under the constraints determined from the system of differential equations. In mathematical interesting, it will find the optimality conditions in which the integrand of the cost functional is defined in various spaces.

In general, a minimization problem of such integral functional is formally formulated as follows.

$$(P_c) \qquad \begin{cases} \text{minimize} & J(x,u) = \int_a^b g(t,x(t),\dot{x}(t),u(t))\,dt \\ \text{subject to} & \dot{x}(t) \in F(t,x(t),u(t)), \\ & u(t) \in U(t) \ \text{ for } t \in [a,b]. \end{cases}$$

Problem (P_c) can be reduced to be an implicit constraint problem (cf, Chen and Lai [4]) as

$$(P) \qquad \begin{cases} \text{minimize} & J(x,u) = \int_a^b L_t(x,\dot{x}))\,dt \\ \text{subject to} & x(t) \in A \subset C^1([a,b],X) \\ & \text{with } L_t(x,\dot{x}) = \inf_{u(t)\in U(t)} g(t,x(t),\dot{x}(t),u(t)), \\ & \text{where } L_t(x,\dot{x}) = L(t,x(t),\dot{x}(t)). \end{cases}$$

In classical variational problem, it is taken $X = R^n$ and

$$A = \{x = C^1([a,b],R^n) | x(a) = \alpha, \quad x(b) = \beta; \quad \alpha,\beta \in R\}.$$

If L has continuous partial derivatives with respect to x and $\dot{x}$, the optimal state x satisfies the Euler-Lagrange equation:

$$\frac{d}{dt} L_{t,\dot{x}}(x,\dot{x}) - L_{t,x}(x,\dot{x}) = 0.$$

The questions arise that how we can relax the space X to a general Banach space, and the space A to a more general function space without the assumption of differentiability on the integrand of (P_c) as well as (P). Early Rockafellar [13] proved the solution of (P_c) satisfies the generalized Euler-Lagrange equation for a convex integrand $g(t,\cdot,\cdot,\cdot) : R^n \times R^n \times R^n \to R$. Clarke [5,7] extended the convexity of g to locally Lipschitzian. Both of them are taken A as a space of absolutely continuous function. Recently, Chen and Lai [1,2] established the Moreau-Rockafeller type theorems for nonconvex, non-locally Lipschitz integrand. Employing these results, we can ask that how about the Euler-Lagrange like equation if the Lagrangian $L_t(\cdot,\cdot)$ in (P) is nonconvex, non- locally Lipschitz, and the constraint space A is replaced by a more general function space. To this end, we let

A_X^p = the space of all absolutely continuous functions, $a : [a,b] \to X$ such that

$$x(t) = x(a) + \int_a^b v(\tau)\, d\tau$$

with

$$\dot{x} = v \in L^p([a,b], X)$$

where X is a Banach space. Further the generalized subgradient $\partial^\uparrow L_t(\cdot,\cdot)$ of the Lagrangian $L_t(x,\dot{x})$ is defined and discussed. In this note, we will get an extended Euler- Lagrange inclusion. Precisely, we will have that if z is an optimal solution of (P) then there exists an absolutely continuous function $\phi : [a,b] \to X^*$, the separable dual space of X, such that

$$(\dot{\varphi}(t), \varphi(t)) \in \partial^\uparrow L_t(z(t), \dot{z}(t)) \text{ for a.a. } t \in [a,b]. \tag{1.1}$$

This extends the classical Euler-Lagrange equation. Actually if $L_t(\cdot,\cdot)$ is differentailable then (1.1) is reduced to the Euler-Lagrange equation:

$$\frac{d}{dt} L_{t,\dot{x}}(x,\dot{x}) = L_{t,x}(x,\dot{x}). \tag{1.2}$$

We would like to explore some basic idea and extend the integral functional in an implicit optimization problem as next section.

