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遅れを持つある微分方程式系の漸近安定性について

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In this paper, we study asymptotic stability of the zero solution of system

(1) $\dot{x}(t)=a\,x(t)+\mathrm{B}\,x(t-r), \qquad r>0,$

where B is an n×n matrix.

The necessary and sufficient condition for the zero solution of the scalar differential-difference equation

$$\dot{x}(t)=a\,x(t)+b\,x(t-r), \qquad r>0.$$

to be asymptotically stable is well-known. (See [1], [2].) Recently in [3], Hara and Sugie gave stability criteria for the system

$$\dot{x}(t)=\mathrm{B}\,x(t-r),$$

and also in [4], Godoy and dos Reis discussed stability for the 2-dimensional system

$$\dot{x}(t)=-\lambda\,x(t)+\lambda\,\mathrm{B}\,x(t-1).$$

Our purpose is to give a necessary and sufficient condition for the zero solution of system (1) to be asymptotically stable. It is an extension of the above results ([1]～[4]).

The zero solution of (1) is asymptotically stable if and only if all roots of

(2) $|\lambda\mathrm{I}-a\mathrm{I}-\mathrm{B}e^{-\lambda r}|=0$

have negative real parts ([2]). This characteristic equation is equivalent to

$$|(\lambda - a)e^{\lambda r} I - B| = 0.$$

Therefore, λ is a root of (2) if and only if λ is a root of

(3) $\qquad \lambda = a + (\alpha + \beta i)e^{-\lambda r},$

where $\alpha + \beta i$ are a eigenvalue of B. We can find a $\theta \in (-\pi, \pi]$ such that

$$\alpha + \beta i = b e^{i\theta}, \qquad b = \sqrt{\alpha^2 + \beta^2}.$$

So, equation (3) may be written as equation

(4) $\qquad \lambda = a + b e^{-\lambda r + i\theta}$

associated with θ. Then λ is a root of (4) if and only if the conjugate of λ is a root of the equation

$$\lambda = a + b e^{-\lambda r - i\theta}$$

associated with $-\theta$. Hence, for given $\theta \in [-\pi, \pi]$, all roots of equation (4) associated with θ have negative real parts if and only if all roots $\lambda = x + yi$ with $y \geqq 0$ of (4) associated with $\pm\theta$ have negative real parts. Therefore, in what follows, we consider only the roots with nonnegative imaginary parts.

We first discuss real roots of (4).

Lemma 1. Let $b > 0$. Then characteristic equation (4) has a real root only when $\theta = 0$ or $\theta = \pm\pi$, and the following hold.

(a) If $\theta = 0$, then (4) has one and only one real root. Moreover, this root is negative if and only if $a < -b$.

(b) If $\theta = \pm\pi$, then (4) has at most two real roots. Also, there are three cases as follows. First, (4) has no real root if and only if $e^{ar-1} < br$. Second, (4) has only negative roots if and only if $ar < br \leqq e^{ar-1} < 1$. Finally, (4) has a nonnegative root if and only if $1 < ar < br \leqq e^{ar-1}$ or

$a \geqq b$.

Proof. Suppose $\lambda = x + yi$ is a real root of (4). Then $y = 0$ and so (4) implies $b e^{-xr} \sin\theta = 0$, which yields either $\theta = 0$ or $\theta = \pm\pi$. First, we consider the case $\theta = 0$. Then (4) is reduced to

$$x = a + b e^{-xr}.$$

Put $h(t) = t - (a + b e^{-tr})$. Then $h(t)$ is a strictly increasing function from $(-\infty, \infty)$ onto $(-\infty, \infty)$, and hence (4) has only one real root. Since $h(0) = -(a + b)$, there exists a negative root of (4) if and only if $a + b < 0$. Second, we consider the case $\theta = \pm\pi$. Then, (4) is reduced to

