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ASYMPTOTIC MEANS OF BOUNDED SEQUENCES IN BANACH SPACES

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1. Introduction. Always X, Y are Banach spaces and $\mathscr{U}$, $\mathscr{V}$ are non-principal ultrafilters on $\mathbb{N}$, the set of natural numbers. For a pair of a norm-bounded sequence (x_n) in X and a non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$, denote $\tau_X(x) = \lim_{n,\mathscr{U}} \|x_n - x\|$ for $x \in X$. In other words, $\tau_X(x) = \int_{\mathbb{N}} \|x_n - x\| \lambda(dn)$ for $x \in X$, where λ is a purely finitely additive 0-1 measure on $2^{\mathbb{N}}$ defined by $\lambda(A) = 1$ if $A \in \mathscr{U}$, $\lambda(A) = 0$ otherwise. Krivine and Maurey [5] called such a functional a type on X. We here call τ_X an *asymptotic mean* of (x_n) along $\mathscr{U}$ on X. Let Y be a closed linear subspace of a Banach space X and (x_n) a bounded sequence in Y. We call the set $M(x_n, \mathscr{U}, Y) = \{a \in Y : \tau_Y(y) \geq \tau_Y(a)$ for all $y \in Y\}$ an *asymptotic center* of (x_n) along $\mathscr{U}$ with respect to Y. If Y is separable, then the set $M(x_n, \mathscr{U}, Y)$ coincides with the asymptotic center in the sense of Lim [6] of a subsequence (x_{n_k}) of (x_n) with respect to Y. For a bounded sequence (x_n) in X, we set $\omega(x_n) = \bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\{x_k : k \geq n\}$. For any relatively weakly compact sequence (x_n) in X, $w\text{-}\lim_{n,\mathscr{U}} x_n$ denotes the weak-limit of (x_n) along a non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$. Similarly, for any bounded sequence (f_n) in the dual space X^*, $w^*\text{-}\lim_{n,\mathscr{U}} f_n$ denotes the weak*-limit of (f_n) along a non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$.

The duality mapping of a Banach space is a possibly multi-valued mapping F_X from X into its dual space X^* which assigns to each $x \in X$ a subset of X^* defined by

$$F_X(x) = \{f \in X^* : f(x) = \|x\|^2 = \|f\|^2\}.$$

A Banach space X is said to be *uniformly Gâteaux differentiable* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $y \in S_X$ uniformly as x varies over S_X, where $S_X = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be *uniformly convex* if there exists a function δ such that $0 < \delta(\epsilon)$ if $0 < \epsilon \le 2$ and such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \epsilon$ then $\left\|\frac{x+y}{2}\right\| \le 1 - \delta(\epsilon)$.

In this note, we shall consider the following three properties in a Banach space X :

Property (I). For every relatively weakly compact sequence (x_n) in X and every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$, $M(x_n, \mathscr{U}, X)$ intersects $\omega(x_n)$.

Property (M). For every relatively weakly compact sequence (x_n) in X and every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$, we have

$$\lim_{n,\mathscr{U}} \|x_n - x\| \ge \lim_{n,\mathscr{U}} \|x_n - a\|, \qquad \text{for all } x \in X,$$

where a is the weak-limit of (x_n) along $\mathscr{U}$. That is, a is a minimizer of the asymptotic mean τ_X defined by $\tau_X(x) = \lim_{n,\mathscr{U}} \|x_n - x\|$, $x \in X$, or $w\text{-}\lim_{n,\mathscr{U}} x_n = a \in M(x_n, \mathscr{U}, X)$

Property (C). For every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$ and every bounded sequence (x_n) with $w\text{-}\lim_{n,\mathscr{U}} x_n = 0$, there exists a sequence (f_n) such that $f_n \in F_X(x_n)$ and the weak*-limit of (f_n) along $\mathscr{U}$ is 0.

In this note, We are concerned with the relations between these three properties.

Implications (C) $\Longrightarrow$ (M) $\Longleftrightarrow$ (I) hold (see Theorem 3). These properties are not isomorphic invariants. In fact, even Hilbert space can be renormed so that it does not have property (I) and so, neither properties (M) or (C). But these all are hereditary, i.e., every closed linear subspace of X has the property whenever the space X does.