2. Preliminaries and Definition

Let X and Y be separable Banach spces, and

$$F : [a,b] \times X \times X \to 2^X \quad \text{a multimapping,}$$
$$g : [a,b] \times X \times X \times Y \to (-\infty, +\infty] \quad \text{integrable on } t \in [a,b].$$

Denote $A^p_X = A^p([a,b], X)$ the space of all X-valued absolutely continuous mappings $x : [a,b] \to X$ such that

$$x(t) = x(a) + \int_a^b v(\tau)\, d\tau$$

with

$$\dot{x} = v \in L^p([a,b], X).$$

We supply the norm form A^p_X by

$$|\,\|\, x \,\|\,| = \|x(t)\|_X + \|\dot{x}\|_p \quad \text{for } x \in A^X_p, \quad 1 \le p \le \infty\,,$$

where $\|x\|_X$ is the norm of X.

Consider an optimal control problem as the form:

$$\begin{aligned} \text{minimize} \quad & J(x,u) = \int_a^b g(t, x(t), x(t), u(t))\, dt \\ \text{subject to} \quad & u(t) \in U(t) \subset Y \quad \text{for a.a. } t \in [a,b] \\ & \text{and } x \in A^p_X, \quad 1 \le p < \infty, \text{ such that} \\ & \dot{x}(t) \in F(t, x(t)u(t))\,, \end{aligned}$$

where $U(t)$ stands for the control space at t.

For a function $f : X \to (-\infty, +\infty)$, the *generalized directional derivative* of f (see Hiriart-Urruty [8, Def. 6], see also Rockafellar [11 §2 and 12 §4]) is defined by

$$f^{\uparrow}(x; v) = \lim_{\epsilon \downarrow 0} \limsup_{\substack{(y,\alpha) \to x_f \\ \lambda \to 0}} \inf_{d \in v + \varepsilon B} \frac{f(y + \lambda d) - \alpha}{\lambda}\,, \tag{2.1}$$

where $(y,\alpha) \to x_f$ means that $(y,\alpha) \in epi f$ such that $(y,\alpha) \to (x, f(x))$, and B is the unit open ball of O in X.

If f is l.s.c. at x then (2.1) becomes

$$f^{\uparrow}(x;v) = \lim_{\epsilon\downarrow 0} \limsup_{\substack{y\to x_f \\ \lambda\to 0}} \inf_{d\in v+\varepsilon B} \frac{f(y+\lambda d)-\alpha}{\lambda}, \tag{2.2}$$

where $y \to x_f$ means that $y \to x$ and $f(y) \to f(x)$. The *Clarke's directional derivative* of f at $x \in X$ in the direction $v \in X$ in the direction $v \in X$ is defined by

$$f^0(x;v) = \limsup_{y\to x\ \lambda\downarrow 0} \frac{f(y+\lambda v)-f(y)}{\lambda}. \tag{2.3}$$

If f is locally Lipschitz at x, then

$$f^{\uparrow}(x;v) = f^0(x;v) \ \text{ for any } v \in X. \tag{2.4}$$

Furhtermore, if f is convex and locally Lipschitz at $x \in X$, then

$$f^{\uparrow}(x;v) = f^0(x;v) = f'(x;v) = \lim_{\lambda\downarrow 0} \frac{f(x+\lambda d)-f(x)}{\lambda}. \tag{2.5}$$

We define the *Rockafellar generalized subgradient* of f at x by

$$\partial^{\uparrow} f(x) = \{z \in X^* |\ \langle z, v\rangle \le f^{\uparrow}(x;v),\ v \in X\}, \tag{2.6}$$

and the *Clarke generalized subgradient* of f at x by

$$\partial^0 f(x) = \{z \in X^* |\ \langle z, v\rangle \le f^0(x;v),\ v \in X\}, \tag{2.7}$$

If f is locally Lipschitz then

$$\partial^{\uparrow} f(x) = \partial^0 f(x) \neq \phi. \tag{2.8}$$

Further, if f is convex at x, then

$$\partial^{\uparrow} f(x) = \partial f(x). \tag{2.9}$$

where $\partial f(x) = \{z \in X^* |\ \langle z, y - x\rangle \leq f(y) - f(x),\ y \in X\}$ is the usual subdifferential of f at x.