$$x = a - b e^{-xr}.$$

Put $k(t) = t - (a - b e^{-tr})$. Then, since $k(t)$ is decreasing on $(-\infty, \frac{1}{r}\log(br))$ and increasing on $(\frac{1}{r}\log(br), \infty)$, (4) has at most two real roots. Also, $k(t)$ tends to ∞ as t $\to \pm\infty$. This implies that when $k(0) = b - a \leqq 0$, there exists a nonnegative root of (4). Since $k(t)$ attains its minimum $\frac{1}{r}\log\frac{br}{e^{ar-1}}$ at $t = \frac{1}{r}\log(br)$, if $ar < br \leqq e^{ar-1} < 1$, then $\frac{1}{r}\log(br) < 0$ and $\frac{1}{r}\log\frac{br}{e^{ar-1}} \leqq 0$, so that each real root of (4) is negative. If $1 < ar < br \leqq e^{ar-1}$, then (4) has a nonnegative root. If $e^{ar-1} < br$, then (4) has no real root. Now, noting that for each a, any one of $ar = e^{ar-1} = 1$, $ar < e^{ar-1} < 1$ or $1 < ar < e^{ar-1}$ holds, we have the conclusion of Lemma 1.

We next consider the distribution of roots of (4) with positive imaginary parts. In what follows, we assume $b > 0$ and introduce differentiable functions defined in $(0, \infty)$

$$f(\phi) = ar - \phi\cot(\phi - \theta)$$

and

$$g(\phi) = \log \frac{-br\sin(\phi - \theta)}{\phi} \quad .$$

Put $x = \frac{1}{r}f(\phi)$ and $y = \frac{1}{r}\phi$. Then x and y fulfill the system of equations

$$x = a + be^{-xr}\cos(yr - \theta),$$

$$y = \quad -be^{-xr}\sin(yr - \theta),$$

if and only if there exists a ϕ such that $f(\phi) = g(\phi)$, because

$$xr = \log \frac{-br\sin(yr - \theta)}{yr}$$

for $y > 0$. Therefore the following remark holds.

Remark. Let $\lambda = \frac{1}{r}f(\phi) + i\frac{1}{r}\phi$ and $\phi > 0$. Then λ is a root of (4) if and only if $f(\phi) = g(\phi)$.

We need to find the domain D on which $f(\phi)$ and $g(\phi)$ are both defined. Put, when $0 < \theta \leqq \pi$,

$$I_0(\theta) = (0, \theta),$$

$$I_n(\theta) = (\theta + (2n-1)\pi, \ \theta + 2n\pi),$$

and when $-\pi \leqq \theta \leqq 0$,

$$I_0(\theta) = (\theta + \pi, \theta + 2\pi),$$

$$I_n(\theta) = (\theta + (2n+1)\pi, \ \theta + (2n+2)\pi),$$

where n is any positive integer. Then it is easy to see that $g(\phi)$ is defined only on $\bigcup_{n=0}^{\infty} I_n(\theta)$, and so $D = \bigcup_{n=0}^{\infty} I_n(\theta)$. In what follows, we denote the set $\{\phi \in I_n(\theta) \mid f(\phi) = 0\}$ by $Z_n(f, \theta)$ and each $\phi \in Z_n(f, \theta)$ by ϕ_n. Also, for the sake of convenience, we may write (c_n, d_n) instead of $I_n(\theta)$.

Lemma 2. Let ϕ^* be a constant in $(0, \theta - \frac{\pi}{2})$, determined by $2\phi^* = \sin2(\phi^* - \theta)$. Then the following hold.

(a) If any one of (i) $0<\theta\leqq\frac{\pi}{2}$ and $a\geqq 0$, (ii) $\frac{\pi}{2}<\theta<\pi$ and $ar>\cos^2(\phi^*-\theta)$, or (iii) $\theta=\pm\pi$ and $ar\geqq 1$ holds, then $Z_0(f,\theta)$ is empty.

(b) If $\frac{\pi}{2}<\theta<\pi$ and $0<ar<\cos^2(\phi^*-\theta)$, then $Z_0(f,\theta)$ contains just two elements.

(c) Except for the above cases, $Z_0(f,\theta)$ is a singleton.

(d) For any positive integer n, $Z_n(f,\theta)$ is a singleton.

