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A localization lemma and its applications*

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Abstract

In this article, we give alternative proofs of two famous facts, the Poincaré-Hopf index theorem and the compatibility of two definitions of the degree of a divisor on a compact Riemann surface, and define a generalization of the tangential index [Br] and [Ho] and prove its index theorem by the method of the localization of the Chern class of a virtual bundle. The tangential index and its index formula was ordinary defined and proved by M.Brunella [Br] for a curve and a singular foliation on a compact complex surface and the author reproved it for a compact curve and a singular foliation on a complex surface [Ho].

1 Introduction

Let X be a C^{∞} manifold of dimension m and E a C^{∞} complex vector bundle of rank n. We consider the Chern class $c(E)\in H^{*}(X;\mathbf{C})$ of E. Note that we use the complex number field $\mathbf{C}$ as the coefficient of the cohomology groups although in fact $c(E)$ itself is in $H^{*}(X;\mathbf{Z})$, since we use the Chern-Weil theory for the construction of Chern classes. If E has a global section $s:X\longrightarrow E$, which is not identically zero, we can make a frame, including s, of the ristriction of E to the complement of the zero set of s. Therefore the top Chern class can be localized to the neighborhood of the zero set of s. This fact have many applications. In this article, we consider a simple generalization of this fact.

Let $\mathcal{V}=\{V_{\alpha}\}$ be an open covering of X such that the vector bundle E has a section $s_{\alpha}:V_{\alpha}\longrightarrow E$ on an open set V_{α}, which is not a zero section. Assume that there exist non-vanishing functions $f_{\alpha\beta}$ on $V_{\alpha}\cap V_{\beta}$ such that

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$s_\beta = s_\alpha f_{\alpha\beta}$ and the system $\{f_{\alpha\beta}\}$ is a cocycle. We denote by F the line bundle defined by $\{f_{\alpha\beta}\}$. Then we consider the Chern class of the virtual bundle $E - F$. It is localized to the neighborhood of each connected component of the union of the zero set of each s_α. Then we can define the index of E by F and get its index formula.

In section 2, we consider a localization lemma and, as examples, the Poincaré-Hopf index formula and the compatibility of two definitions of the degree of a divisor on a compact Riemann surface. Although the Čech-de Rham cohomology theory and its integration theory play important roles in this article, we refer to [BT], [Leh1], [Leh2], [LS] and [Su] for the details of these theories. In section 3, a generalization of the tangential index [Br] and [Ho] are defined and we prove its index formula. This index can be considered to represent how a variety and a one dimentional singular foliation intersect, and it is a kind of indices relative not only to a singular foliation but also to a variety. The tangential index is defined by M.Brunella [Br] for a curve and a singular foliation on a compact complex surface. We generalize it for a variety and a dimension one singular foliation on X.

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2 Localization lemma

Let X be a C^∞ manifold of dimension m, E a complex vector bundle of rank n, $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ an open covering of X and $s_\alpha : V_\alpha \longrightarrow E$ a C^∞ section of E on each V_α. We can assume that E is trivial on each V_α if necessary taking a refinement of $\mathcal{V}$. Moreover we assume the following condition.

Assumption 2.1 *For any α, $\beta \in A$ such that $V_\alpha \cap V_\beta \neq \emptyset$, there exsits a non-vanishing C^∞ function $f_{\alpha\beta} : V_\alpha \cap V_\beta \longrightarrow \mathbf{C}^*$ such that $s_\beta = s_\alpha f_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ and the system $\{f_{\alpha\beta}\}$ forms a cocycle.*

We denote by F the line bundle which is defined by this cocycle $\{f_{\alpha\beta}\}$. Then there exists a bunble map $f : F \longrightarrow E$ such that

(1) $f(F_p) \subset E_p$ for all $p \in E$

(2) there exist a subset $S \subset X$ such that $f_p : F_p \to E_p$ is injective for $p \in X - S$.

We call S the set of singularities of the line bundle F. Let $S = \coprod_{\lambda\in\Lambda} T_\lambda$ be the decomposition to connected components. We assume that each T_λ is compact. Take an open set U_λ for each λ such that $U_\lambda \supset T_\lambda$ and $U_\lambda \cap U_\mu = \emptyset$

for $\lambda \neq \mu$. Then $\mathcal{U} = \{U_0, (U_\lambda)_{\lambda\in\Lambda}\}$, where $U_0 = X - S$, is an open covering of X.