2. The spaces $c_0, \ell_p, 1 \leq p < +\infty$. As mentioned above, the following is easily verified :

PROPOSITION 1. *All of properties (I), (M) and (C) are hereditary.*

Next we note the following two facts (see [1]):

(a) If $X = c_0$, then for every relatively weakly compact sequence (x_n) in X and every non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ the following holds :

$$\lim_{n,\mathcal{U}} \|x_n - x\|_\infty = \max(\|x - a\|_\infty, \lim_{n,\mathcal{U}} \|x_n - a\|_\infty), \quad \text{for all } x \in X,$$

where $a = w\text{-}\lim_{n,\mathcal{U}} x_n$.

(b) If $X = \ell_p, 1 \leq p < +\infty$, then for every bounded sequence (x_n) in X and every non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ the following holds :

$$\lim_{n,\mathcal{U}} \|x_n - x\|_p = (\|x - a\|_p^p + \lim_{n,\mathcal{U}} \|x_n - a\|_p^p)^{\frac{1}{p}}, \quad \text{for all } x \in X,$$

where $a = w^*\text{-}\lim_{n,\mathcal{U}} x_n$.

In virtue of (a) and (b) we have the following.

PROPOSITION 2. *The spaces c_0 and ℓ_p, $1 \leq p < +\infty$ have property (M).*

REMARK. The space ℓ_∞ does not have property (I), and hence does not have property (M). In fact, let (e_n) be the usual unit vector basis of ℓ_∞. Then clearly we see that $w\text{-}\lim_{n\to\infty} e_n = 0$ and so $\omega(e_n) = \{0\}$. While it is easily verified that $M(e_n, \mathscr{U}, \ell_\infty) = \{x \in \ell_\infty : x = (\xi_n),\ \|x\|_\infty \leq 1/2,\ \lim_{n,\mathscr{U}} \xi_n = 1/2\}$. Consequently, we have $\omega(e_n) \cap M(e_n, \mathscr{U}, \ell_\infty) = \emptyset$.

Properties (C), (M) and (I) have the following relations.

THEOREM 3. *In any Banach space X, the following implications hold :*

$$(C) \Longrightarrow (M) \Longleftrightarrow (I).$$

PROOF. The implication (M) $\Longrightarrow$ (I) is obvious. To show (C) $\Longrightarrow$ (M), suppose that X has property (C). Let (x_n) be a relatively weakly compact sequence and $\mathscr{U}$ a non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$. Let $a = w^*\text{-}\lim_{n,\mathscr{U}} x_n$. Since $w\text{-}\lim_{n,\mathscr{U}} (x_n - a) = 0$, there is a sequence (f_n) in X^* such that $f_n \in F_X(x_n - a)$ and $w^*\text{-}\lim_{n,\mathscr{U}} f_n = 0$. Then for every $x \in X$ we have

$$\|x_n - x\|^2 - \|x_n - a\|^2$$

$$
\begin{aligned}
&\geq 2(\|x_n - x\|\|x_n - a\| - \|x_n - a\|^2) \\
&\geq 2(f_n(x_n - x) - f_n(x_n - a)) \\
&= 2f_n(a - x).
\end{aligned}
$$

Applying $\lim\limits_{n,\mathscr{U}}$, we get the following inequality :

$$\lim_{n,\mathscr{U}} (\|x_n - x\|^2 - \|x_n - a\|^2) \geq 0.$$

Thus we obtain

$$\lim_{n,\mathscr{U}} \|x_n - x\| \geq \lim_{n,\mathscr{U}} \|x_n - a\|, \qquad \text{for all } x \in X.$$

Finally, we prove (I) $\Longrightarrow$ (M). Suppose (I) holds and let $(x_n)_{n=1}^{\infty}$ be relatively weakly compact and $\mathscr{U}$ an ultrafilter on $\mathbb{N}$. Denote $a = w\text{-}\lim\limits_{n,\mathscr{U}} x_n$ and suppose that $a \notin M(x_n, \mathscr{U}, X)$. We want to obtain a contradiction to (I). Note that $M(x_n, \mathscr{U}, X)$ is a closed, convex set in X and, by assumption (I), it is nonempty. We may find a weak neighbourhood V of a such that its weak closure $\overline{V}$ does not intersect $M(x_n, \mathscr{U}, X)$. Let A be the subset of $\mathbb{N}$ such that

$$A = \{n \in \mathbb{N} \ : x_n \in V\}.$$

Then A is an element of $\mathscr{U}$ as we have assumed that $a = w\text{-}\lim\limits_{n,\mathscr{U}} x_n$. Consider now the subsequence $(x_n)_{n\in A}$ of $(x_n)_{n\in N}$. The ultrafilter $\mathscr{U}$ on $\mathbb{N}$ defines an ultrafilter $\tilde{\mathscr{U}}$ on A.