Proof Suppose that $-\pi<\theta\leqq 0$ and $n=0$, or that $-\pi\leqq\theta\leqq\pi$ and n is any positive integer. Then $f(\phi)$ is a strictly increasing function from $I_n(\theta)$ onto $(-\infty,\infty)$, and so $Z_n(f,\theta)$ is a singleton. Next, suppose $0<\theta\leqq\frac{\pi}{2}$. Since $f(0)=ar$, $f(\phi)$ is a strictly increasing function from $I_0(\theta)=(0,\theta)$ onto (ar,∞). Hence, if $ar\geqq 0$, then $Z_0(f,\theta)$ is empty. On the other hand, if $ar<0$, then $Z_0(f,\theta)$ is a singleton. When $\frac{\pi}{2}<\theta<\pi$, an elementary calculation shows

$$\min_{0<\phi<\theta} f(\phi)=ar-\cos^2(\phi^*-\theta).$$

Since $f(0)=ar$, and since $f(\phi)$ is strictly decreasing on $(0,\phi^*]$ and strictly increasing on $[\phi^*,\theta)$, the following ((a) ~ (c)) are satisfied:

(a) If $ar>\cos^2(\phi^*-\theta)$, then $Z_0(f,\theta)$ is empty.

(b) If $0<ar<\cos^2(\phi^*-\theta)$, then $Z_0(f,\theta)$ contains two elements.

(c) If $ar\leqq 0$ or $ar=\cos^2(\phi^*-\theta)$, then $Z_0(f,\theta)$ is a singleton.

Finally, suppose $\theta=\pm\pi$. Then $f(\phi)$ tends to $ar-1$ as $\phi\to+0$. Since $f(\phi)$ is a strictly increasing function from $I_0(\theta)=(0,\pi)$ onto $(ar-1,\infty)$, if $ar\geqq 1$, then $Z_0(f,\theta)$ is empty. Also, if $ar<1$, then $Z_0(f,\theta)$ is a singleton. Thus the proof is completed.

Lemma 3. For every nonnegative integer n, $I_n(\theta)$ contains one and only one ϕ such that $f(\phi) = g(\phi)$, except for the case that $\theta = \pm\pi$, n=0 and $br \leqq e^{ar-1}$ hold.

Proof. We shall divide the proof by three cases. Case I: $0 < \theta < \pi$ and n = 0. Since $f(0) = ar$ and $f(\phi) \to \infty$ as $\phi \to -0$, and since $g(\phi)$ is a strictly decreasing function from $(0, \theta)$ onto $(-\infty, \infty)$, there exists a $\bar{\phi} \in I_0(\theta)$ such that

$$f(\bar{\phi}) = g(\bar{\phi}).$$

Case II: $\theta = \pm\pi$ and n=0. Let $\tilde{g}(\phi)$ be the numerator of $g'(\phi) = \frac{\phi\cot\phi - 1}{\phi}$. Then $\tilde{g}(\phi)$ tends to 0 as $\phi \to +0$, and it's derivative is negative on $(0, \pi)$. Hence $g'(\phi) < 0$ on $(0, \pi)$, and so $g(\phi)$ is a strictly decreasing function from $I_0(\theta) = (0, \pi)$ onto $(-\infty, \log br)$. Since $f(\phi)$ is an increasing function from $(0, \pi)$ onto $(ar - 1, \infty)$, there exists a $\bar{\phi} \in I_0(\theta)$ such that $f(\bar{\phi}) = g(\bar{\phi})$ if and only if

$$br > e^{ar-1}.$$

Case III: $n \in N$, or $-\pi < \theta \leqq 0$ and n=0. It is easy to show that

$$\frac{g(\phi)}{f(\phi)} \to 0 \quad \text{as } \phi \to c_n + 0,$$

where $I_n(\theta) = (c_n, d_n)$, and so there exists a $\delta_n > 0$ such that

$$f(\phi) < g(\phi) < 0 \qquad \text{on } (c_n, c_n + \delta_n).$$

Since $f(\phi) \to \infty$ and $g(\phi) \to -\infty$ as $\phi \to d_n - 0$, there exists a $\bar{\phi} \in I_n(\theta)$ such that

$$f(\bar{\phi}) = g(\bar{\phi}).$$

Finally, it will be proved that $I_n(\theta)$ contains only one $\bar{\phi}$ such that $f(\bar{\phi}) = g(\bar{\phi})$. Differentiating $f(\phi)$ and $g(\phi)$, we have

$$f'(\phi) - g'(\phi) = \frac{\phi^2 - \phi\sin 2(\phi - \theta) + \sin^2(\phi - \theta)}{\phi\sin^2(\phi - \theta)}.$$