We consider the Čech-de Rham cohomology group $H^*(A^\bullet(\mathcal{U}))$ associated with this open covering $\mathcal{U}$. Note that this cohomology is isomorphic to the de Rham cohomology (see [BT]). The n-th Chern class $c_n(E-F)$ of the virtual bundle $E-F$ has a representative $(\sigma_n^0, (\sigma_n^\lambda)_\lambda, (\sigma_n^{0\lambda})_\lambda)$ in the Čech-de Rham cohomology group $H^{2n}(A^\bullet(\mathcal{U}))$ of degree $2n$, where σ_n^0 and σ_n^λ are $2n$-closed forms which are representatives of $c_n(E-F)$ on U_0 and U_λ, respectively, in the de Rham cohomology group and $\sigma_n^{0\lambda}$ is a $(2n-1)$-form on $U_0 \cap U_\lambda$ such that $d\sigma_n^{0\lambda} = \sigma_n^\lambda - \sigma_n^0$. Note that we can construct σ_n^0, σ_n^λ and $\sigma_n^{0\lambda}$ from connections of E and F, using the Chern-Weil theory, and the Čech-de Rham cohomology class represented by these forms is independent on the choice of the connections.

Lemma 2.2 (localization) *Let $j^* : H^{2n}(X, X-S; \mathbf{C}) \longrightarrow H^{2n}(X; \mathbf{C})$ be natural map. Then there exists $c \in H^{2n}(X, X-S; \mathbf{C})$ such that $j^*(c) = c_n(E-F)$.*

Proof. Since F can be considered a subbundle of E on U_0, there exists the decomposition $F \oplus E'$ of E. The system $\{s_\alpha\}$ forms a frame of F. Let ∇_0^F be a trivial connection of F respect to the frame, $\nabla_0^{E'}$ a conncetion of E' on U_0 and κ the curvature matrix of the connection $\nabla_0 = \nabla_0^F \oplus \nabla_0^{E'}$ of E on U_0. Then $\sigma_n^0 = \det\kappa = 0$. We can construct σ_n^λ and $\sigma_n^{0\lambda}$ from ∇_0 and conncetions of E and F on U_λ.

Therefore the representative of $c_n(E-F)$ in the Čech-de Rham cohomology is $\sigma = (0, (\sigma_n^\lambda)_\lambda, (\sigma_n^{0\lambda})_\lambda)$. This is a $2n$-cocyle in the Čech-de Rham complex relative to $X-S$. Let τ be a $2n$-form on X corresponding to σ. Then $c = [\tau] \in H^{2n}(X, X-S; \mathbf{C})$ and $j^*(c) = c_n(E-F)$.

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We denote $c \in H^{2n}(X, X-S\ ;\ \mathbf{C})$ by $c_n(E; F)$. This is a localization of $c_n(E-F)$.

If X is compact, there exists following commutative diagram.

$$\begin{array}{ccc}
H^{2n}(X, X-S; \mathbf{C}) & \xrightarrow{A} & H_{m-2n}(S; \mathbf{C}) = \bigoplus_{\lambda\in\Lambda} H_{m-2n}(T_\lambda; \mathbf{C}) \\
{\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle i_*} \\
H^{2n}(X; \mathbf{C}) & \xrightarrow{\frown [X]} & H_{m-2n}(X; \mathbf{C}),
\end{array}$$

where A is the Alexander duality, i the natural inclusion and $[X]$ the fundamental class of X.

Definition 2.3 *We define an index* $I(E,F;T_\lambda) \in H_{m-2n}(T_\lambda;\mathbf{C})$ *of* E *by* F *at* T_λ *by*

$$A(c) = (I(E,F;T_\lambda))_{\lambda\in\Lambda}.$$

Remark 2.4 *We can define the index* $I(E,F;T_\lambda)$ *if* S *is compact.*

From the commutativity of the above diagram, we have following theorem.

Theorem 2.5 *If* X *is compact, we have*

$$\sum_{\lambda\in\Lambda} i_* I(E,F;T_\lambda) = c_n(E-F) \frown [X].$$

In the rest of this section, we assume that X is compact and S consists only of isolated points. Since each T_λ consists of a point p_λ under this assumption, we can take a sufficiently small open neighborhood U_λ of p_λ. Then we can assume each σ_n^λ is 0 without loosing generalities. Hence the localized Chern class $c_n(E;F)$ has a representative $(0,0,(\sigma_n^{0\lambda})_\lambda)$. So it is important to write the $(2n-1)$-form $\sigma_n^{0\lambda}$ explicitly. We have to mention the Bochner-Martinelli kernel for the purpose of writing $\sigma_n^{0\lambda}$ clearly.