Note that $M((x_n)_{n\in A}, \tilde{\mathscr{U}}, X) = M((x_n)_{n\in N}, \mathscr{U}, X)$ and that $\omega((x_n)_{n\in A}) \subseteq \overline{V}$ as $\{x_n : n \in A\} \subseteq V$. Hence $M((x_n)_{n\in A}, \tilde{\mathscr{U}}, X) \cap \omega((x_n)_{n\in A}) = \emptyset$, a contradiction to property (I) which completes the proof. $\square$

REMARK. The converse implication (M) $\Longrightarrow$ (C) is not in general valid (see Theorem 15). But if a Banach space X is uniformly Gâteaux differentiable, then (M) $\Longrightarrow$ (C) holds as Theorem 5 shows.

THEOREM 4. *Let $\mathscr{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and (x_n) a sequence in c_0 such that $w\text{-}\lim_{n,\mathscr{U}} x_n = 0$. For each $n \in \mathbb{N}$ take $f_n \in F_{c_0}(x_n)$. Then we have $w^*\text{-}\lim_{n,\mathscr{U}} f_n = 0$. In particular, c_0 has property (C).*

To prove this, we need the following lemma.

LEMMA 5. Let $x = (\xi_k) \in c_0$ and $f = (\eta_k) \in F_{c_0}(x)$. Then the following holds :

$$\{k \in \mathbb{N} \ : \eta_k \neq 0\} \subseteq \{k \in \mathbb{N} \ : |\xi_k| = \|x\|_\infty\}.$$

PROOF. From $f(x) = \|x\|_\infty^2 = \|f\|_1^2$ we see easily that

$$\sum_{k=1}^{\infty} (\|x\|_\infty |\eta_k| - \xi_k \eta_k) = 0.$$

Consequently, we get $\|x\|_\infty |\eta_k| = \xi_k \eta_k$, for all $k \in \mathbb{N}$. □

PROOF OF THEOREM 4. Let $x_n = (\xi_k^{(n)})$ and $f_n = (\eta_k^{(n)})$. Set $A_n = \{k \in \mathbb{N} : \|x\|_\infty = |\xi_k^{(n)}|\}$ and $k_n = \min A_n$. Then we have $\lim_{n,\mathscr{U}} k_n = +\infty$. For, if not, then there is an $N \in \mathbb{N}$ such that $k_n \leq N$ for all $n \in \mathbb{N}$. We may assume without loss of generality that $\lim_{n,\mathscr{U}} \|x_n\|_\infty > c > 0$ for some c, and so there exists $A \in \mathscr{U}$ such that $\|x_n\|_\infty > c$ for all $n \in A$. Set $B_k = \{n \in A : |\xi_k^{(n)}| > c\}$ for each $1 \leq k \leq N$. Then $A = \bigcup_{k=1}^{N} B_k$. Since $A \in \mathscr{U}$, $B_k \in \mathscr{U}$ for some $1 \leq k \leq N$. Thus $|\xi_k^{(n)}| > c$ for all $n \in B_k$ and hence $\lim_{n,\mathscr{U}} |\xi_k^{(n)}| \geq c > 0$, which contradicts that $w\text{-}\lim_{n,\mathscr{U}} x_n = 0$. Consequently, $\lim_{n,\mathscr{U}} k_n = +\infty$. On the other hand, by the previous lemma, since $\{k \in \mathbb{N} : \eta_k^{(n)} \neq 0\} \subseteq A_n$, we have $k_n \leq \min\{k \in \mathbb{N} : \eta_k^{(n)} \neq 0\}$. This means that $\lim_{n,\mathscr{U}} \eta_k^{(n)} = 0$ eventually for each $k \in \mathbb{N}$. Hence we have $w^*\text{-}\lim_{n,\mathscr{U}} f_n = 0$. □