Put $F(\phi)=\phi^2-\phi\sin 2(\phi-\theta)+\sin^2(\phi-\theta)$. Then

$$F'(\phi)=2\phi\{1-\cos 2(\phi-\theta)\}>0 \quad \text{on } I_n(\theta),$$

and so $F(\phi)$ is strictly increasing on $I_n(\theta)$. Since $F(0)=0$ and since

$$F(\theta+(2n+1)\pi)=\{\theta+(2n+1)\pi\}^2\geqq 0$$

for any n, it follows that

$$f'(\phi)-g'(\phi)>0 \quad \text{on } I_n(\theta).$$

Therefore, for every nonnegative integer n, $f(\phi)-g(\phi)$ is strictly increasing on $I_n(\theta)$. This implies that $f(\phi)-g(\phi)$ vanishes at only one ϕ in $I_n(\theta)$. Thus the proof is completed.

Lemma 4. Let $f(\bar{\phi})=g(\bar{\phi})$ and $\bar{\phi}\in I_n(\theta)$, $n\geqq 0$. Assume that either of the following conditions (a) or (b) holds:

(a) $Z_0(f,\theta)$ contains a ϕ_0 such that $g(\phi_0)<0$ and $f(\phi)<0$ on (c_0,ϕ_0), where $I_0(\theta)=(c_0,d_0)$.

(b) $Z_0(f,\theta)$ contains two elements ϕ_0,ϕ_0' such that $\phi_0'<\phi_0$ and $g(\phi_0)<0<g(\phi_0')$.

Then $f(\bar{\phi})<0$. Therefore, all imaginary roots of (4) associated with θ have negative real parts.

Proof. Suppose there exists a $\bar{\phi}\in I_0(\theta)$ such that $f(\bar{\phi})=g(\bar{\phi})$. By Lemma 3, such a $\bar{\phi}$ is unique. According to the proof of Lemma 3, the function $f(\phi)-g(\phi)$ is strictly increasing on $I_0(\theta)$, and so $\bar{\phi}<\phi_0$, because $f(\phi_0)-g(\phi_0)>0$. If (a) holds, then it is obvious that $f(\bar{\phi})<0$. On the other hand, if (b) holds, then it follows from Lemma 2 that $\frac{\pi}{2}<\theta<\pi$ and $0<ar<\cos^2(\phi^*-\theta)$. Since $g(\phi_0')>0$ for some $\phi_0'\in(c_0,\phi_0)$, clearly

$$f(\phi_0')-g(\phi_0')<0<f(\phi_0)-g(\phi_0)$$

and hence

$$\phi_0' < \bar{\phi} < \phi_0.$$

Then it is easily seen that $f(\bar{\phi})<0$. Thus the conclusion of Lemma 4 is valid for n=0. In order to show that $f(\bar{\phi})<0$ for $n \geqq 1$, we first consider the case $a \neq 0$. Since

$$f(\phi_0+2n\pi)=f(\phi_0)-2n\pi\cot(\phi_0-\theta)$$
$$=-2n\pi\cot(\phi_0-\theta),$$

it follows that

$$f(\phi_0+2n\pi)>0 \quad \text{when } \phi_0\in(\theta-\frac{\pi}{2},\theta)\cup(\theta+\frac{3}{2}\pi,\theta+2\pi)$$

and

$$f(\phi_0+2n\pi)<0 \quad \text{when } \phi_0\in(0,\theta-\frac{\pi}{2})\cup(\theta+\pi,\theta+\frac{3}{2}\pi).$$

