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The local analytical triviality of a complex analytic singular foliation

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Abstract

A singular foliation on a complex manifold M is defined as an integrable coherent subsheaf E of the tangent sheaf of M. In this talk we discuss the existence of the "leaf (integral submanifold)" of E at each point of M (Theorem (3.3)). The dimensions of the leaves are not constant on M in general, so the singular set $S(E)$, which is in fact an analytic subset of M, is given as the set of points where the dimension of the leaf of E is not maximal. As an application of the existence of the leaves, we can show that the structure of the foliation E is locally analytically trivial along its each leaf (Theorem (3.4)). This kind of triviality was studied by P.Baum ([B]) for the point p such that p is a non-singular point of $S(E)$ and $\dim_p S(E) = \dim E(p) = \operatorname{rank} E - 1$. D.Cerveau also took up a similar problem from another viewpoint in [C] (for the real case, see [N], [Ss] and [St]). We generalize and arrange their theory, and add some new results.

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1 Complex analytic singular foliations

At first, we recall some generalities about complex analytic singular foliations on complex manifolds. The notation in the following is originary due to T.Suwa. For further details, see [B], [BB] and [Sw].

Let M be a (connected) complex manifold of (complex) dimension n, and let $\mathcal{O}_M$, Θ_M and Ω_M denote, respectively, the sheaf of holomorphic functions on M, the tangent sheaf and the cotangent sheaf of M.

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Let E be a coherent subsheaf of Θ_M. Note that, in this case, E is *coherent* if and only if E is locally finitely generated, since Θ_M is locally free. We set

$$S(E) = \{p \in M \mid (\Theta_M/E)_p \text{ is not } (\mathcal{O}_M)_p\text{-free}\},$$

and call it the *singular set* of E. Each point p of $S(E)$ is called a *singular point* of E. If we restrict E to a sufficiently small coordinate neighborhood U with coordinates $(z_1, z_2, \ldots, z_n)$, we can express E on U explicitly as follows:

$$E = (v_1, v_2, \ldots, v_s), \qquad v_i = \sum_{j=1}^{n} f_{ij}(z) \frac{\partial}{\partial z_j}, \qquad 1 \leq i \leq s, \tag{1.1}$$

where $f_{ij}(z)$ are holomorphic functions defined on U, and s is a non-negative integer. Then the singular set $S(E)$ is given on U by

$$S(E) \cap U = \{p \in U \mid \operatorname{rank}(f_{ij}(p)) \text{ is not maximal}\}.$$

A coherent subsheaf E of Θ_M is said to be *integrable* (or *involutive*) if for any point p of M,

$$[E_p, E_p] \subset E_p \tag{1.2}$$

holds (where $[\ ,\ ]$ denotes the *Lie bracket* of smooth vector fields). Moreover, we define the *rank* (we sometimes call it *dimension*) of E to be the rank of locally free sheaf $E|_{M-S(E)}$, and denote it rank$E$. Using the notation in (1.1), we can rewrite it as

$$\operatorname{rank}E = \max_{p \in M} \operatorname{rank}(f_{ij}(p)).$$

Definition 1.3 *A* (complex analytic) singular foliation *on M is an integrable coherent subsheaf E of Θ_M.*

It is clear that a singular foliation E induces a non-singular foliation on $M - S(E)$.

Definition 1.4 *Let E be a coherent subsheaf of Θ_M. We say that E is* reduced *if*

$$v \in \Gamma(U, \Theta_M), \ \ v|_{U-S(E)} \in \Gamma(U - S(E), E) \implies v \in \Gamma(U, E)$$

holds for every open set U in M.

By the preceding two definitions, we can consider *"reduced foliations"* in natural sense, i.e., a reduced foliation on M is a coherent subsheaf of Θ_M which is integrable and reduced.

Remark 1.5 We can check the following facts about reduced foliations:

(i) If a singular foliation E is locally free,

$$E \text{ is reduced} \iff \mathrm{codim} S(E) \geq 2 \, .$$

(ii) Let E be a reduced coherent subsheaf of Θ_M. Then E is integrable if (1.2) holds for every point $p \in M - S(E)$.

