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CHERN CLASSES OF LOCAL COMPLETE INTERSECTIONS WITH ISOLATED SINGULARITIES

TATSUO SUWA

Department of Mathematics, Hokkaido University, Sapporo 060, Japan
E-mail: suwa@math.sci.hokudai.ac.jp

For a compact complex manifold X, we have the Chern class $c(X)$, which is the Chern class $c(TX)$ of its tangent bundle TX, in the cohomology $H^*(X)$. For a possibly singular complex algebraic or analytic variety X, there are at least three kinds of Chern classes in the homology $H_*(X)$. Namely, the Chern-Schwartz-MacPherson class $c_*(X)$, the Chern-Mather class $c^M(X)$ and the canonical class or Fulton-Johnson's Chern class $c^{FJ}(X)$. These three classes all reduce to $c(X) \frown [X]$ when the variety has no singularities. We discuss here the relation among them, particularly in the case of local complete intersections with isolated singularities.

We briefly review the above mentioned Chern classes. Let X be a compact subvariety of dimension n in a complex manifold M. The Chern-Mather class $c^M(X)$ is, roughly speaking, the Chern class of the bundle of limiting tangent spaces of the non-singular part of X. To be a little more precise, let $\tilde{\pi} : G_n(TM) \to M$ be the Grassmann bundle of n-planes in TM and $\gamma : X_0 \to G_n(TM)$ the map which assigns the tangent space T_pX to each point p in the non-singular part $X_0 = X \setminus \mathrm{Sing}(X)$ of X. The Nash modification $\tilde{X}$ of X is defined to be the closure of $\mathrm{Im}\,\gamma$ in $G_n(TM)$. It is equipped with the projection $\pi : \tilde{X} \to X$, which is the restriction of $\tilde{\pi}$, as well as the Nash tangent bundle τ, which is the restriction of the tautological bundle $\tilde{\tau}$ over $G_n(TM)$. Then the Chern-Mather class is defined by

$$c^M(X) = \pi_*(c(\tau) \frown [\tilde{X}]),$$

where $c(\tau)$ is the total Chern cohomology class of the vector bundle τ and $[\tilde{X}]$ is the fundamental class of $\tilde{X}$.

The Chern class for a singular variety X in a complex manifold M was first constructed by M.-H. Schwartz [Sc] in the relative cohomology $H^*(M, M \setminus X)$ using radial vector fields. On the other hand, the existence of the theory of Chern homology class as a natural transformation of funtors was conjectured by P. Deligne and A. Grothendieck and was proved by R. MacPherson [M]. Basic ingredients for this theory are the Chern-Mather classes and the "local Euler obstructions". Let

$\mathcal{F}(X)$ be the abelian group of constructible functions on X, which is freely generated by the local Euler obstruction functions Eu_V of reduced, irreducible subvarieties V of X. It is proved in [M] that there exists a morphism $c_* : \mathcal{F}(X) \longrightarrow H_*(X)$ which is functorial and satisfies the extra condition that, if X is non-singular then $c_*(1_X) = c(X) \frown [X]$, where 1_X denotes the characteristic function of X. In fact, c_* is given by $c_*(\sum_V n_V \; \mathrm{Eu}_V) = \sum_V n_V \, c^M(V)$. If we set $c_*(X) = c_*(1_X)$, in [BS], it is shown that it corresponds to the Schwartz class by the Alexander duality $H_*(X) \simeq H^*(M, M \setminus X)$. We call $c_*(X)$ the Chern-Schwartz-MacPherson class of X.