Let $a<0$. When $0<\theta\leqq\pi$, according to the proof of Lemma 2,

$$f(\phi)<0 \quad \text{on } (0,\phi_0)$$

and

$$f(\phi)>0 \quad \text{on } (\phi_0,\theta),$$

and so

$$\theta-\frac{\pi}{2}<\phi_0<\theta,$$

which follows from

$$f(\theta-\frac{\pi}{2})=ar<0.$$

When $-\pi\leqq\theta\leqq 0$, since

$$f(\phi)<0 \qquad \text{on } (\theta+\pi,\phi_0)$$

and

$$f(\phi)>0 \qquad \text{on } (\phi_0,\theta+2\pi),$$

and since $f(\theta+\frac{3}{2}\pi)=ar$, it follows that

$$\theta+\frac{3}{2}\pi<\phi_0<\theta+2\pi.$$

Thus, for any $\theta\in[-\pi,\pi]$ and any positive integer n,

$$f(\theta-\frac{\pi}{2}+2n\pi)=ar<0<f(\phi_0+2n\pi),$$

and hence

$$\theta - \frac{\pi}{2} + 2n\pi < \phi_n < \phi_0 + 2n\pi .$$

This implies

$$0 < \cos(\phi_n - \theta) < \cos(\phi_0 - \theta).$$

On the other hand, letting $a > 0$, we have

$$0 < \phi_0 < \theta - \frac{\pi}{2} \qquad \text{when } 0 < \theta \leqq \pi$$

and

$$\theta + \pi < \phi_0 < \theta + \frac{3}{2}\pi \qquad \text{when } -\pi \leqq \theta \leqq 0.$$

Hence, when $0 < \theta \leqq \pi$, for any positive integer n,

$$f(\phi_0 + 2n\pi) < f(\phi_n) < f(\theta - \frac{\pi}{2} + 2n\pi)$$

and so

$$\phi_0 + 2n\pi < \phi_n < \theta - \frac{\pi}{2} + 2n\pi .$$

This implies

$$\cos(\phi_0 - \theta) < \cos(\phi_n - \theta) < 0,$$

which is valid also when $-\pi \leqq \theta \leqq 0$. Thus, for $a \neq 0$, since

$\phi_n \cot(\phi_n - \theta) = a r$,

$$\begin{aligned} g(\phi_n) &= \log \frac{-b\cos(\phi_n - \theta)}{a} \\ &< \log \frac{-b\cos(\phi_0 - \theta)}{a} \\ &= g(\phi_0) < 0 \end{aligned}$$

and hence

(5) $\qquad g(\phi_n) < f(\phi_n).$

We next consider the case $a = 0$. From Lemma 2, if $0 < \theta \leqq \frac{\pi}{2}$, then $Z_0(f, \theta)$ is empty. So, let $\frac{\pi}{2} < \theta \leqq \pi$. Since

$$f(\theta - \frac{\pi}{2}) = f(\theta - \frac{\pi}{2} + 2n\pi) = a r = 0,$$

the equalities

$$\phi_0 = \theta - \frac{\pi}{2}, \qquad \phi_n = \theta - \frac{\pi}{2} + 2n\pi$$

hold. Then clearly,

$$g(\phi_n) = \log \frac{br}{\phi_n} < \log \frac{br}{\phi_0} = g(\phi_0) < 0,$$

that is,

$$g(\phi_n) < f(\phi_n).$$

Similarly, it is seen that (5) is fulfilled when $-\pi \leqq \theta \leqq 0$. Thus, (5) is valid for any real a. On the other hand, there exists a $\delta_n > 0$ such that

$$f(\phi) < g(\phi) \qquad \text{on } (c_n, c_n + \delta_n).$$

This, together with (5), implies that there exists a $\bar{\phi} \in (c_n, \phi_n)$ such that

$$f(\bar{\phi}) = g(\bar{\phi}) < 0,$$

because $f(\phi) < 0$ for $\phi \in (c_n, \phi_n)$. Now the proof is completed.

The above lemmas verify the following theorem.

Theorem. All roots of (4) have negative real parts if and only if any one of the following conditions holds.

(a) $a < 0$ and $b = 0$.

(b) $\theta = 0$ and $a < -b < 0$.

(c) $0 < \theta \leqq \pi$, $a < 0 < b$ and $b\cos(\phi - \theta) < |a|$ for ϕ such that $\phi\cot(\phi - \theta) = ar$ and $\max\{0, \theta - \frac{\pi}{2}\} < \phi < \theta$.

(d) $\frac{\pi}{2} < \theta \leqq \pi$ and $a = 0 < br < \theta - \frac{\pi}{2}$.

(e) $\theta = \pi$, $0 < ar < \min\{br, 1\}$ and $b\cos\phi < a$ for ϕ such that $\phi\cot\phi = ar$ and $0 < \phi < \frac{\pi}{2}$.