THEOREM 6. *In a uniformly Gâteaux differentiable Banach space, property (M) is equivalent to property (C). In particular, $\ell_p, 1 < p < +\infty$ have property (C).*

PROOF. (C) $\Longrightarrow$ (M) has already proved in Theorem 3. To show the converse, suppose that a Banach space X is uniformly Gâteaux differentiable and has property (M). Let $\mathscr{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and (x_n) a bounded sequence with $w\text{-}\lim_{n,\mathscr{U}} x_n = 0$ in X. Without loss of generality, we may assume that $\lim_{n,\mathscr{U}} \|x_n\| > 0$. Let $\tau_X(x) = \lim_{n,\mathscr{U}} \|x_n - x\|$ for $x \in X$. By assumption, $\tau_X(x) \geq \tau_X(0)$ for all $x \in X$. Since X is uniformly Gâteaux differentiable, the convex function τ_X is also Gâteaux differentiable, and hence the Gâteaux derivative $\tau_X'(0)$ at the origin is 0. For $x \in X$, we have

$$\begin{aligned} 0 &= \langle \tau_X'(0), -x \rangle \\ &= \lim_{t \to 0} \frac{\tau_X(-tx) - \tau_X(0)}{t} \\ &= \lim_{t \to 0} \lim_{n,\mathscr{U}} \frac{\|x_n + tx\| - \|x_n\|}{t} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n,\mathscr{U}} \lim_{t\to 0} \frac{\|x_n + tx\| - \|x_n\|}{t} \\
&= \lim_{n,\mathscr{U}} \frac{f_n(x)}{\|x_n\|} \\
&= \frac{\lim\limits_{n,\mathscr{U}} f_n(x)}{\lim\limits_{n,\mathscr{U}} \|x_n\|},
\end{aligned}
$$

where $f_n = F_X(x_n)$. Thus we get $w^*\text{-}\lim\limits_{n,\mathscr{U}} f_n = 0$. The last assertion is obvious from Proposition 2. □

The proofs of the following proposition is easy.

PROPOSITION 7. *If X is reflexive, then for every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$ and every bounded sequence (x_n), $M(x_n, \mathscr{U}, X) \neq \emptyset$.*

PROPOSITION 8. *If X is a uniformly convex Banach space, then for every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$ and every bounded sequence (x_n), $M(x_n, \mathscr{U}, X)$ is a singleton set.*

In view of Proposition 2 and Proposition 8 we have the following.

PROPOSITION 9. *Let $1 < p < +\infty$. Then for every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$ and every bounded sequence (x_n) in ℓ_p, $M(x_n, \mathscr{U}, \ell_p) \subseteq \omega(x_n)$. In particular, the spaces ℓ_p, $1 < p < +\infty$ have property (I).*

Let (x_n) be a bounded sequence in a closed linear subspace Y of a Banach space X and $\mathscr{U}$ a non-principal ultrafilter on $\mathbb{N}$. We define the set $C(x_n, \mathscr{U}, Y)$ by

$$C(x_n, \mathscr{U}, Y) = \{a \in Y : \exists\, (f_n) \text{ in } Y^* \text{ such that}$$
$$f_n \in F_Y(x_n - a) \text{ and } w^*\text{-}\lim_{n,\mathscr{U}} f_n = 0\}.$$

THEOREM 10. *For every bounded sequence (x_n) in a Banach space X and every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$, $C(x_n, \mathscr{U}, X) \subseteq M(x_n, \mathscr{U}, X)$.*

This is obvious from the proof of Theorem 3. □

COROLLARY 11. If $w\text{-}\lim_{n,\mathscr{U}} x_n = 0$, then for any sequence (f_n) with $f_n \in F_X(x_n)$, $0 \in C(f_n, \mathscr{U}, X^*)$. In particular, $0 \in M(f_n, \mathscr{U}, X^*)$.