Next, let us represent singular foliations in terms of holomorphic 1-forms. It is not so difficult to rewrite it from the viewpoint of its "dual", but there are several points which require a little care.

Definition 1.6 *Let F be a coherent subsheaf of Ω_M. Then we set*

$$S(F) = \{p \in M \mid (\Omega_M/F)_p \text{ is not } (\mathcal{O}_M)_p \text{ -free}\},$$

and call it the singular set *of F. Each point in $S(F)$ is often called a* singular point *of F.*

Definition 1.7 *A coherent subsheaf F of Ω_M is said to be* integrable *when*

$$dF_p \subset \Omega_p \wedge F_p$$

holds for every point $p \in M$. Moreover, the rank of F is defined to be the rank of the locally free sheaf $F\,|_{M-S(F)}$, and denoted $\mathrm{rank} F$.

Definition 1.8 *A* (complex analytic) singular foliation *on M is an integrable coherent subsheaf F of Ω_M.*

Definition 1.9 *Let $F(\subset \Omega_M)$ be a coherent subsheaf of Ω_M. We say that F is* reduced *if*

$$\omega \in \Gamma(U, \Omega_M), \;\; \omega|_{U-S(F)} \in \Gamma\,(U - S(F),\, F) \implies \omega \in \Gamma(U, F)$$

holds for every open set U in M.

In the following we describe the relation between the two definitions, (1.3) and (1.8).

Definition 1.10 *For singular foliations $E \subset \Theta_M$ and $F \subset \Omega_M$, we set*

$$E^a = \{\omega \in \Omega_M \mid \langle v, \omega \rangle = 0 \quad \text{for all} \quad v \in E\} \, ,$$

$$F^a = \{v \in \Theta_M \mid \langle v, \omega \rangle = 0 \quad \text{for all} \quad \omega \in F\} \, ,$$

where $\langle \;,\; \rangle$ denotes the natural pairing between a vector field and a 1-form. Then $E^a(\subset \Omega_M)$ and $F^a(\subset \Theta_M)$ define reduced singular foliations on M. We call E^a (resp. F^a) the annihilator *of E (resp. F). Furthermore, $(E^a)^a$ (resp. $(F^a)^a$) is called the* reduction *of E (resp. F).*

Remark 1.11 Note that $S(E^a) \subset S(E)$ and $S(F^a) \subset S(F)$ hold.

If we use the notation in (1.10), a singular foliation $E \subset \Theta_M$ (resp. $F \subset \Omega_M$) is reduced if and only if $(E^a)^a = E$ (resp. $(F^a)^a = F$). In this way we can make any singular foliation reduced by taking its reduction. If we consider only reduced foliations, then the two definitions of singular foliation stated above are equivalent, and in this occasion, moreover, there is no difference between the singular set in terms of vecter fields and that in terms of 1-forms.

2 Singular set of a singular foliation

Next, let us summarize the basic properties of the singular set of a singular foliation. Hereafter, we assume $E(\subset \Theta_M)$ to be a singular foliation on a complex manifold M and set $r = \mathrm{rank} E$.

Definition 2.1 *For each point p in M, we set*

$$E(p) = \{v(p) \mid v \in E_p\},$$

where $v(p)$ denotes the evaluation of the vector field germ v at p. Note that $E(p)$ is a sub-vector space of the tangent space T_pM.

Definition 2.2 *For an integer k with $0 \leq k \leq r$, we set*

$$L^{(k)} = \{p \in M \mid \dim_{\mathbf{C}} E(p) = k\},$$
$$S^{(k)} = \{p \in M \mid \dim_{\mathbf{C}} E(p) \leq k\},$$

and set $L^{(-1)} = S^{(-1)} = \emptyset$ for convenience. Clearly we have

$$L^{(k)} = S^{(k)} - S^{(k-1)}, \qquad S^{(k)} = \bigcup_{i=0}^{k} L^{(i)}$$

for $k = 0, 1, 2, \ldots, r$.