Definition 2.6 *We call following* $(n,n-1)$*-form* β_n *on* $\mathbf{C}^n$ *the Bochner-Martinelli kernel;*

$$\beta_n = C_n \sum_{i=1}^{n} (-1)^{i-1} \frac{\bar{z}_i d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n}{\| z \|^{2n}},$$

where

$$C_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n}.$$

Remark 2.7 *Let* $S^{2n-1} \subset \mathbf{C}^n$ *be a* $(2n-1)$*-sphere centered at the origin* 0*. Then the Bochner-Martinelli kernel* β_n *is real on* S^{2n-1} *and a generator of the cohomology group* $H^{2n-1}(S^{2n-1};\mathbf{C})$*:*

$$\int_{S^{2n-1}} \beta_n = 1$$

Theorem 2.8 *Assume that* $p_\lambda \in V_\alpha$ *for some* α*. Then we have*

$$\sigma_n^{0\lambda} = -s_\alpha^* \beta_n,$$

To prove this theorem, the Chen-Weil theory and the integration along the fiber are needed. Here the proof is omitted.

Hereafter we assume that X is oriented. We introduce the integration on the Čech-de Rham cohomology group.

Let R_λ be a closed neighborhood of p_λ such that $R_\lambda \subset U_\lambda$ for each λ. Put $R_0 = X - \bigcup_{\lambda\in\Lambda} \mathrm{int} R_\lambda$ and $R_{0\lambda} = R_0 \cap R_\lambda$. R_0 and R_λ are oriented as submanifolds of X for each λ and $R_{0\lambda}$ as the boundary of R_0; $R_{0\lambda} = \partial R_0 = -\partial R_\lambda$. We call a family $\mathcal{R} = \{R_0, (R_\lambda)_\lambda, (R_{0\lambda})_\lambda\}$ a system of honey-comb cells adapted to the open covering $\mathcal{U}$.

Then we can define the integration on the Čech-de Rham cohomology group $H^m(A^\bullet(\mathcal{U}))$ associated with $\mathcal{U}$ when X is compact. For any $\sigma = [(\sigma_0, (\sigma_\lambda)_\lambda, (\sigma_{0\lambda})_\lambda)] \in H^m(A^\bullet(\mathcal{U}))$, we define the integration by

$$\int_X \sigma = \int_{R_0} \sigma_0 + \sum_{\lambda\in\Lambda} \int_{R_\lambda} \sigma_\lambda + \sum_{\lambda\in\Lambda} \int_{R_{0\lambda}} \sigma_{0\lambda}.$$

This definition is well-defined and compatible with the integration on the de Rham cohomology group;

$$\int_X \sigma = \int_X \tau,$$

where τ is a $2n$-form on X corresponding to σ. See [Leh1], [Leh2], [LS] and [Su] for the details and more general definitions.

Then we describe examples.

Corollary 2.9 *Let C be a compact Riemann surface, $D = \{(U_i, f_i)\}$ a Cartier divisor on C, $D' = \sum_{i=1}^n n_i p_i$ the Weil divisor corresponding to D. Then*

$$\int_C c_1([D]) = \sum_{i=1}^n n_i,$$

where $[D]$ is the line bundle associated with D.

Proof. We can assume that each point p_i in D' is contained in U_i and not contained in other U_j. There exists a coordinate z_i on each U_i such that $z_i(p_i) = 0$ and $f_i(z_i) = z_i^{n_i}$. Then $\mathcal{U} = \{U_0, (U_i)_i\}$, where $U_0 = C - \{p_1, p_2, \cdots, p_n\}$ is an open covering of C. Let $\mathcal{R}$ be a system of honey-comb cell adapted to $\mathcal{U}$. Note that each f_i is a section of $[D]$ on U_i. From theorem(2.5) and (2.8), we have

$$\begin{aligned}
\int_C c_1([D]) &= \sum_{i=1}^n \int_{R_{0i}} -f_i^*\beta_1 \\
&= \sum_{i=1}^n \frac{1}{2\pi\sqrt{-1}} \int_{S^1_{p_i}} \frac{df_i}{f_i} = \sum_{i=1}^n \frac{1}{2\pi\sqrt{-1}} \int_{S^1_{p_i}} \frac{n_i dz}{z} \\
&= \sum_{i=1}^n n_i,
\end{aligned}$$

where $S^1_{p_i}$ is a 1-sphere in $\mathbf{C}$ centerd at p_i and oriented naturally.