THEOREM 12. *If X is a uniformly Gâteaux differentiable Banach, then for every bounded sequence (x_n) in X and every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$, $C(x_n, \mathscr{U}, X) = M(x_n, \mathscr{U}, X)$.*

PROOF. The inclusion $C(x_n, \mathscr{U}, X) \subseteq M(x_n, \mathscr{U}, X)$ has already been given in Theorem 9. To show the converse, let $a \in M(x_n, \mathscr{U}, X)$. Define $\tau_X(x) = \lim_{n,\mathscr{U}} \|x_n - x\|, x \in X$. Without loss of generality we may assume that $\tau_X(a) > 0$. In the same way as in the proof of Theorem 6, since X is uniformly Gâteaux differentiable, we see that the convex function τ_X is Gâteaux differentiable, and so the Gâteaux derivative $\tau_X'(a)$ of τ_X at a is 0. Hence we have

$$0 = \langle \tau_X'(a), -x \rangle = \frac{\lim_{n,\mathscr{U}} f_n(x)}{\tau_X(a)}, \quad \text{for all } x \in X,$$

where $f_n = F_X(x_n - a)$. Thus we have $w^*\text{-}\lim_{n,\mathscr{U}} f_n = 0$ and hence $a \in C(x_n, \mathscr{U}, X)$. □

LEMMA 13. *Let (x_n) be a bounded sequence in a Banach space X. Then (x_n) converges weakly to an $a \in X$ if and only if for every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$, $a = w\text{-}\lim_{n,\mathscr{U}} x_n$.*

PROOF. The necessity is obvious. To show the sufficiency, assume that (x_n) does not converge weakly to a. Then there exist a subsequence (x_{n_k}) of (x_n) and a weakly open subset U containing a such that $x_{n_k} \in U^c$ for every $k \in \mathbb{N}$. Let $\mathscr{U}$ be non-principal ultrafilter on $\mathbb{N}$ containing the set $A = \{n_k\}$. By hypothesis, $a = w\text{-}\lim_{n,\mathscr{U}} x_n$ and so, for some infinite subset $B \subseteq A$ with $B \in \mathscr{U}$, it follows that $x_n \in U$ for every $n \in B$, which is a contradiction. □

THEOREM 14. *Let X be a uniformly Gâteaux differentiable and uniformly convex Banach space with property (M). If (x_n) is a bounded sequence in X such that $(\|x_n - x\|)_{n=1}^{\infty}$*

converges for each $x \in X$, then (x_n) converges weakly and the weak-limit is a minimizer of $\tau_X(x) = \lim_{n\to\infty} \|x_n - x\|$, $x \in X$. In particular, $\ell_p, 1 < p < +\infty$ have such a property.

PROOF. Since $(\|x_n - x\|)_{n=1}^{\infty}$ is a convergent sequence, by Proposition 8, there exists $a \in X$ such that $M(x_n, \mathscr{U}, X) = \{a\}$ for every non-principal ultrafilter $\mathscr{U}$. And since X has property (M), we have $a = w\text{-}\lim_{n,\mathscr{U}} x_n$ for every $\mathscr{U}$. Hence it follows from the Lemma 13 that $a = w\text{-}\lim_{n\to\infty} x_n$. The last assertion is clear from that $\ell_p, 1 < p < +\infty$ have property (M). □

EXAMPLE. Let (e_n) be the usual unit vector basis of ℓ_∞, i.e., $e_n = (0, 0, \cdots, 0, \overset{n}{1}, 0, \cdots)$ and $\mathscr{U}$ a non-principal ultrafilter on $\mathbb{N}$. Then we have the following :

(1) If $X = c_0$, then $\omega(e_n) = \{0\}$, $M(e_n, \mathscr{U}, c_0) = \{x \in c_0 : \|x\|_\infty \leq 1\}$ and $C(e_n, \mathscr{U}, c_0) = \{a \in c_0 : a = (\xi_n), \|a\|_\infty \leq 1, \{n \in \mathbb{N} : \|e_n - a\|_\infty = 1 - \xi_n\} \in \mathscr{U}\}$. Hence, in this case, $\omega(e_n) \subsetneq C(e_n, \mathscr{U}, c_0) \subsetneq M(e_n, \mathscr{U}, c_0)$.

(2) If $X = \ell_\infty$, then $\omega(e_n) = \{0\}$, $M(e_n, \mathscr{U}, \ell_\infty) = \{x \in \ell_\infty : x = (\xi_n), \|x\|_\infty \leq 1/2, \lim_{n,\mathscr{U}} \xi_n = 1/2\}$ and $C(e_n, \mathscr{U}, \ell_\infty) = \{a \in \ell_\infty : a = (\xi_n), \|a\|_\infty \leq 1/2, \{n \in \mathbb{N} : \xi_n = 1/2\} \in \mathscr{U}\}$. Hence, in this case, $C(e_n, \mathscr{U}, \ell_\infty) \subsetneq M(e_n, \mathscr{U}, \ell_\infty)$ and $\omega(e_n) \cap M(e_n, \mathscr{U}, \ell_\infty) = \emptyset$.