Remark 2.3 *$L^{(k)}$ and $S^{(k)}$ are analytic sets for every integer k with $0 \leq k \leq r$.*

By the remark stated above, we get the *natural filtration* which consists of analytic sets:

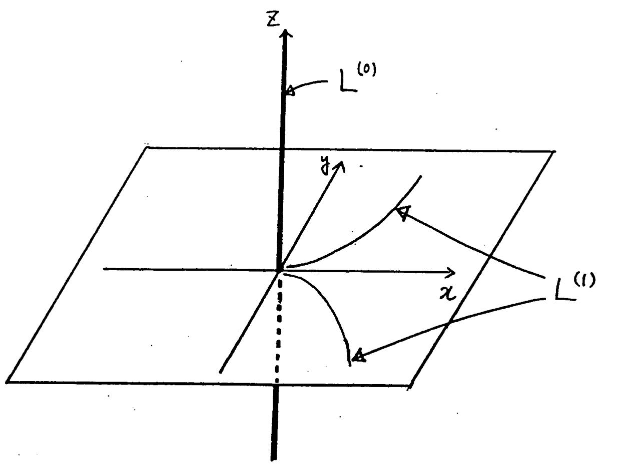
$$\begin{array}{ccccccccccccc} S^{(r)} & \supset & S^{(r-1)} & \supset & S^{(r-2)} & \supset & \cdots\cdots & \supset & S^{(1)} & \supset & S^{(0)} & \supset & S^{(-1)}. \\ \| & & \| & & & & & & & & & & \| \\ M & & S(E) & & & & & & & & & & \emptyset \end{array} \tag{2.4}$$

This filtration seems to give us information only about the "dimension" of the space $E(p)$ at p. However, the local structure of each $S^{(k)}$ appearing in (2.4) also controlls the "direction" of $E(p)$ at each point $p \in S^{(k)}$.

Example 2.5 Let v_1, v_2, v_3 be holomorphic vector fields on $M = \mathbf{C}^3 = \{(x,y,z)\}$ defined by

$$(2.6) \qquad \begin{cases} v_1 = 3y^2\dfrac{\partial}{\partial x} + 2x\dfrac{\partial}{\partial y} \\ v_2 = (x^2-y^3)\dfrac{\partial}{\partial y} + 3y^2 z\dfrac{\partial}{\partial z} \\ v_3 = (x^2-y^3)\dfrac{\partial}{\partial x} - 2xz\dfrac{\partial}{\partial z}. \end{cases}$$

Let $E(\subset \Theta_M)$ be the coherent subsheaf generated by v_1, v_2, v_3. We can easily check that E is integrable, so E defines a singular foliation on $\mathbf{C}^3$. Since the rank of E is two, all $S^{(k)}$ appearing in (2.4) are given by $S(E) = S^{(1)} = \{xz = yz = x^2 - y^3 = 0\} = \{x = y = 0\} \cup \{z = x^2 - y^3 = 0\}$ and $S^{(0)} = \{x = y = 0\}$.



$$S^{(1)} = L^{(0)} \cup L^{(1)}$$

$$S^{(0)} = L^{(0)}$$

Let us observe the analytic set $S^{(1)}$ in the preceding example. For any point p belonging to $S^{(1)}$, the dimension of the space $E(p)$ should be one or zero by the definition of $S^{(1)}$. However we can obtain more information about $E(p)$ from just looking at the local structure of $S^{(1)}$. In fact, the direction of $E(p)$ is always "tangential " to $S^{(1)}$, in other words, $E(p)$ is always contained in the tangent cone of $S^{(1)}$ at p. This property can be stated precisely as follows.

Theorem 2.7 (Tangency Lemma) *Let k be an integer with $0 \le k \le r$ and p a point in $S^{(k)}$. Then we have*

$$E(p) \subset C_p S^{(k)},$$

where $C_p S^{(k)}$ denotes the tangent cone of $S^{(k)}$ at p.

