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An Automorphic Forms on the Expanded Symmetric Domain of Type IV (Introductory Talk)

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Today, I will talk about the construction of an automorphic form on the expanded symmetric domain of type IV, from a given Jacobi form of index 1. Today's talk is a brief explanation of my pre-print. For details, see the pre-print, which can be down-loaded at the following address:

http://www.kurims.kyoto-u.ac.jp/~hiroki/index.html

(RIMS pre-print series No. 1147).

Oda and Gritsenko constructed an automorphic form on the classical symmetric domain of type IV from a given Jacobi form of index 1. Applying this idea to the expanded domain, we can construct an automorphic form from a given Jacobi form. To apply this idea, we have to lift a Jacobi form to the expanded domain. I constructed a function to enable this lifting.

First, I explain about the expanded domain. The 14 exceptional unimodular singularities were introduced by Arnold. The deformation theory of these singularities was studied by Brieskorn, Dolgacev, Nikulin, Looijenga and Pinkham, in connection with the period domain for $K3$ surfaces, which is classically well known to be the symmetric domain of type IV by Pyatetski-Shapiro. Looijenga considered the expansion of the symmetric domain of type IV in connection with the triangle singularity. On the other hand, Saito studied the period domain for the integrals of primitive forms. He showed that this period domain associated to the exceptional singularities is the expansion of the classical domain of type IV. This expanded domain coincides with Looijenga's expanded domain. And now, this domain is much concerned with the string theory.

Let L be the following even integral unimodular lattice, with quadratic form S:

$$L := \overbrace{\underbrace{L_0}_{\substack{\text{negative}\\ \text{definite}}} \oplus \underbrace{II_{1,1}}_{\langle e_1, e_{-1}\rangle}}^{L_1} \oplus \underbrace{II_{1,1}}_{\langle e_2, e_{-2}\rangle} ,$$

where L_0 is an even integral negative definite lattice, $II_{1,1}$ is a hyperbolic plane, and $\langle e_i, e_j\rangle = \delta_{i,-j}$. Let

$$V := L \otimes_{\mathbb{Z}} \mathbb{R}$$

be a base space of L. We fix one coordinate

$$S = \begin{array}{c} \begin{pmatrix} & & 1 \\ & S_1 & \\ 1 & & \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \ e_2 \\ \leftarrow \ L_1 \\ \leftarrow \ e_{-2} \end{array}$$

$$= \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & -S_0 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \begin{array}{l} \leftarrow \ e_2 \\ \leftarrow \ e_1 \\ \leftarrow \ L_0\,, \\ \leftarrow \ e_{-1} \\ \leftarrow \ e_{-2} \end{array}$$

where S_0 is the positive definite matrix. Then the classical symmetric domain of type IV is,

$$\begin{aligned} & \left\{ W \subset V \;\middle|\; \dim_{\mathbb{R}} W = l+2,\ W < 0 \right\} \\ = \ & \mathrm{P}_{\mathbb{C}} \left\{ v \in \mathbb{C}^{l+4} = L \otimes_{\mathbb{Z}} \mathbb{C} \;\middle|\; S\{v\} > 0,\ S[v] = 0 \right\}_0 \\ = \ & \left\{ w = \begin{pmatrix} \omega \\ \xi \\ \tau \end{pmatrix} \in \mathbb{C}^{l+2} = L_1 \otimes_{\mathbb{Z}} \mathbb{C} \;\middle|\; S_1[\mathrm{Im} w] > 0 \right\}_0 =: \mathcal{H}_0^{l+2}, \end{aligned}$$

where

$$S\{v\} := {}^t\bar{v} S v \quad , \quad S[v] := {}^t v S v$$

and the mark " $_0$ " means the connected component. On the other hand, the expanded domain is,

$$\text{Saito's domain} \left\{ \begin{aligned} & \left\{ \varphi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \;\middle|\; \mathrm{Ker}(\varphi) < 0 \right\} \\ = & \left\{ v \in \mathbb{C}^{l+4} = L \otimes_{\mathbb{Z}} \mathbb{C} \;\middle|\; S\{z\} - |S[z]| > 0 \right\}_0 \longleftarrow \text{Looijenga's domain} \\ \xrightarrow{\mathrm{P}_{\mathbb{C}}} & \left\{ Z = \begin{pmatrix} t \\ w \end{pmatrix} \in \mathbb{C}^{l+3} = \mathbb{C} \times (L_1 \otimes_{\mathbb{Z}} \mathbb{C}) \;\middle|\; S_1[\mathrm{Im} w] > \frac{|t| - \mathrm{Re} t}{2} \right\}_0 =: \mathcal{B}_0^{l+3} \end{aligned} \right.$$

