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THE WORLD OF RATIONAL GORENSTEIN SINGULARITIES

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§1. An entrance to the world.

We are going to the world of rational Gorenstein singularities from the view point of classification theory of algebraic varieties. In the classification theory of 3-folds, canonical singularities and terminal singularities are very important. We recall the definitions here.

Definition. A normal variety X has *canonical* (resp. *terminal*) singularities if the following two conditions hold:

(1) There exists a integer r such that the Weil divisor rK_X is Cartier divisor.
(2) For a resoluion $f : Y \to X$ and the exceptional prime divisors E_is, the following formula holds.

$$rK_Y = f^*(rK_X) + \Sigma a_i E_i$$

where $a_i \geq 0$ (resp. $a_i > 0$).

Definition. In the above definition, we call the smallest number r *index* and a_i *discrepancy* at E_i.

Remark. These singularities are very familier with well known singularities in two dimension. If the dimension of the variety X is two, then terminal singularity is non-singular and canonical singularities are same as rational double points of type A_n, D_n and E_n.

Now we will see three dimensional case. Terminal singularities are classified completely by Mori [Mo, cf.Re1]. If the index $r = 1$, then they are isolated hypersurface

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compound DuVal singularities. If r is greater than 2, then they are cyclic quotient of the above singularities. At this moment, there is no classifications for canonical singularities. But we have the following fact:

Theorem. *In any dimension, canonical singularity of index 1 is rational Gorenstein.*

It is very convenient to understand canonical singularity in some sense.

§2. Resolution of singularities.

Now we know the existence of the resolution in general and we will introduce special resolution here:

Definition. The resolution of the singularities $f : Y \to X$ is crepant if and only if there is no discrepancy at any exceptional prime divisors.

From this resolution, we can obtain terminal singularities in Y. Naturally if the singularity is two dimensional rational double point, then the crepant resolution is minimal resolution.

We have too many rational Gorenstein singularities in general, then we will see only quotient singularities here. For this, we have following fact:

Fact. *The quotient singularity $X = \mathbb{C}^n/G$ has rational Gorenstein singularities if and only if the group G is a finite subgroup of $SL(n, \mathbb{C})$ without quasi refrections.*

We don't know the existence of the crepant resolution in general. Morover we will consider the following conjecture which came from Vafa's formula in superstring theory:

Conjecture[HH]. *Let X be the quotient of $\mathbb{C}^n$ by the finite subgroup G of $SL(n, \mathbb{C})$ and $f : Y \to X$ crepant resolution. Then the topological Euler number is the number of the conjugacy classes of the group G.*

Remark. This conjecture is for local topology, but we can see similarly for global topology. Moreover we can consider mathematical meaning of the above formula for Euler numbers. In two dimesional case, we can see it as McKay correspondence and in higher dimensional case we have some results for the explanation.

To consider this conjecture we will see some examples.

(1) $n = 2$ The singularities are rational double points and they have minimal resolutions. The conjecture holds for them. We can also see this from McKay correspondence and we have the following formula from it:

$$h_2(Y) = \#\{\text{conjugacy class of } G \text{ but not identity}\}$$

$$h_0(Y) = 1 = \#\{\text{identity}\}$$

As the global example, we have Kummer surface which is obtained as a minimal resolution of the quotient by the involution of an Abelian surface M. Original quotient has 16 singularities of type A_1. By the minimal resolution, we get the Euler number is 24 and it is also computed as the orbifold Euler number:

$$e_V(X) = \Sigma_{gh=hg} e(M^g \cap M^h)$$

where the summation runs over the pair (g, h) in the acting group G which is commutative and $e(M^g \cap M^h)$ is the topological Euler number of the common component of two fixed parts.

By the way, if you consider the finite linear group which is not subgroup of $SL(2, \mathbb{C})$, then the conjecture does not hold.

(2) $n = 3$ In this case, the singularities are canonical but not terminal. The existence of the crepant resolution is shown by some people from 1987 to 1996. (cf.[Ma1] [Ro1] [Ma2] [Ma3] [Ro2] [It1] [It2] [Ro3]) These proofs were depend on the classification of the finite subgroups in $SL(3, \mathbb{C})$ and there are some papers on this conjecture. And there is no complete proof for the existence without the classification.