It can be shown that if f is finite at $x \in X$, then

$$\partial^{\uparrow} f(x) = \phi \quad \text{if and only if} \quad f^{\uparrow}(x; 0) = -\infty\,.$$

Otherwise $f^{\uparrow}(x;0) = 0$ and $\partial^{\uparrow} f(x) \neq \phi$, it follows that

$$f^{\uparrow}(x; v) = \sup\{\langle z, v\rangle | z \in \partial^{\uparrow} f(x)\} \;\text{ for all } v \in X. \tag{2.10}$$

The following properties are not hard to see

(i) If f is finite at x, then $f^{\uparrow}(x;0) = 0$ and

$$\partial^{\uparrow} f^{\uparrow}(x;0) = \partial f^{\uparrow}(x;0) = \partial^{\uparrow} f(x).$$

(ii) If $f$is continuous differentiable at x, then

$$\partial^{\uparrow} f(x) = \{Df(x)\}.$$

Indeed

(i) , since $v \longrightarrow f^{\uparrow}(x; v)$ is l.s.c. and sublinear on $v \in X$, it is convex. Evidently $f^{\uparrow}(x;0) = 0$, we have

$$\begin{aligned}
\partial^{\uparrow} f(x) &= \{z \in X^* |\ \langle z, v\rangle \leq f^{\uparrow}(x; v) \;\text{ for all } v \in X\} \\
&= \{z \in X^* |\ \langle z, v\rangle \leq f^{\uparrow}(x; v) - f^{\uparrow}(x;0) \;\text{ for all } v \in X\} \\
&= \partial f^{\uparrow}(x;0) \\
&= \partial^{\uparrow} f^{\uparrow}(x;0)\,.
\end{aligned}$$

(ii) , if f has a continuous derivative at x, it is locally Lipschitz at x, then

$$f^{\uparrow}(x; v) = f^0(x; v) = \langle Df(x), v\rangle \;\text{ for all } v \in X.$$

This implies that $\partial^{\uparrow} f(x) = \{Df(x)\}$.

3. Lagrange Problem

Recall a control problem on A^p_X:

$$(P_c)\qquad \begin{cases} \text{minimize} & J(x;u) = \int_a^b g(t,x(t),\dot{x}(t),u(t)))\,dt \\ \text{subject to} & x(t) \in A^p_X, \quad 1 \le p \le \infty \\ & \dot{x}(t) \in F(t,x(t),u(t)) \\ & u(t) \in U(t) \subset Y \;\; \text{for a.a. } t \in [a,b], \end{cases}$$

where Y is another Banach space, $U(t)$ is a convex compact subset of Y, and $F(t,\cdot,\cdot) : X \times X \to 2^X$ is a multimapping.

Problem (P_c) can be reformulated by an unsonstrained problem as

$$(P_1)\qquad \begin{cases} \text{minimize} & \int_a^b h(t,x(t),\dot{x}(t),u(t))\,dt \\ \text{subject to} & x \in A^p_X, \quad 1 \le p \le \infty \end{cases}$$

where

$$\begin{aligned} h(t,x(t),\dot{x}(t),u(t)) &= (t,x(t),\dot{x}(t),u(t)) + I_{GrF(t,\cdot,\cdot)}(x(t),\dot{x}(t),u(t)) \\ &\quad + I_{U(t)}(u(t)), \\ GrF(t,\cdot,\cdot) &= \{(x,v,u) \in X \times X \times Y \,|\, v \in F(t,x,u)\} \\ &\quad \text{the graph of } F(t,\cdot,\cdot), \end{aligned}$$

and $I_K(\cdot)$ is the indicator function of the set K.