The canonical class or Fulton-Johnson's Chern class $c^{FJ}(X)$ is defined in terms of the Segre class of X in general, and is relatively easy to understand when X is a local complete intersection. Thus let X be a local complete intersection in a complex manifold M. Then the normal bundle to the non-singular part of X is canonically extended to a vector bundle N_X over the whole X. More precisely, let $\mathcal{I}_X$ be the ideal sheaf of X in the structure sheaf $\mathcal{O}_M$ of M and $\mathcal{O}_X = \mathcal{O}_M/\mathcal{I}_X$, then the vector bundle N_X is identified with the normal sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X)$, which is locally free in this case. For such X, we have the virtual tangent bundle $TM|_X - N_X$, whose total Chern cohomology class is given by $c(TM|_X - N_X) = c(TM|_X) \cdot c(N_X)^{-1}$. Then Fulton-Johnson's Chern class in this case is given by

$$c^{FJ}(X) = c(TM|_X - N_X) \frown [X].$$

Here we consider varieties X satisfying a little stronger condition. Namely, we assume that there exist a holomorphic vector bundle $E \longrightarrow M$ of rank k over M and a holomorphic section s of E so that $X = s^{-1}(0)$. We further assume that the ideal sheaf $\mathcal{I}_X$ is locally generated by the local components of s. Thus X is a local complete intersection and the restriction $E|_X$ coincides with the normal bundle N_X. For example, this condition is satisfied in the following cases, with a naturally given vector bundle E:

Examples. 1) X a hypersurface in M ($k = 1$). In this case, we may take as E the line bundle determined by the divisor X.
2) X a (projective algebraic) complete intersection in the projective space $\mathbb{CP}^{n+k}$. This means that the ideal I_X of homogeneous polynomials vanishing on X is generated by k homogeneous polynomials $P_1, \ldots, P_k$. In this case, we may take as E the bundle $H^{d_1} \oplus \cdots \oplus H^{d_k}$, where H denotes the hyperplane bundle and d_i the degree of P_i for $i = 1, \ldots, k$.

Theorem I [Su]. *Let X be a compact variety of dimension n as above with isolated singularities $p_1, \ldots, p_r$. Then we have*

$$c_*(X) = c^{FJ}(X) + (-1)^{n+1} \sum_{i=1}^{r} \mu_i,$$

where μ_i is the Milnor number of X at p_i.

This together with various known formulas imply the following.

Theorem II [OSY]. *Let X be as in Theorem I. Then we have*

$$c^M(X) = c^{FJ}(X) + (-1)^{n+1} \sum_{i=1}^{r} m_n(X, p_i),$$

where $m_n(X, p_i)$ is the n-th polar multiplicity of X at p_i in the sense of [Ga].

Now we recall Milnor numbers and polar multiplicities. Let X be a complete intersection variety in $\mathbb{C}^{n+k}$ with an isolated singularity at the origin. We list [Mi], [H], [Lê], [Gr] and [Lo] as general references for the Milnor number of such a singularity (cf. also [E1], [E2]). Let $n = \dim X$ and suppose that the germ $(X, 0)$ is given as the zero set $f^{-1}(0)$ of an analytic map-germ $f = (f_1, \ldots, f_k) : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0)$. For general $t \in \mathbb{C}^k$, the inverse image $X_t = f^{-1}(t)$ is non-singular and the intersectin $X_t \cap B_\varepsilon$ with a sufficiently small ball B_ε about 0 is called the Milnor fiber. It is known that it has a homotopy type of a bouquet of spheres of dimension n. The number of spheres appearing is the Milnor number $\mu_0(X)$ of X at 0. There is an algebraic formula for this ([Lê], [Gr]). We set, for $i = 1, \ldots, k$,

$$a_i = \dim_{\mathbb{C}} \mathcal{O}_{n+k}/(J(f_1, \ldots, f_i), f_1, \ldots, f_{i-1}),$$

where $J(f_1, \ldots, f_i)$ is the Jacobian ideal of the map $(f_1, \ldots, f_i) : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^i, 0)$. Then we have

$$\mu_0(X) = \sum_{i=1}^{k} (-1)^{k-i} a_i.$$

Also, the n-th polar multiplicity of T. Gaffney [Ga] is defined by

$$m_n(X, 0) = \dim_{\mathbb{C}} \mathcal{O}_{n+k}/(J(f_1, \ldots, f_k, \ell), f_1, \ldots, f_k),$$

where ℓ is a general linear function ℓ on $\mathbb{C}^{n+k}$. Note that it is equal to the sum $\mu_0(X) + \mu_0(X \cap H)$, where H is the hyperplane defined by ℓ. We recall that there are polar multiplicities $m_i(X, 0)$ of Lê-Teissier for $i = 0, \ldots, n-1$.