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As the second example, we consider the Poincaré-Hopf index formula for a dimension one reduced singular foliation.

Definition 2.10 *A dimension one singular foliation $\mathcal{F}$ on a complex manifold X is determined by a triple $(\{V_\alpha\}, v_\alpha, e_{\alpha\beta})$ such that*

(1) *$\{V_\alpha\}$ is an open covering of X and, for each α, v_α is a holomorphic vector field on V_α,*

(2) *for each pair (α, β), $e_{\alpha\beta}$ is a non-vanishing holomorphic function on $V_\alpha \cap V_\beta$ which satisfies the cocyle condition, $e_{\alpha\gamma} = e_{\alpha\beta}e_{\beta\gamma}$ on $V_\alpha \cap V_\beta \cap V_\gamma$,*

(3) *$v_\beta = v_\alpha e_{\alpha\beta}$ on $V_\alpha \cap V_\beta$.*

The cocycle $\{e_{\alpha\beta}\}$ defines a line bundle which is called the holomorphic tangent bundle of $\mathcal{F}$.

Note that this definition is adapted to the assumption (2.1) if we regard the holomorphic tangent bundle TX as a C^∞ complex vector bundle E. The singular set of a foliation is defined similary. A dimension one singular foliation is said to be reduced if its singular set consists only of isolated points.

Corollary 2.11 (Poincaré-Hopf) *Let X be a compact complex manifold of complex dimenson n, $\mathcal{F} = (\{V_\alpha\}, v_\alpha, e_{\alpha\beta})$ a reduced dimension one singular foliation and F a holomorphic tangent bundle of $\mathcal{F}$. Then we have*

$$\sum_{p\in S} PH(v,p) = \int_X c_n(TX - F),$$

where S is the singular set of $\mathcal{F}$ and $PH(v,p)$ is the Poincaré-Hopf index of v at p.

Proof. Note that the Poincaré-Hopf index $PH(v,p)$ is written as

$$\int_{S_p^{2n-1}} v^*\beta_n = \int_{R_{0\lambda}} \sigma_n^{0\lambda},$$

for some λ. Hence this fomula is an obvious corollary of the theorem(2.5) and (2.8).

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Remark 2.12 *The oridinal Poincaré-Hopf index formula is*

$$\sum_{p \in S} PH(v,p) = \chi(X),$$

where v is a vector field on X. If there exists a global vector field with only isolated zero points, the tangent bundle F is trivial and we get the classical formula, using the fact $\int_X c_n(X) = \chi(X)$, from the theorem(2.11). *This formula is a special case of the Baum-Bott residue theorem* [BB].

Note that some other formulas, for example, the Riemann-Hurwitz formula, can be proved in this way.

3 Tangential index

Let X be a complex manifold of dimension $n+k$, $V \subset X$ a strong locally complete intersection (SLCI) of dimension n (See [LS] for the definition of SLCI), $\mathcal{V} = \{V_\alpha\}$ an open covering of X and $V' = \mathrm{Reg}(V) = V - \mathrm{Sing}(V)$ a regular part of V. Since V is an SLCI, there exists a C^∞ complex vector bundle $\tilde{N}$ on a neighborhood U of V in X such that the restriction $\tilde{N}|_{V'}$ is the normal bundle $N_{V'}$ of V'.

Assumption 3.1 *There exists a bundle map $\tilde{\pi} : TX|_U \longrightarrow \tilde{N}$ such that a diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & TV' & \longrightarrow & TX|_{V'} & \xrightarrow{\ \pi\ } & N_{V'} & \longrightarrow & 0 \quad (exact) \\
 & & & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} & & \\
 & & & & TX|_U & \xrightarrow[\tilde{\pi}]{} & \tilde{N} & &
\end{array}$$

is commutative.

The above assumption(3.1) is satisfied, for example, when an SLCI V is defined by a holomorphic section s of a holomorphic vector bundle E; $V = s^{-1}(0)$. In this case, E is isomorphic to $\tilde{N}$.