Recall $L_t(x,x) = L(t,x(t),\dot{x}(t))$. Assume that the infmum

$$L_t(x,\dot{x}) = \inf_{u(t)\in U(t)} h(t,x(t),\dot{x}(t),u(t))$$

is attained. Then (P_1) is reduced to problem:

$$(P)\qquad \begin{cases} \text{minimize} & J(x) = F(x,\dot{x}) = \int_a^b L_t(x,\dot{x})\,dt \\ \text{subject to} & x \in A^p_X\,. \end{cases}$$

This problem (P) is called the *generalized Lagrange problem.*

If the integrand $L_t(\cdot,\cdot)$ in (P) is non-convex, non-locally Lipschitz but pseudo locally Lipschitz, then under some natural conditions, the generalized subdifferential operator $\partial^{\uparrow}$ acting on the integral functional $F(x,\dot{x})$

is satisfying the Moreau Rockafellar type theorem:

$$\partial^{\uparrow} F(x,\dot{x}) \subset \int_a^b \partial^{\uparrow} L_t(x,\dot{x})\,dt$$

where $x \in A_X^p$, $1 \leq p < \infty$, (see Chen and Lai [2, Theorem 4.1], cf also [1, Theorem 3.1]). Employing this theorem, we can established the optimality condition for a local solution of problem (P).

For convenience, in (P) we say that the integrand $f_t : X \to R \cup \{+\infty\}$ is pseudo locally Lipschitz at $z(t) \in X$ in the direction $v \in X$ if there exist a neighborhood W in the neighborhood system of v and functions

$$k_l \in L^q([a,b], R^+), \quad k_2 \in L^p([a,b], R^+), \quad \frac{1}{p} + \frac{1}{q} = 1$$

such that $\lambda \in (0, \bar{\lambda})$ and

$$\frac{f_t(x+\lambda w) - f_t(x)}{\lambda} \leq k_1(t)w + k_2(t)$$

for all $w \in W$ and $x \in \{x \in X |\ \|x - z(t)\|_X \leq \varepsilon\}$, for some $\bar{\lambda} > 0$ and $\varepsilon > 0$.

Now we can state some results for the optimality condition of problem (P).

4. Euler-Lagrange Inclusion

Theorem 1. *Suppose that $z \in A_X^p$, $1 \leq p < \infty$, is a local solution of problem (P) and assume that*

(i) for each $(s,v) \in X \times X$, $L_t(s,v)$ is measurable in $t \in [a,b]$ and for each t, $L_t(\cdot,\cdot)$ is continuous on $X \times X$,

(ii) $L_t(\cdot,\cdot)$ is pseudo local Lipschitz at $(z(t),\dot{z}(t)) \in X \times X$ in any direction $(s,v) \in IntL_t^{\uparrow}(z,\dot{z};\cdot,\cdot)$ and

$$\cap_{t\in[a,b)} IntDomL_t^{\uparrow}(z,\dot{z};\cdot,\cdot) \neq \phi,$$

(iii) $L_t^{\uparrow}(z,\dot z;\cdot,\cdot)=0$ *for all* $t\in[a,b]$,

(iv) either the normal cone $N_{Dom\mathcal{L}}(0,0)=\{0,0\}$ *or* $Dom\mathcal{L}=DomL_t^{\uparrow}(z,\dot z;\cdot,\cdot)$ *has positive measure for* t *in some subset of* $[a,b]$.