For the proof of Theorem I, we use the following formula.

Theorem III [SS2]. *Let X be as in Theorem I. Then the Euler-Poincaré characteristic $\chi(X)$ of X is given by*

$$\chi(X) = \int_X c_n(TM|_X - N_X) + (-1)^{n+1} \sum_{i=1}^{r} \mu_i.$$

Remarks. 1) When X is non-singular, the formula in Theorem III reduces to

$$\chi(X) = \int_X c_n(X),$$

which is the "Gauss-Bonnet" theorem.
2) When $n = k = 1$, we have $\int_X c_1(TM|_X - N_X) = -K_M \cdot X - X \cdot X$, where K_M denotes the canonical bundle of M. Thus the formula in Theorem III becomes the classical "adjunction formula" for singular curves in a complex surface ([K]).
3) If X is a complete intersection in $M = \mathbb{CP}^{n+k}$, N_X is determined by its multidegree $(d_1, \ldots, d_k)$ (see Example 2) above) and we have

$$\begin{aligned} c(TM|_X - N_X) \frown [X] &= \left((1+h)^{n+k+1} \cdot \prod_{i=1}^{k} \frac{d_i\, h}{1 + d_i\, h}\right) \frown [\mathbb{CP}^{n+k}] \\ &= \left((1+h)^{n+k+1} \cdot \prod_{i=1}^{k} \frac{d_i}{1 + d_i\, h}\right) \frown [\mathbb{CP}^{n}], \end{aligned}$$

where h denotes the first Chern class of the hyperplane bundle and $\mathbb{CP}^n$ a linear subspace of dimension n. This together with Theorem I or II give the Chern-Schwartz-MacPherson class $c_*(X)$ or the Chern-Mather class $c^M(X)$ of X.

Theorem III is proved as an application of the study of various kinds of indices of a vector field on a singular variety. For a vector field v on a singular variety X, we consider the "Schwartz index", the "GSV-index" and the " virtual index" at the singularity of v. All these reduce to the usual Poincaré-Hopf index when the singularity of v is in the regular part of X, so we compare them when it is in the singular part of X.

M.-H. Schwartz defined an index for "radial" vector fields on a singular variety X ([Sc], [BS]). This definition can be extended to vector fields which are not radial and we call the corresponding index the Schwartz index of a vector field. For a global vector field on a compact variety X, the sum of the Schwartz indices gives $\chi(X)$.

The GSV-index, which is introduced in [Se], [GSV] and [SS1] is defined for an isolated singularity of a vector field on a local complete intersection X in a complex manifold M and it takes into account the topology of X as well as the way X is embedded in M. The difference of the Schwartz index and the GSV-index is given in terms of the Milnor number of X at the singularity, which, together with the formula for the sum of the Schwartz indices, give a formula for the sum of the GSV-indices of a global vector field.

The virtual index is introduced in [LSS]. It is defined by differential geometric method and for this, besides X being a local complete intersection, we need the additional condition described above (in fact we could develope a similar theory under a weaker condition). This index is defined equally well at singularities which may not be isolated and, in the global situation, the sum of the virtual indices gives the top Chern class of the virtual tangent bundle $TM|_X - N_X$. Thus using the fact that, if the singularity is isolated, the virtual index coincides with the GSV-index ([LSS]) and also noting that there is always a global vector field with isolated singularities on X, we obtain Theorem III.

Final remarks. 1) A. Parusiński [P] defined a generalized Milnor number for each compact connected component of the singular set of a hypersurace and proved a formula for the sum of these numbers, which coincides with the one in Theorem III, if the singularities are isolated.

2) For hypersurfaces with arbitrary singularities, P. Aluffi [A] has obtained a formula similar to the one in Theorem I.

3) Our method outlined above applies in more general settings such as the case of non-isolated singularities of local complete intersections, and it will be treated in a forthcoming work (cf. the article of D. Lehmann in this volume).

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