(3) If $X = \ell_1$, then $\omega(e_n) = \emptyset$, and $C(e_n, \mathscr{U}, \ell_1) = M(e_n, \mathscr{U}, \ell_1) = \{0\}$, and so $\omega(e_n) \cap M(e_n, \mathscr{U}, \ell_1) = \emptyset$.

(4) If $X = \ell_p, 1 < p < +\infty$, then $\omega(e_n) = M(e_n, \mathscr{U}, \ell_p) = C(e_n, \mathscr{U}, \ell_p) = \{0\}$.

We have already observed that ℓ_∞ does not have property (M), and so it does not have property (C). We show that ℓ_1 does not have property (C), either.

THEOREM 15. *The space ℓ_1 does not have property (C).*

PROOF. For any pair $i < j$ $(i, j \in \mathbb{N})$, we define $y_{ij} \in \ell_1$ by

$$y_{ij} = (\xi_k^{ij})_{k\in\mathbb{N}}$$

$$\xi_k^{ij} = \begin{cases} \frac{1}{2} & \text{if } k = i, \\ -\frac{1}{2} & \text{if } k = j, \\ 0 & \text{if } k \neq i, j. \end{cases}$$

Let $n = n(i,j) = \frac{(j-2)(j-1)}{2} + i$, $1 \leq i < j$ and $x_n = x_{n(i,j)} = y_{ij} = (\xi_k^{ij})$. Let $\mathscr{V}$ be a non-principal ultrafilter on $\mathbb{N}$. Set $U_V = \{n(i,j) : i < j; i, j \in V\}$ for each $V \in \mathscr{V}$. Then the family $\mathscr{B} = \{U_V : V \in \mathscr{V}\}$ forms a filter base on $\mathbb{N}$. Let $\mathscr{U}$ be a non-principal ultrafilter on $\mathbb{N}$ which contains $\mathscr{B}$. Define the purely finitely additive 0-1 measure λ on the power set of $\mathbb{N}$ by

$$\lambda(A) = \begin{cases} 1 & , A \in \mathscr{U}, \\ 0 & , A \notin \mathscr{U}. \end{cases}$$

Let A be any subset of $\mathbb{N}$. Then since $\mathscr{V}$ is an ultrafilter, $A \in \mathscr{V}$ or $A^c \in \mathscr{V}$. If $A \in \mathscr{V}$, then

$$\begin{aligned} \lim_{n,\mathscr{U}} \langle x_n, \chi_A \rangle &= \int_{\mathbb{N}} \sum_{k\in A} \xi_k^{(n)} \, \lambda(dn) \\ &= \int_{U_A} \sum_{k\in A} \xi_k^{(n)} \, \lambda(dn) \\ &= 0. \end{aligned}$$

If $A^c \in \mathscr{V}$, then

$$\begin{aligned}\lim_{n,\mathscr{U}}\langle x_n, \chi_A\rangle &= \int_{\mathbb{N}} \sum_{k\in A} \xi_k^{(n)}\ \lambda(dn)\\ &= \int_{U_{A^c}} \sum_{k\in A} \xi_k^{(n)}\ \lambda(dn)\\ &= 0.\end{aligned}$$

Thus for every subset A of $\mathbb{N}$ we have $\lim_{n,\mathscr{U}}\langle x_n, \chi_A\rangle = 0$. Hence $w\text{-}\lim_{n,\mathscr{U}} x_n = 0$. Let $0 < \varepsilon_n < 1$, $\varepsilon_n \downarrow 0$ and $z_n = \varepsilon_n e_1 + (1-\varepsilon_n)x_n$. Then $w\text{-}\lim_{n,\mathscr{U}} z_n = 0$. Now let f_n be any element of $F_{\ell_1}(z_n)$. Noting that $\|z_n\|_1 = 1$, we see that $f_n(e_1) = 1$, for every $n \in \mathbb{N}$. Consequently, we w^*-$\lim_{n,\mathscr{U}} f_n \neq 0$. The proof is complete. □