Then there exists an absolutely continuous function $\alpha:[a,b]\to X^*$ *such that*

$$(\dot\alpha(t),\alpha(t))\in\partial^{\uparrow}L_t(z,\dot z)\ \text{ for a.a. } t\in[a,b].$$

Proof. Applying Chen and Lai [1, Theorem 3.1], the conditions (i)-(iv) would imply

$$\partial^{\uparrow}F(z,\dot z)\subset\int_a^b\partial^{\uparrow}L_t(x,\dot x)dt.$$

It follows that for any $w=(w_1,w_2)\in\partial^{\uparrow}(z,\dot z)$, there exist absolutely continuous functions α and β such that

(1) $(\beta(t),\alpha(t))\in\partial^{\uparrow}L_t(z,\dot z)$ for a.a. $t\in[a,b]$,
and for any $s\in A_X^p$ and $v=\dot s\in L^p([a,b],x)$,

(2) $\langle(w_1,w_2),(s,v)\rangle=\int_a^b\{\langle\beta(t),s(t)\rangle+\langle\alpha(t),v(t)\rangle\}\,dt.$

As z is a local minimum of (P), F attains the local minimum at $(z,\dot z)$ and so

$$(0,0)\in\partial^{\uparrow}F(z,\dot z).$$

Taking $(w_1,w_2)=(0,0)$, (2) turns to

$$\int_a^b\langle\beta(t),s(t)\rangle dt=-\int_a^b\langle\alpha(t),v(t)\rangle\,dt. \tag{4.1}$$

As $s\in A_X^p$, $v=\dot s\in L^p([a,b],X)$, we have

$$s(t)=s(a)+\int_a^t v(\tau)\,d\tau.$$

Choosing

$$v(t)=u\chi_{[a,\tau]}(t),\quad u\in X$$

and so

$$\begin{aligned} s(t) &= \int_a^t u\chi_{[a,\tau]}(l)\,dl + s(a) \\ &= \begin{cases} u(t-a)+s(a) & \text{if } t<\tau \\ u(\tau-a)+s(a) & \text{if } t\geq\tau. \end{cases} \end{aligned}$$

Substituting such s and v in (3), we have

$$\int_a^\tau \langle \beta(t), u(t-)+s(a)\rangle\,dt + \int_\tau^b \langle \beta(t), u(\tau-a)+s(a)\rangle\,dt$$
$$= -\int_a^\tau \langle \alpha(t), u\rangle\,dt.$$

Differentiating the above identity with respect to τ, we obtain

$$\int_\tau^b \langle \beta(t), u\rangle\,dt = \langle -\alpha(\tau), u\rangle\,.$$

It follows that

$$-\alpha(\tau) = \int_\tau^b \beta(t)\,dt \quad \text{for } \tau\in[a,b].$$

Hence $\beta(t)=\dot\alpha(t)$ for a.a. $t\in[a,b]$, and (1) follows that

$$(\dot\alpha(t), \alpha(t)) \in \partial^\uparrow L_t(z,\dot z)\,.$$

□

We say that a function $f_t(\cdot)$ is quasi locally Lipschitz at $z\in E\subset L^p([a,b],X)$ if there is a function $k\in L^q([a,b],R^+)$ such that for $t\in[a,b]$,

$$|f_t(s_1)-f_t(s_2)| \leq k(t)||s_1-s_2||_X, \text{ for } s_1,s_2\in N_{z_0}$$

where $N_{z_0}=\{x\in X \mid \; ||x-z(t)||_X<\varepsilon_0\}$ for some $\varepsilon_0>0$.

In Theorem 1, if the pseudo locally Lipschitz of L is replaced by the quasi locally Lipschitz, then we have

Theorem 2. *If z is an optimal solution of (P), and assume that $L_t(z,\dot z)$ is measurable in t and satisfies the quasi locally Lipschitz at $(z,\dot z)$:*

$$|L_t(s_1,v_1)-L_t(s_2,v_2)| \leq k(t)||s_1-s_2, v_1-v_2)||$$

for all (s_1, v_1), $(s_2, v_2) \in (z(t), \dot{z}(t)) + \varepsilon B_{X \times X}$, *the* ε*-neighborhood of* $(z, \dot{z})$ *where* B *is an open ball and* $\varepsilon > 0$ *is arbitary. Then there is an absolutely continuous function* $\alpha \in A_X^1$ *such that*

$$(\dot{\alpha}(t), \alpha(t)) \in \partial^0 L_t(z, \dot{z}) \text{ for a.a. } t \in [a, b]$$

where $\partial^0 L_t(z, \dot{z})$ *is the Clarke generalized subgradient of* $L_t(\cdot, \cdot)$ *at* $(z, \dot{z})$.