Remark 2.8 Theorem (2.7) was proved by P.Baum under the hypotheses that E is reduced, and p is a non-singular point of $S^{(k)}$ (see [B]). For the case of real singular foliations, see [N], [Ss] and [St].

This theorem is drawn as a corollary of a theorem by D.Cerveau ([C]), but we directly obtain a stronger result than (2.7) when E is reduced. For the precise proof of the following proposition, which is originally due to T.Suwa, we refer to [Y].

Proposition 2.9 ((Strong) Tangency Lemma) *Suppose $E(\subset \Theta_M)$ is reduced and p is a point of M. Let v be a germ in E_p and let $\{\varphi_t = \exp tv\}$ be the local 1-parameter group of transformations induced by v. For all t sufficiently close to 0, we have*

$$(\varphi_t)_* E_p = E_{\varphi_t(p)},$$

where $(\varphi_t)_*$ denotes the differential map of φ_t.

We can check that proposition (2.9) is stronger than theorem (2.7) as follows. Take a germ $v \in E_p$ and set $\varphi_t = \exp tv$. Suppose $\varphi_t(p) \notin S^{(k)}$ for some t. Then we have

$$\dim E(p) \leq k < \dim E(\varphi_t(p)),$$

which contradicts proposition (2.9). So we have $\varphi_t(p) \in S^{(k)}$ for all t sufficiently close to 0. Hence

$$v(p) = \lim_{t\to 0} \frac{\varphi_t(p) - p}{t}$$

is in the tangent cone $C_p S^{(k)}$ of $S^{(k)}$ at p.

3 Main Results

Let E be a singular foliation of rank r on M. We have already recalled that E induces a non-singular foliation on $M - S(E)$, so if a point $p \in M$ does not belong to $S(E)$, it is clear that there exists an integral submanifold (of dimension r) passing through p. As an application of theorem (2.7), we can show that there also exist integral submanifolds on the singular set $S(E)$, whose dimensions are lower than r.

In order to prove the existence of the integral submanifolds on $S(E)$, we have to take a stratification since the singular set $S(E)$ is not a smooth submanifold of M in general. However we must be careful in the choice of the stratification, because if we take a stratification too much fine, then the space $E(p)$ is not always contained in the tangent space of the stratum at p. As a "good" stratification of $S(E)$, we adopt here the famous method of the *natural Whitney stratification* which is due to H.Whitney. For the generalities of the Whitney stratification, see [W].

Lemma 3.1 *Let $E(\subset \Theta_M)$ be a singular foliation on a complex manifold M and S an analytic subset of M. Suppose that $E(p) \subset C_pS$ holds for every point $p \in S$ (C_pS denotes the tangent cone of S at p). Let $\mathcal{S}$ be the natural Whitney stratification of S. Then we have $E(p) \subset T_pX$ for every point $p \in S$ where $X(\in \mathcal{S})$ is the stratum passing through p.*

We can prove this lemma using theorem (2.7) and the way of construction of the natural Whitney stratification. For the precise proof, we refer to [MY].

The following corollary is an immediate consequence from (2.7) and (3.1).

Corollary 3.2 *Let $E(\subset \Theta_M)$ be a singular foliation of rank r on a complex manifold M. Let k be an integer with $0 \leq k \leq r$ and $\mathcal{S}^{(k)}$ the natural Whitney stratification of $S^{(k)}$. Then for any stratum $X \in \mathcal{S}^{(k)}$ and each point $p \in X$ we have $E(p) \subset T_pX$.*

If we use this corollary, it is not so difficult to show the existence of the integral manifolds on the singular set of a singular foliation E.