(f) $\frac{\pi}{2} < \theta < \pi$, $0 < ar < \cos^2(\phi^* - \theta)$ and $-b\cos(\phi - \theta) < a < -b\cos(\phi' - \theta)$ for ϕ and ϕ' such that $\phi\cot(\phi - \theta) = \phi'\cot(\phi' - \theta) = ar$ and $0 < \phi' < \phi < \theta - \frac{\pi}{2}$, where ϕ^* is the same one as in Lemma 2.

Proof. (Necessity) Suppose all roots of (4) have negative real parts. We first show that $Z_0(f,\theta)$ contains a ϕ_0 satisfying

(6) $g(\phi_0) < 0,$

whenever $Z_0(f,\theta)$ is not empty. If it is false, then

$$g(\phi_0) \geqq 0 \qquad \text{for } \phi_0 = \max\{\phi \mid \phi \in Z_0(f,\theta)\}.$$

Since $f(\phi)$ is strictly increasing on (ϕ_0, d_0), and since $g(\phi) \to -\infty$ as $\phi \to d_0 - 0$, $I_0(\theta)$ contains a $\bar{\phi}$ such that

$$f(\bar{\phi}) = g(\bar{\phi}) \geqq 0.$$

This implies that there exists a root of (4) with nonnegative real part, a contradiction. Therefore, there exists a $\phi_0 \in Z_0(f,\theta)$ which satisfies (6). For such a ϕ_0, clearly

$$\phi_0 \cot(\phi_0 - \theta) = ar,$$

and so

$$g(\phi_0) = \log \frac{-b\cos(\phi_0 - \theta)}{a},$$

whenever $a \neq 0$. From the above, if $Z_0(f,\theta)$ is not empty, then

(7) $$\log \frac{-b\cos(\phi_0 - \theta)}{a} < 0.$$

Now we divide the proof by six cases as follows. Case I : $b = 0$. It is trivial that $a < 0$. Case II : $\theta = 0$ and $b > 0$. Lemma 1 implies $a < -b < 0$. Case III : $0 < \theta \leqq \pi$ and $a < 0 < b$. It follows from Lemma 2 that $Z_0(f,\theta)$ is not empty, and hence from (7)

$$b\cos(\phi_0 - \theta) < |a|.$$

Also, the proof of Lemma 4 shows that ϕ_0 belongs to the interval $(\max\{0, \theta - \frac{\pi}{2}\}, \theta)$. Case IV : $0 < \theta \leqq \pi$ and $a = 0 < b$. If $0 < \theta \leqq \frac{\pi}{2}$, then $f(\phi) > 0$ for $\phi \in I_0(\theta)$. Hence, from Lemma 3, there exists a $\bar{\phi} \in I_0(\theta)$ such that

$$f(\bar{\phi}) = g(\bar{\phi}) \geqq 0,$$

which is a contradiction. Thus θ must belong to $(\frac{\pi}{2}, \pi]$. Then it follows from Lemma 2 that $Z_0(f, \theta)$ is a singleton. Also, it is clear from the assumption on a that $Z_0(f, \theta) = \{\theta - \frac{\pi}{2}\}$. This and (6) imply

$$\log \frac{br}{\theta - \frac{\pi}{2}} < 0,$$

and hence

$$br < \theta - \frac{\pi}{2}.$$

Case V: $\theta = \pi$, $a > 0$ and $b > 0$. Let $br > e^{ar-1}$. Then by Lemma 3, there exists a $\bar{\phi} \in I_0(\pi) = (0, \pi)$ such that

$$f(\bar{\phi}) = g(\bar{\phi}).$$

Since all roots have negative real parts, $f(\bar{\phi}) < 0$. Then, since $f(\phi) \to \infty$ as $\phi \to \pi - 0$, $Z_0(f, \theta)$ is not empty. Hence

$$\frac{b\cos\phi_0}{a} = \frac{-b\cos(\phi_0 - \pi)}{a} < 1$$

and so

$$b\cos\phi_0 < a,$$

where $0 < \phi_0 < \frac{\pi}{2}$. Moreover, the inequality $ar < 1$ follows from Lemma 2. Now, let $br \leqq e^{ar-1}$. Then, by Lemma 1, (4) has real roots. Since these roots must be negative, the inequalities