If $L_t(\cdot, \cdot) \in C^1(X \times X)$, then Theorems 1 and 2 are reduced to the usual Euler-Lagrange equation which we state as follows.

Theorem 3. *Let* z *be a solution of* (P) *and let* $L_t(\cdot, \cdot)$ *be continuous differentiable with respect to* $(s, v) \in X \times X$*. Then*

$$\frac{d}{dt} L_{t,\dot{x}}(z, \dot{z}).$$

<u>**Proof.**</u> If $L_t(\cdot, \cdot) \in C^1(X \times X)$, then

$$L_t^{\uparrow}(z, \dot{z}; v_1, v_2) = L_t^0(z, \dot{z}; v_1, v_2) = \langle DL_t(z, \dot{z}), (v_1, v_2) \rangle$$

for all $(v_1, v_2) \in X \times X$, where $DL_t(z, \dot{z})$ denotes the Frechet derivative of L_t at $(z, \dot{z})$. Hence

$$\partial^{\uparrow} L_t(z, \dot{z}) = \partial^0 L_t(z, \dot{z}) = \{DL_t(z, \dot{z}\} = \{L_{t,x}(z, \dot{z}), L_{t,\dot{x}}(z, \dot{z})\}.$$

By Theorems 1 and 2, we obtain

$$(\dot{\alpha}, \alpha) = (L_{t,x}(z, \dot{z}), L_{t,\dot{x}}(z, \dot{z}))\,.$$

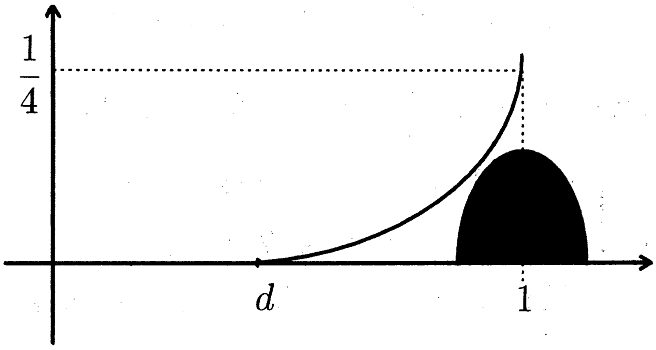
This shows that

$$\frac{d}{dt} L_{t,\dot{x}}(z, \dot{z}) = L_{t,x}(z, \dot{z})\,.$$

As an example to solve the optimal solution from the differential inclusion like in Theorem 2. We give a practical problem as the airplane tak-off or landing in which one will minimize the loss of energy.

Example. Let $a > 0$, $b > 0$ and consider the problem :

$$\text{minimize} \quad J(x) = \int_0^1 [a|x(t)| + b\dot{x}(t)^2]\, dt$$
$$\text{subject to} \quad x \in AC([0,1], R) \quad \text{and} \quad x(0) = 0, \quad x(1) = 1/4.$$



The integrand of $J(x)$ is not smooth, one will minimize $J(x)$ with the given constraint.

If $b = \frac{1}{2}m$, $a = mg$ where m denotes the mass of body and g the gravity, then

$mgx(t)$ is the potential energy obtained and

$\frac{1}{2}m\dot{x}(t)^2$ is the kinetic energy lost.

The problem will minimize the loss of energy when a plane will take off over the time interval $[0, 1]$.