And for Betti numbers are computed as follows:

$$h_{2i} = \#\{\text{conjugacy class of age } i\}$$

for $i = 0, 1$ and 2. The age of the elememnt is computed from the eigen value. For precise definition, see the paper by Reid and the author [IR].

Three dimensional global example is a Calabi-Yau 3-fold. If you take a elliptic curve C with complex multiplicity of order 3 and the the finite group which is isomorphic to the cyclic group of order three. Let M be $C \times C \times C$. Then we obtain a Calabi-Yau 3 fold as a crepant resoution of the quotient space M/G. The original quotient space has 27 fixed points which are isolated sigularities of type $1/3(1,1,1)$. And the Euler number and also orbifold Euler number are 27.

(3) $n = 4$ Some of these singularities are canonical and terminal, then they don't have any crepant resolutions. Moreover if the singularity has two crepant morphism, the topological type of the terminal singularities are not same in general. And we have no classification of these subgroups.

(4) n general

(i) The action of the group is diagonal as follows

$$(x_1, x_2, \cdots, x_n) \longrightarrow (\epsilon x_1, \epsilon x_2, \cdots, \epsilon x_n)$$

where ϵ is n-th root of unity. Then we have a crepant resolution and the unique exceptional divisor is isomorphic to $\mathbb{P}^{n-1}$ and the conjecture also holds.

(ii) $X = \mathbb{C}^{2n}/S_n$ where S_n is symmetric group, that is, n-th symmetric product of $\mathbb{C}^2$. The crepant resolution is obtained by Hilbert-Chow morphism and it is Hilbert scheme of n points on $\mathbb{C}^2$. We will see these fact in the next section. And the conjecture is true for them.[Gö]

(5) If you assume the existence of the crepant resolution, the conjecture holds for any case. It was proved by Batyrev and Dais [BD] in the case of abelian groups and by Batyrev and Kontsevich in general.

§3. Canonical resolution.

In the case $\dim X = 2$, we can construct minimal resolution without classification of the finite subgroups of $SL(2, \mathbb{C})$. First, we recall some properties of Hilbert scheme of n-points on the smooth projective surface S.

The Hilbert scheme $\mathrm{Hilb}^n(S)$ is a projective scheme parametrizing 0-dimensional subschemes of length n of S.

Fact 1. *[Fo] If S is smooth projective surface, then the Hilbert-Chow morphism*

$$\pi : Hilb^N(S) \longrightarrow Symm^n S$$

is a resolution of singularities.

From this fact we can see the Hilbert scheme is smooth and irreducible.

Fact 2. *[Fu(n=2)][Be] The Hilbert-Chow morphism is a crepant resolution.*

Using these fact, we will get the following theorem:

Theorem. *[IN] If the group G is the finite subgroup of $SL(2,\mathbb{C})$ and the order of G is n, then*

$$\phi : Hilb^G(\mathbb{C}^2) \longrightarrow Symm^n(\mathbb{C}^2)^G$$

is a minimal resolution of rational double point, where $Hilb^G(\mathbb{C}^2)$ is unique two dimensional irredusible component of G fixed part of the Hilbert scheme of n points on $\mathbb{C}^2$ dominating $\mathbb{C}^2/G$.

For the proof of this theorem, we have to consider the restriction to the G-fixed part of the Hilbert-Chow morphism. If we take care of the holomophic symplectic form, then we obtain the result.

§4. Recent progress.

The construction of the minimal resolution with Hilbert scheme does not depend on the classifiation of the finite subgroups in $SL(2,\mathbb{C})$. So we can obtain it canonically. If we can do same things in higher dimension, we will be very happy, but it is not so easy because the Hilbert scheme of n-points on $\mathbb{C}^n$ is not smooth in general.

In spite of this difficulty, Nakamura proved that we can construct a crepant resolution with Hilbert scheme if the group G is abelian in $SL(3,\mathbb{C})$ [Na][Re2].

On the other hand, we have another construction of minimal resolution by Kronheimer [Kr] and it is releted with the construction with Hilbert scheme in the sence of Kronheimer and Nakajima [KN]. Moreover we have a result by Sardo-Infirri [Sa] which is 3 dimensional generalization of the construction by Kronheimer.

Recently Nakajima and the author showed that there is a similar description as [KN] for 3 dimensional case and which coincides with the result of [Sa] with a particular parameter. And they will also show you 3 dimensional McKay correspondence. The paper is in preparation now.

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