Theorem 3.3 (Existence of Integral Submanifolds) *There exist integral submanifolds (whose dimensions are lower than r) also on $S(E)$. To be more precise, there is a family $\mathcal{L}$ of submanifolds of M such that $M = \bigcup_{L\in\mathcal{L}} L$ is a disjoint union and that any $L \in \mathcal{L}$ and $p \in L$, we have $E(p) = T_pL$.*

Proof. For each point $p \in M$, take the unique integer k such that $p \in L^{(k)}$ $(= S^{(k)} - S^{(k-1)})$. Let $\mathcal{S}^{(k)}$ be the natural Whitney stratification of $S^{(k)}$ and $X \in \mathcal{S}^{(k)}$ the unique stratum through p. Since $S^{(k-1)}$ is closed in M, $X - S^{(k-1)}$ has the structure of a complex manifold. Corollary (3.2) implies that E induces a non-singular foliation on $X - S^{(k-1)}$ (whose rank must be k). Therefore there exists a family $\mathcal{L}_X$ which consists of k-dimensional complex submanifolds of $X - S^{(k-1)}$ such that $X - S^{(k-1)} = \bigcup_{L\in\mathcal{L}_X} L$ is a disjoint union and that any $L \in \mathcal{L}_X$ and $q \in L$, we have $E(q) = T_qL$. Then it is obvious that

$$\mathcal{L} = \bigcup_{k=0}^{r} \bigcup_{\substack{X\in\mathcal{S}^{(k)} \\ X-S^{(k-1)}=\emptyset}} \mathcal{L}_X$$

is the family of submanifolds of M which satisfies the conditions in the theorem.

Q.E.D.

Each element L of $\mathcal{L}$ is called a *leaf* of E.

Thus, it turns out that M is the disjoint union of the leaves of E. Furthermore, we can show that the structure of a singular foliation E is locally analytically trivial along the leaf at each point p in M. This claim can be expressed precisely as follows in the case that E is reduced.

Theorem 3.4 (Local Analytical Triviality) *Let $E(\subset \Theta_M)$ be a reduced foliation of rank r on a complex manifold M. Let k be an integer with $0 \leq k \leq r$ and p a point in $L^{(k)}$ $(= S^{(k)} - S^{(k-1)})$. Then there exist a neighborhood D of 0 in $\mathbf{C}^{n-k}$, a singular foliation E' on D with $E'(0) = \{0\}$, a neighborhood U_p of p in M and a submersion $\pi : U_p \longrightarrow D$ with $\pi(p) = 0$ such that*

$$E|_{U_p} = \left(\pi^*(E'^a)\right)^a.$$

This theorem is proved by taking a sufficient small neighborhood U_p of p and constructing a good coordinates on U_p. To be more concrete, if we take a small coordinate neighborhood U_p of p and a good coordinates $(z_1, \ldots, z_n)$ on U_p, then $E|_{U_p}$ is generated by the following $k+s$ vector fields:

$$
\begin{aligned}
&\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_k}, \\
v_1 &= \sum_{i=k+1}^{n} a^1{}_i(z_{k+1}, \ldots, z_n)\frac{\partial}{\partial z_i}, \\
&\vdots \\
v_s &= \sum_{i=k+1}^{n} a^s{}_i(z_{k+1}, \ldots, z_n)\frac{\partial}{\partial z_i},
\end{aligned}
\tag{3.5}
$$

where each $a^j{}_i$ is a holomorphic function of $(n-k)$-variables. We refer to [MY] for the precise proof of theorem (3.4).

Remark 3.6 The fact that $E|_{U_p}$ is generated by the $k+s$ vector fields of the form (3.5) holds without assuming E is reduced ([C]). From [MY] and [Y], we have an independent proof of this in the reduced case (see prop (2.9) and the comments right before it).

Remark 3.7 Let us recall the singular foliation E on $\mathbf{C}^3$ given in example (2.5). For any point p of $L^{(1)}$, the leaf of E passing through p is $L^{(1)}$ itself. Theorem (3.4) tells us that E is locally analytically trivial at p along $L^{(1)}$. On the other hand, if we consider a point q of $L^{(0)} - \{0\}$, the leaf of E passing through q consists of one point q, so we cannot obtain any information from theorem (3.4) about the structure of singular foliation E near q. For the problem of the triviality along this type of singular set, see [Y].