Solution. If z is an optimal trajecting, we denote a neighborhood of $(z, \dot{z})$ by

$$N = \{(s, v) : |s - z(t)| < 1, \ |v - \dot{z}(t)| < 1\}$$

then the Lagrangian

$$L(t, s, v) = a|s| + bv^2$$

is convex and locally Lipschitz. Indeed for any (s_1, v_1), (s_2, v_2) in N,

$$|L(t, s_1, v_1) - L(t, s_2, v_2)|$$
$$\leq a|s_1 - s_2| + b|v_1 + v_2||v_1 - v_2|$$

$$\leq a|s_1 - s_2| + 2b(1 + \dot{z}(t))|v_1 - v_2|$$
$$\leq [a + 2b(1 + \dot{z}(t))]\|(s_1 - s_2, v_1 - v_2)\|.$$

Here $k(t) = a + 2b(1 + \dot{z}(t)) \in L^1[0,1]$. By Theorem 2, there exists an absolutely continuous function $\alpha : [0,1] \to R$ such that

$$(\dot{\alpha}(t), \alpha(t)) \in \partial L(t, z(t), \dot{z}(t)).$$

Since L_x and $L_{\dot{x}}$ exist except $z(t) = 0$, it follows that

$$(\dot{\alpha}(t), \alpha(t)) \in \begin{cases} \{(L_x, L_{\dot{x}})\} = \{a, 2b\dot{z})\} & \text{if } z(t) > 0 \\ \{(L_x, L_{\dot{x}})\} = \{-a, 2b\dot{z})\} & \text{if } z(t) < 0 \\ \partial L(t, z, \dot{z}) = \{(s, 2b\dot{z}) | s \in [-a, a]\} & \text{if } z(t) = 0. \end{cases}$$

Hence

(i) if $z(t) > 0$ then $\dot{\alpha}(t) = a$ and $\alpha(t) = 2b\dot{z}$
$\Rightarrow \ddot{z} = a/2b$;

(ii) if $z(t) < 0$ then $\dot{\alpha}(t) = -a$ and $\alpha(t) = 2b\dot{z}$
$\Rightarrow \ddot{z} = -a/2b$;

(iii) if $z(t) = 0$ then $\dot{\alpha}(t) \in [-a, a]$ and $\alpha(t) = 2b\dot{z}$
$\Rightarrow \ddot{z} = [-a/2b,\ a/2b]$.

The case (ii) does not happen since $z(t) \geq 0$. Thus we may assume that there exists $d \in [0, 1)$ such that $z(t) = 0$ for $0 \leq t \leq d$ and $z(t) > 0$ for $d < t \leq 1$.

1. If $d = 0$, we solve the equation :

$$\begin{cases} \ddot{z}(t) = a/2b, & 0 \leq t \leq 1 \\ z(0) = 0 \text{ and } z(1) = 1/4 \end{cases}$$

and get the solution

$$\begin{cases} z(t) = (a/4b)t^2 + (1/4 - a/4b)\,t \\ z(t) \geq 0 \text{ only if } a < b. \end{cases}$$

2. If $0 < d < 1$, $z(t) = 0$ for $0 \le t \le d$, then we solve the equation :

$$\begin{cases} \ddot{z}(t) = a/2b \quad \text{for} \quad d \le t \le 1 \\ z(d) = 0 \quad \text{and} \quad z(1) = 1/4 \end{cases}$$

and get the solution

$$z(t) = (a/4b)(t-d)^2 \quad \text{where} \quad 0 < d = 1 - \sqrt{\frac{b}{a}} \quad \text{with} \quad b < a\,.$$

Consequently,

1. If $b \ge a$, the optimal solution of (P) is

$$z(t) = \frac{a}{4b}t^2 + (\frac{1}{4} - \frac{a}{4b})t$$

with optimal value

$$J(z) = \frac{a}{8} + \frac{b}{16} - \frac{a^2}{48b}\,.$$

2. If $b < a$, the optimal solution of (P) is

$$z(t) = \frac{a}{4b}(t-d)^2, \quad 0 < d = 1 - \sqrt{\frac{b}{a}}$$

with optimal value

$$J(z) = \frac{1}{6}\sqrt{ab}.$$

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