The group $G_{\mathbb{R}} := (O(S,\mathbb{R}))_0$ acts on these domains. The action of $G_{\mathbb{R}}$ on $\mathcal{B}_0^{l+3}$ is as follows:

$$Z \mapsto g(Z) = \left(\begin{pmatrix} \frac{t}{J(g,Z)^2} \\ \left(\frac{\frac{1}{2}g_{i,0}(t-S_1[w])+\sum_{j=1}^{l+2} g_{i,j}w_j+g_{i,l+3}}{J(g,Z)} \right)_{1\le i\le l+2} \end{pmatrix} \right)$$

$$\left(\quad g = \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l+3} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l+3} \\ \vdots & \vdots & \ddots & \vdots \\ g_{l+3,0} & g_{l+3,1} & \cdots & g_{l+3,l+3} \end{pmatrix} \in G_{\mathbb{R}} \; , \; Z = \begin{pmatrix} t \\ \\ w \\ \\ \end{pmatrix} = \begin{pmatrix} t \\ w_1 \\ \vdots \\ w_{l+2} \end{pmatrix} \in \mathcal{B}_0^{l+3} \quad \right),$$

where

$$J(g,Z) := \frac{1}{2}g_{l+3,0}(t - S_1[w]) + \sum_{j=1}^{l+2} g_{l+3,j}w_j + g_{l+3,l+3}$$

is the denominator. Hence we can define automorphic forms on these domains with respect to

$$\Gamma := O(S,\mathbb{Z}) \cap G_{\mathbb{R}}.$$

We define $\mathbb{M}_k$ as a set of all automorphic forms of weight k. Let $\mathrm{P}_{\mathbb{R}}$ be a parabolic subgroup of $G_{\mathbb{R}}$, preserving the isotropic plane $\langle e_1, e_2 \rangle$, and $\mathrm{P}_{\mathbb{Z}} := \mathrm{P}_{\mathbb{R}} \cap \Gamma$. Similarly to the classical theory, we can define expanded Jacobi forms from the following Fourier expansions of automorphic forms on the expanded domain:

$$\mathbb{M}_k(*,\Gamma) \ni F(*) = \sum_{m=0}^{\infty} \overbrace{\varphi_m(*)\mathrm{e}(m\omega)}^{\mathrm{P}_{\mathbb{Z}}-\text{invariant}}$$

$$\Big(\quad \mathrm{e}(*) := \exp(2\pi\sqrt{-1}*) \quad \Big).$$

We denote $\mathbb{J}_{k,m}$ as a set of all Jacobi forms of weight k and index m.

In the pre-print, we survey (1) the lifting of Jacobi forms, and (2) the construction of automorphic forms, where the following diagram becomes commutative:

$$\begin{array}{ccc} \mathbb{J}_{k,1}(\mathcal{H}_0^{l+2}, \mathrm{P}_{\mathbb{Z}}) \ni \varphi & \xrightarrow{\text{Gritsenko}} & * \in \mathbb{M}_k(\mathcal{H}_0^{l+2}, \Gamma) \\ {\scriptstyle (1)}\big\downarrow & & \big\uparrow{\scriptstyle t=0} \\ \mathbb{J}_{k,1}(\mathcal{B}_0^{l+3}, \mathrm{P}_{\mathbb{Z}}) \ni \tilde{\varphi} = \varphi \times f_1 & \xrightarrow[(2)]{} & F(Z) \in \mathbb{M}_k(\mathcal{B}_0^{l+3}, \Gamma) \end{array}$$

(1) The lifting of Jacobi forms

To lift Jacobi forms, it is enough to find the function f_1 such that

$$\begin{cases} f_1(t,\tau) = \exp\left(-\pi\sqrt{-1}\frac{ct}{c\tau+d}\right) f_1\left(\frac{t}{(c\tau+d)^2}, \frac{a\tau+b}{c\tau+d}\right) \\ f_1(0,\tau) = 1 \end{cases}$$

Let

$$f_1(t,\tau) = 1 + \sum_{j=1}^{\infty} \frac{2(\pi\sqrt{-1})^j}{j!(j-1)!} \left(\frac{d^j}{d\tau^j} \log \eta(\tau)\right) t^j.$$

The series f_1 converses absolutely and locally uniformly on $\mathbb{C} \times \mathbb{H}$. To prove that this f_1 satisfies above conditions, we use the fact that $\eta(\tau)^{24}$ is an automorphic form of weight 12 with respect to $SL_2(\mathbb{Z})$ and that $\eta(\tau)$ has no zeros on $\mathbb{H}$.