Let $f_1^\alpha, f_2^\alpha, \cdots, f_k^\alpha$ be defining functions of V on V_α;

$$V \cap V_\alpha = \{f_1^\alpha = f_2^\alpha = \cdots = f_k^\alpha = 0\}.$$

We can take coordinates $(x_1^\alpha, x_2^\alpha, \cdots, x_{n+k}^\alpha)$ on V_α such that $x_{n+i}^\alpha = f_i^\alpha$ for $i = 1, 2, \cdots, k$. Then

$$\pi\frac{\partial}{\partial x_{n+1}^\alpha}, \pi\frac{\partial}{\partial x_{n+2}^\alpha}, \cdots, \pi\frac{\partial}{\partial x_{n+k}^\alpha}$$

form a frame of $N_{V'} = (TX|_{V'})/TV'$. We can assume $\tilde{N}$ is trivial on each $V_\alpha \cap U$ and there exists a frame $\{e_1^\alpha, e_2^\alpha, \cdots, e_k^\alpha\}$ of $\tilde{N}$ such that

$$e_i^\alpha|_{V'} = \pi \frac{\partial}{\partial x_{n+i}^\alpha}$$

for each i. This frame is said to be associated with $\{f_1^\alpha, f_2^\alpha, \cdots, f_k^\alpha\}$.

Let $\mathcal{F} = \{(V_\alpha, v_\alpha, e_{\alpha\beta})\}$ be a dimension one singular foliation and F the holomorphic tangent bundle of $\mathcal{F}$.

Assumption 3.2 *The SLCI V is not invariant by $\mathcal{F}$, i.e. $v_\alpha(f_{\alpha,i}) \notin I(V \cap V_\alpha)$, where $I(V \cap V_\alpha)$ is the ideal of holomorphic functions vanishing on $V \cap V_\alpha$ and generated by the defining functions of V on V_α.*

Take a frame $\{e_i^\alpha\}$ of $\tilde{N}$ associated with $\{f_i^\alpha\}$. Then we get

$$\tilde{\pi}(v_\alpha)|_V = \sum_{i=1}^{k} v_\alpha(f_{\alpha,i})e_i^\alpha.$$

Let

$$T_\alpha = \mathrm{Sing}V \cup \{p \in V' \cap V_\alpha \mid v_\alpha(p) \in T_pV'\}.$$

Then we have

$$\begin{aligned} \tilde{\pi}(v_\alpha)|_V &= \tilde{\pi}(v_\beta)|_V e_{\beta\alpha} \\ T_\alpha &= \{p \in V \cap V_\alpha \mid \tilde{\pi}(v_\alpha)(p) = 0\}. \end{aligned}$$

So $T = \cup_\alpha T_\alpha$ is well-defined. T is the set of tangential points of $\mathcal{F}$ and V. Let $T = \coprod_{\lambda \in \Lambda} T_\lambda$ be the decomposition to connected components and we assume that T is compact. Then there exists generalized tangential index.

Theorem 3.3 (tangential index) *There exists index*

$$I(N, F; T_\lambda) \in H_{2(n-k)}(T_\lambda; \mathbf{C})$$

of N by F at T_λ. Moreover if V is compact,

$$\sum_{\lambda \in \Lambda} i_* I(N, F; T_\lambda) = c_k(N - F) \frown [V]$$

Theorem 3.4 *If $n = k$ and T consists only of isolated points, then*

$$I(N, F; p) = \int_L (\pi(v))^* \beta_k,$$

where $p \in T$ and L is a link of V at p with a usual orientation; $L = \{f_1 = f_2 = \cdots = f_k = 0, |v(f_1)|^2 + |v(f_2)|^2 + \cdots + |v(f_k)|^2 = \varepsilon\}$ for a sufficientry small $\varepsilon > 0$ and $d\arg v(f_1) \wedge d\arg v(f_2) \wedge \cdots \wedge d\arg v(f_k) > 0$.

These two theorems are corollaries of theorem(2.5) and (2.8), respectively. Apply these theorems to a virtual bundle $\tilde{N} - F$ and V.

This index can be considered to represent how tangent a variety and on dimensional singular foliation. If $n = k = 1$ then we have

$$I(N, F; p) = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{dv(f)}{v(f)}.$$

This coincides with an intersection number $(v(f), f)_p$ at p (See [GH] Chapter 5) and the original tangential index [Br] and [Ho].

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