$$ar < br \leqq e^{ar-1} < 1$$

hold. On the other hand, since

$$ar = \frac{\phi_0}{\sin\phi_0}\cos\phi_0 > \cos\phi_0$$

it follows that

$$b\cos\phi_0 < \frac{\cos\phi_0}{r} < a.$$

Case VI: $0 < \theta < \pi$, $a > 0$ and $b > 0$. By Lemma 3, there exists a $\bar{\phi} \in I_0(\theta)$ such that

$$f(\bar{\phi}) = g(\bar{\phi}).$$

If $0 < \theta \leqq \frac{\pi}{2}$, then $f(\phi) > 0$ on $I_0(\theta) = (0, \theta)$. Hence

$$f(\bar{\phi}) = g(\bar{\phi}) > 0,$$

a contradiction. So, θ must belong to $(\frac{\pi}{2}, \pi)$. For such a θ, if $ar \geqq \cos^2(\phi^* - \theta)$, then $f(\phi) \geqq 0$ on $I_0(\theta)$. Hence there exists a $\bar{\phi} \in I_0(\theta)$ such that

$$f(\bar{\phi}) = g(\bar{\phi}) \geqq 0,$$

a contradiction. Thus, $ar < \cos^2(\phi^* - \theta)$. Then, from Lemma 2, $Z_0(f, \theta)$ cotains two elements ϕ_0 and ϕ_0', and they satisfy

$$0 < \phi_0' < \phi_0 < \theta - \frac{\pi}{2}.$$

Since $f(\phi) > 0$ on $(0, \phi_0') \cup (\phi_0, \theta)$, and since $g(\phi)$ is strictly decreasing on $I_0(\theta)$, it follows that

$$g(\phi_0) < 0 < g(\phi_0').$$

This implies

$$\log \frac{-br\sin(\phi_0 - \theta)}{\phi_0} < 0 < \log \frac{-br\sin(\phi_0' - \theta)}{\phi_0'},$$

that is

$$-b\cos(\phi_0 - \theta) < a < -b\cos(\phi_0' - \theta).$$

Thus, the proof of necessity is completed.

(Sufficiency) Suppose any one of conditions (a) through (f) holds. When (a) holds, $\lambda = a < 0$ is the unique root of (4). When (b) holds, by Lemma 1, real root of (4) is negative. On the other hand, according to Lemma 2, $Z_0(f, \theta)$ has only one $\phi_0 \in I_0(0) = (\pi, 2\pi)$. Since $b\cos\phi_0 < b < |a|$,

$$g(\phi_0) = \log \frac{-b\cos\phi_0}{a} < 0.$$

Also, it is clear that $f(\phi) < 0$ on (π, ϕ_0). Hence Lemma 4 assures that all roots of (4) have negative real parts. When (c) holds, by Lemma 2, $Z_0(f, \theta)$ and $Z_0(f, -\theta)$ are both singletons. Since

$$\theta - \frac{\pi}{2} < \phi_0 < \theta \qquad \text{for } \phi_0 \in Z_0(f, \theta),$$

it is obvious that

$$(8) \qquad g(\phi_0) = \log\frac{-b\cos(\phi_0 - \theta)}{a} < 0.$$

Let $Z_0(f, -\theta) = \{\hat{\phi}_0\}$. Then it follows that

$$\hat{\phi}_0 = ar\tan(\hat{\phi}_0 + \theta)$$

and of course

$$\phi_0 = ar\tan(\phi_0 - \theta).$$

Now, note that

$$\frac{3}{2}\pi - \theta < \hat{\phi}_0 < 2\pi - \theta$$

or

$$-\frac{\pi}{2} < \hat{\phi}_0 + \theta - 2\pi < 0,$$

and consider the zeros of the functions $\phi - ar\tan(\phi - \theta)$ and $\phi - ar\tan(\phi + \theta)$. Then it is easily seen that

$$-\frac{\pi}{2} < \hat{\phi}_0 + \theta - 2\pi < \phi_0 - \theta < 0,$$

which yields

$$b\cos(\hat{\phi}_0 + \theta) < b\cos(\phi_0 - \theta).$$

This implies

$$b\cos(\hat{\phi}_0 + \theta) < |a|,$$

so that the inequality

$$g(\hat{\phi}_0) < 0$$

holds. Since $f(\phi) < 0$ for $\phi < \phi_0$ in $I_0(\theta)$ and for $\phi < \hat{\phi}_0$ in $I_0(-\theta)$, it follows from Lemma 4 that all imaginary roots of (4) associated with $\pm\theta$ have negative real parts. On the other hand, from Lemma 1, (4) has real roots only when $\theta = \pi$ and