(2) The construction of automorphic forms

Now we define an embedding of the Hecke operator of $SL_2(\mathbb{Z})$ to the Hecke operator of $\mathrm{P}_{\mathbb{Z}}$

$$p : \mathrm{H}\left(SL_2(\mathbb{Z}); M_2^+(\mathbb{Z})\right) \to \mathrm{H}\left(\mathrm{P}_{\mathbb{Z}}, \mathrm{P}_{\mathbb{Q}}\right)$$

by

$$SL_2(\mathbb{Z}) \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} SL_2(\mathbb{Z}) \mapsto \mathrm{P}_{\mathbb{Z}} \begin{pmatrix} \alpha & & & & \\ & \beta & & & \\ & & E_l & & \\ & & & \beta^{-1} & \\ & & & & \alpha^{-1} \end{pmatrix} \mathrm{P}_{\mathbb{Z}},$$

where $\mathrm{P}_{\mathbb{Q}} := \mathrm{P}_{\mathbb{R}} \cap GL_{l+4}(\mathbb{Q})$. Let

$$T_-(m) := p\left(\sum_{\substack{\alpha,\beta \in \mathbb{N} \\ \alpha\beta = m, \alpha|\beta}} SL_2(\mathbb{Z}) \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} SL_2(\mathbb{Z})\right).$$

Then we can write the part (2). For any given $\varphi \in \mathbb{J}_{k,1}(\mathcal{H}_0^{l+2}, \mathrm{P}_{\mathbb{Z}})$, let

$$\tilde{\varphi} = \varphi f_1 \in \mathbb{J}_{k,1}(\mathcal{B}_0^{l+3}, \mathrm{P}_{\mathbb{Z}}),$$

and

$$F(Z) := \underbrace{\left(\tilde{A}_{(0,0)}(t) + \sum_{n=1}^{\infty} \sum_{\substack{a \in \mathbb{N} \\ a|n}} a^{k-1} A_{(0,0)}(a^2 t) \mathrm{e}(n\tau)\right)}_{\text{Eisenstein series of } \tilde{A}_{(0,0)}(t) \text{ with respect to } SL_2(\mathbb{Z})} + \sum_{m=1}^{\infty} \frac{1}{m} \left(\tilde{\varphi}|_k T_-(m)\right)(t,\xi,\tau) \mathrm{e}(m\omega),$$

where

$$\tilde{\varphi}(t,\xi,\tau) = \sum_{\substack{n\in\mathbb{N}\cup\{0\},u\in L_0\\ 2n\geq S_0[u]}} A_{(n,u)}(t)\mathbf{e}(-{}^tuS_0\xi + n\tau) \in \mathbb{J}_{k,1}(\mathcal{B}_0^{l+3},\mathrm{P}_{\mathbb{Z}}),$$

$$A_{(0,0)}(t) = \sum_{n=0}^{\infty} b_n t^n,$$

and

$$\tilde{A}_{(0,0)}(t) := \sum_{n=0}^{\infty} b_n \frac{(k+2n-1)!\zeta(k+2n)}{(2\pi\sqrt{-1})^{k+2n}} t^n.$$

Let

$$F_{\mathbb{Z}} := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma \;\middle|\; T \in SO(L_1,\mathbb{Z}) \right\}.$$

To prove the automorphic property of $F(Z)$, it is enough to show that the following three invariances:

(a) $\mathrm{P}_{\mathbb{Z}}$-invariance
The first part of $F(Z)$ is $SL_2(\mathbb{Z})$-invariant. In the case $m = 0$, $SL_2(\mathbb{Z})$-invariant equals to $\mathrm{P}_{\mathbb{Z}}$-invariant. So, $F(Z)$ is $\mathrm{P}_{\mathbb{Z}}$-invariant.

(b) $(\omega \leftrightarrow \tau)$-invariance
By the direct calculation, we have

$$F(Z) = \tilde{A}_{(0,0)}(t) + \sum_{\substack{{}^t(n,u,m)\in L_1\setminus\{0\}\\ S_1[{}^t(n,u,m)]\geq 0}} \sum_{\substack{a\in\mathbb{N}\\ a|(n,u,m)}} a^{k-1} A_{(\frac{mn}{a^2},\frac{u}{a})}(a^2t)\mathbf{e}\left(m\omega - {}^tuS_0\xi + n\tau\right).$$

So, $F(Z)$ is invariant with respect to exchanging variables ω and τ.

(c) $F_{\mathbb{Z}}$-invariance
In above equation, we remark that $\mathbf{e}\,(m\omega - {}^tuS_0\xi + n)$ equals to $\mathbf{e}\,((n,u,m)S_1w)$. Similarly to the classical theory of Jacobi forms, we can easily get the fact that $A_{(n,u)}(t)$ only depends on the value $2n - S_0[u]$. Hence $A_{(\frac{mn}{a^2},\frac{u}{a})}$ only depends on the value $\frac{1}{a^2}S_1[{}^t(n,m,u)]$. It means that $F(Z)$ is $F_{\mathbb{Z}}$-invariant.

Hence $F(Z)$ becomes an automorphic form, formally. $F(Z)$ converges on some domain, which is smaller than $\mathcal{B}_0^{l+3}$. But this domain of convergence includes the fundamental domain of $\mathcal{B}_0^{l+3}$ with respect to the group $\mathrm{P}_{\mathbb{Z}}$, so we can expand the domain of definition of $F(Z)$ to $\mathcal{B}_0^{l+3}$. Hence we can regard $F(Z)$ as an automorphic form on the expanded domain.

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