As an application of theorem (3.4), we can show the following proposition.

Proposition 3.8 *If a singular foliation $E(\in \Theta_M)$ is reduced, then* $\operatorname{codim} S(E) \geq 2$.

Remark 3.9 For the converse of this proposition, we have counterexamples. However, under the assumption that E is locally free, the converse is also true (cf. remark (1.5)).

Proof of (3.8). Suppose that E is reduced and $\operatorname{codim} S(E) = 1$. Set $\dim_{\mathbf{C}} M = n$ and $\operatorname{rank} E = r$. First we choose a point $p \in S(E)$ such that $p \notin \operatorname{Sing}(S(E))$ and $\dim_p S(E) = n-1$. Take a sufficiently small neighborhood U of p and coordinates $(z_1, \ldots, z_n)$ on U such that $U \cap S(E) = \{z_n = 0\}$ and $p = (0, \ldots, 0)$. We set $k = \max\{\dim_{\mathbf{C}} E(q) \mid q \in U \cap S(E)\}$, then clearly $0 \leq k \leq r-1$.

Next, choose a point q in $U \cap S(E)$ such that $\dim_{\mathbf{C}} E(q) = k$. For simplicity, we 'shift' the coordinates $(z_1, \ldots, z_n)$ on U so that $q = (0, \ldots, 0)$. Since $S^{(k-1)}$ is a closed set, we can take a neighborhood $U_q(\subset U)$ of q so that $U_q \cap S^{(k-1)} = \emptyset$. Then we have $U_q \cap S(E) = U_q \cap L^{(k)}$. Applying theorem (3.4) (or (3.5)), we can retake U_q and $(z_1, \ldots, z_n)$ so that $E|_{U_q}$ is generated by the following $k+s$ holomorphic vector fields:

$$\tag{3.10} \begin{aligned} &\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_k}, \\ v_1 &= \sum_{i=k+1}^{n} a^1{}_i(z_{k+1}, \ldots, z_n)\frac{\partial}{\partial z_i}, \\ &\vdots \\ v_s &= \sum_{i=k+1}^{n} a^s{}_i(z_{k+1}, \ldots, z_n)\frac{\partial}{\partial z_i}. \end{aligned}$$

If $s = 0$ then E gives a non-singular foliation on U_q. This contradicts $q \in S(E)$, so we have $s \geq 1$. On the other hand, $U_q \cap S(E) = U_q \cap L^{(k)}$ implies that $\dim_{\mathbf{C}} E(x) = k$ holds for every point $x \in U_q \cap S(E)$, therefore all $a^j{}_i$ appearing in (3.10) satisfy $a^j{}_i(z_{k+1}, \ldots, z_{n-1}, 0) \equiv 0$. For $i = k+1, \ldots, n$, we represent $a^1{}_i$ as

$$a^1{}_i(z_{k+1}, \ldots, z_n) = z_n{}^{\alpha_i} \cdot b_i(z_{k+1}, \ldots, z_n)$$

where $\alpha_i \in \mathbf{Z}$ and b_i are holomorphic functions such that $b_i(z_{k+1}, \ldots, z_{n-1}, 0) \not\equiv 0$. Note that α_i and b_i are uniquely determined and $\alpha_i \geq 1$. We set $\alpha = \min\{\alpha_i\}$, and define a holomorphic vector field $\tilde{v_1}$ on U_q by

$$\tilde{v_1} = \sum_{i=k+1}^{n} z_n{}^{\alpha_i - \alpha}\, b_i(z_{k+1}, \ldots, z_n)\frac{\partial}{\partial z_i} \left(= \frac{1}{z_n{}^{\alpha}} v_1\right).$$

Then we have $\tilde{v_1}|_{U_q - S(E)} \in E|_{U_q - S(E)}$, but $\tilde{v_1} \notin E|_{U_q}$ since $\tilde{v_1} \not\equiv 0$. This contradicts that E is reduced.

Q.E.D.

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