$$br \leqq e^{ar-1}$$

hold. Then, since $a < 0$,

$$ar < br \leqq e^{ar-1} < 1.$$

Hence Lemma 1 assures that real roots of (4) are negative. When (d) holds, according to Lemma 2, $Z_0(f, \pm\theta)$ are both singletons. Also,

$$\phi_0 = \theta - \frac{\pi}{2} \quad \text{for } \phi_0 \in Z_0(f, \theta)$$

and

$$\phi_0 = \frac{3}{2}\pi - \theta \quad \text{for } \phi_0 \in Z_0(f, -\theta).$$

Since $\theta - \frac{\pi}{2} \leqq \frac{\pi}{2} \leqq \frac{3}{2}\pi - \theta$, it follows that

$$br < \theta - \frac{\pi}{2} \leqq \frac{3}{2}\pi - \theta,$$

which yields

$$g(\phi_0) = \log \frac{br}{\phi_0} < 0$$

for $\phi_0 \in Z_0(f, \pm\theta)$. Moreover, it is clear that $f(\phi) < 0$ on (c_0, ϕ_0). Hence all imaginary roots of (4) associated with $\pm\theta$ have negative real parts. On the other hand, from Lemma 1, (4) has real roots only when $\theta = \pi$ and

$$br \leqq e^{ar-1}$$

hold. Then, since $a = 0$,.

$$br \leqq e^{ar-1} < 1,$$

and so real roots of (4) are negative. When (e) holds, by Lemma 2, $Z_0(f, \pm\pi)$ are singletons and

$$\phi_0 \in (0, \frac{\pi}{2}) \quad \text{for } \phi_0 \in Z_0(f, \pm\pi),$$

because $f(\frac{\pi}{2}) > 0$. Hence

$$g(\phi_0) = \log \frac{b\cos\phi_0}{a} < 0.$$

This implies that all imaginary roots of (4) associated with $\pm\pi$ have negative real parts. On the other hand, if there exist real roots of (4), then

$$br \leqq e^{ar-1}$$

follows from Lemma 1, and hence

$$ar < br \leqq e^{ar-1} < 1,$$

because $ar < 1$. It follows again from Lemma 1 that all real roots of (4) are negative. Finally, suppose (f) holds. Then from Lemma 2, $Z_0(f, \theta)$ contains two elements ϕ_0 and ϕ_0' with

$$0 < \phi_0' < \phi_0 < \theta - \frac{\pi}{2}.$$

Hence (8) and

$$g(\phi_0') = \log \frac{-b\cos(\phi_0' - \theta)}{a} > 0$$

hold. Then, Lemma 4 assures that all imaginary roots of (4) associated with θ have negative real parts. On the other hand, it follows from Lemma 2 that $Z_0(f, -\theta)$ contains only one $\hat{\phi}_0$ which satisfies

$$\pi - \theta < \hat{\phi}_0 < \frac{3}{2}\pi - \theta.$$

In the analogous way to the case of (c),

$$0 < \phi_0 - (\theta - \pi) < \hat{\phi}_0 + (\theta - \pi) < \frac{\pi}{2},$$

and so

$$\begin{aligned} -\cos(\hat{\phi}_0 + \theta) &= \cos(\hat{\phi}_0 + \theta - \pi) \\ &< \cos(\phi_0 - \theta + \pi) \\ &= -\cos(\phi_0 - \theta), \end{aligned}$$

which implies

$$-b\cos(\hat{\phi}_0 + \theta) < -b\cos(\phi_0 - \theta) < a.$$

Then it is easy to show the inequality

$$g(\hat{\phi}_0) < 0.$$

Thus, it follows from Lemma 4 that all imaginary roots of (4) associated with $\pm\theta$ have negative real parts. From Lemma 1, it is clear that (4) has no real root. Now the proof is completed.

The following result is an immediate consequence of Theorem.

Corollary. The zero solution of system (1) is asymptotically stable if and only if every eigenvalue $b\, e^{i\theta}$ of B with $\theta \geqq 0$ satisfies one of conditions (a) through (f) in Theorem.

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