3. The spaces $L_p, 1 \leq p < +\infty$. Brezis and Lieb [1] showed the following: If $X = L_p[0,1]$, $1 \leq p < +\infty$, then for every bounded sequence (x_n) in X which converges a.e. to a function a on $[0,1]$ and every non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$ the following holds :

$$\lim_{n,\mathscr{U}} \|x_n - x\|_p = (\|x-a\|_p^p + \lim_{n,\mathscr{U}} \|x_n - a\|_p^p)^{\frac{1}{p}}, \quad \text{for all } x \in X.$$

Thus under the same hypothesis as above, we have

$$(*) \qquad \lim_{n,\mathscr{U}} \|x_n - x\|_p \geq \lim_{n,\mathscr{U}} \|x_n - a\|_p, \quad \text{for all } x \in X.$$

If the above hypothesis "converges a.e." is replaced by the hypothesis "converges weakly", then $(*)$ does not hold except for the case $p \neq 2$ as following shows.

THEOREM 16. *The spaces $L_p[0,1], 1 \leq p < +\infty, p \neq 2$ do not have property (I).*

PROOF. Let $1 \leq p < +\infty, p \neq 2$. Let ϕ be a periodic real-valued function of period 1 such that

$$\phi(t) = \begin{cases} 1 & ,0 \leq t < \frac{2}{3}, \\ -2 & ,\frac{2}{3} \leq t < 1. \end{cases}$$

We let $x_n(t) = \phi(nt)$. Then $w\text{-}\lim\limits_{n\to+\infty} x_n = 0$, and so $\omega(x_n) = \{0\}$. Let $\mathscr{U}$ be any non-principal ultrafilter on $\mathbb{N}$. Define $\tau_p(x) = \lim\limits_{n,\mathscr{U}} \|x_n - x\|_p$ for all $x \in L_p[0,1]$. In particular, for any constant function $\alpha \in \mathbb{R}$, $\tau_p(\alpha) = \lim\limits_{n,\mathscr{U}} \|x_n - \alpha\|_p = \left(\int_0^1 |\phi(t) - \alpha|^p dt\right)^{\frac{1}{p}}$. Set $\varphi_p(\alpha) = \tau_p(\alpha)^p, \alpha \in \mathbb{R}$. Then $\varphi_p : \mathbb{R} \to [0,+\infty)$ is differentiable at 0, and its derivative is $\varphi_p'(0) = -p\int_0^1 |\phi(t)|^{p-1}\text{sign}(\phi(t))dt$. By the definition of ϕ, $\varphi_p'(0) \neq 0$ if $p \neq 2$. This means that 0 is not a minimizer of τ_p, except for the case $p \neq 2$. Thus we have $\omega(x_n) \cap M(x_n, \mathscr{U}, L_p) = \emptyset$. Consequently, $L_p[0,1], 1 \leq p < +\infty, p \neq 2$ do not have property (I). □

Thus the Property (I) or (M) is independent of uniform convexity or uniform Gâteaux differentiability.

COROLLARY 17. *The spaces $L_p[0,1]$, $1 \leq p < +\infty$, $p \neq 2$ can not be isometricaly embedded in ℓ_p.*

THEOREM 18. *Let (S, Σ, μ) be a nonnegative measure space and $1 < p < \infty, p \neq 2$.*

Then $L_p(S, \Sigma, \mu)$ has property (I) if and only if $L_p(S, \Sigma, \mu)$ is isometrically isomorphic to $\ell_p(\Gamma)$ where $\operatorname{card} \Gamma \leq \aleph_0$.

PROOF. The sufficiency is obvious from Theorem 9. To show the necessity, assume that μ is not purely atomic, i.e., S contains a subset $S_0 \in \Sigma$ with $\mu(S_0) > 0$ such that $\mu|_{S_0}$ has no atoms. Then by [2, Theorem 9, p.127], the space $L_p(S, \Sigma, \mu)$ contains a subspace isometrically isomorphic to $L_p[0,1]$. But $L_p[0,1]$ does not have property (I) as shown in Theorem 16, and so $L_p(S, \Sigma, \mu)$ does not have property (I), either, which is a contradiction. Thus μ is purely atomic. Again applying [2, Theorem 9, p.127], we see that $L_p(S, \Sigma, \mu)$ is isometric to $\ell_p(\Gamma)$ where $\operatorname{card} \Gamma \leq \aleph_0$. □

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