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A GENERALIZATION OF PROCESSES AND STABILITIES IN ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction

The concept of processes discussed in [1–3] and [9] is a useful tool in the study of mathematical analysis for some phenomena whose dynamics is described by equations that contain the derivative with respect to the time variable. Indeed, Hale [3] derived some stability properties for processes and applied those to get stability results for some kinds of equations, including functional differential equations, partial differential equations and evolution equations. As will be seen by an example in Section 3, however, the concept of processes does not fit in with the study of ρ-stability in the stability problems for functional differential equations with infinite delay. Here ρ-stability means that the solution remains small with respect to the metric ρ, which induces the compact open topology, if the initial function is small with respect to ρ. The ρ-stability is a useful tool in the study of the existence of almost periodic solutions for almost periodic systems [13].

To overcome the difficulty stated above, in this paper we generalize the concept of processes and get a more extended concept which is called quasi-processes. In Section 2, we discuss some stability properties for quasi-processes and obtain some equivalence relations concerning with stabilities for quasi-processes in connection with those for "limiting" quasi-processes (Theorem 1), which is a generalization of [3, Theorem 3.7.4]. In Section 3, we treat abstract functional differential equations with infinite delay defined on some fading memory space $\mathcal{B}$ and, corresponding to $\mathcal{B}$-metric and ρ-metric, we construct

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processes and quasi-processes associated with functional differential equations, respectively. Finally, establishing that processes and quasi-processes fit in with the analysis of $\mathcal{B}$-stability problems and ρ-stability problems, respectively, we obtain some equivalence relationships between these stabilities for the original equation and those for limiting equations (Theorems 2 and 3).

2. Quasi-processes and some stability properties

In this section, we shall introduce the concept of quasi-processes which is a generalization of processes and deduce some results on stability properties for quasi-processes. Suppose $\mathcal{X}$ is a metric space with metric d and let $w : R^+ \times R^+ \times \mathcal{X} \mapsto \mathcal{X}, R^+ := [0, \infty)$, be a function satisfying the following properties for all $t, \tau, s \in R^+$ and $x \in \mathcal{X}$:

(p1) $w(0, s, x) = x$;

(p2) $w(t + \tau, s, x) = w(t, \tau + s, w(\tau, s, x))$.

Let $\mathcal{Y}$ be a nonempty closed set in $\mathcal{X}$. We call the mapping w *a $\mathcal{Y}$-quasi-process* on $\mathcal{X}$ or simply *a quasi-process* on $\mathcal{X}$, if w satisfies the condition,

(p3) the restricted mapping $w : R^+ \times R^+ \times \mathcal{Y} \mapsto \mathcal{X}$ is continuous,

together with the conditions (p1) and (p2). In case of $\mathcal{Y} = \mathcal{X}$, the concept of quasi-processes is identical with that of processes investigated in [1–3] and [9]. We emphasize that the concept of processes does not fit in with the study of ρ-stabilities in functional differential equations in contrast with the concept of quasi-processes, as will be seen in the next section.

Denote by W the set of all quasi-processes on $\mathcal{X}$. For $\tau \in R^+$ and $w \in W$, we define the translation $\sigma(\tau)w$ of w by

$$(\sigma(\tau)w)(t, s, x) = w(t, \tau + s, x), \qquad (t, s, x) \in R^+ \times R^+ \times \mathcal{X},$$

and set $\gamma_\sigma^+(w) = \bigcup_{t \geq 0} \sigma(t)w$. Clearly $\gamma_\sigma^+(w) \subset W$. We denote by $H_\sigma(w)$ all functions $\chi : R^+ \times R^+ \times \mathcal{X} \mapsto \mathcal{X}$ such that for some sequence $\{\tau_n\} \subset R^+$, $\{\sigma(\tau_n)w\}$ converges to χ pointwise on $R^+ \times R^+ \times \mathcal{X}$, that is, $\lim_{n\to\infty}(\sigma(\tau_n)w)(t, s, x) = \chi(t, s, x)$ for any $(t, s, x) \in R^+ \times R^+ \times \mathcal{X}$. The set $H_\sigma(w)$ is considered as a topological space with the pointwise convergence, and it is called the hull of w.

Consider a $\mathcal{Y}$-quasi-process w on $\mathcal{X}$ satisfying

(p4) $\qquad H_\sigma(w) \subset W.$

Clearly, $H_\sigma(w)$ is invariant with respect to the tanslation $\sigma(\tau), \tau \in R^+$. For any $t \in R^+$, we consider a function $\pi(t) : \mathcal{X} \times H_\sigma(w) \mapsto \mathcal{X} \times H_\sigma(w)$ defined by

$$\pi(t)(x, \chi) = (\chi(t, 0, x), \sigma(t)\chi)$$

for $(x, \chi) \in \mathcal{X} \times H_\sigma(w)$. $\pi(t)$ is called *the skew product flow of the quasi-process* w, if the following property holds true:

(p5) $\pi(t)(y, \chi)$ is continuous in $(t, y, \chi) \in R^+ \times \mathcal{Y} \times H_\sigma(w)$.

The skew product flow $\pi(t)$ is said to be $\mathcal{Y}$ *-strongly asymptotically smooth* if, for any nonempty, closed, bounded set $B \subset \mathcal{Y} \times H_\sigma(w)$, there exists a compact set $J \subset \mathcal{Y} \times H_\sigma(w)$ with the property that $\{\pi(t_n)(y_n, \chi_n)\}$ has a subsequence which approaches to J whenever sequences $\{t_n\} \subset R^+$ and $\{(y_n, \chi_n)\} \subset B$ satisfy $\lim_{n\to\infty} t_n = \infty$ and $\pi(t)(y_n, \chi_n) \in B$ for all $t \in [0, t_n]$. In case of $\mathcal{Y} = \mathcal{X}$, the $\mathcal{Y}$-strong asymptotic smoothness of $\pi(t)$ implies the asymptotic smoothness of $\pi(t)$ introduced in [2–3]. In the next section, we shall see that the $\mathcal{Y}$-strong asymptotic smoothness of $\pi(t)$ is ensured when w is a quasi-process generated by some functional differential equations.

Now we suppose that $H_\sigma(w)$ is sequentially compact and we discuss relationships between some stability properties for the quasi-process w and those for the "limiting" quasi-processes $\chi \in \Omega_\sigma(w)$; here $\Omega_\sigma(w)$ denotes the ω-limit set of w with respect to the translation semigroup $\sigma(t)$. A continuous function $\mu : R^+ \mapsto \mathcal{Y}$ is called *an integral* of the quasi-process w on R^+, if $w(t, s, \mu(s)) = \mu(t+s)$ for all $t, s \in R^+$ (cf. [2, p.80]). In the following, we suppose that there exists an integral $\mu : R^+ \mapsto \mathcal{Y}$ of the quasi-process w on R^+ such that the set $O^+(\mu) = \{\mu(t) : t \in R^+\}$ is contained in $\mathcal{Y}$ and is relatively compact in $\mathcal{X}$. From (p5) we see that $\pi(\delta)(\mu(s), \sigma(s)w) = (w(\delta, s, \mu(s)), \sigma(s+\delta)w) = (\mu(s+\delta), \sigma(s+\delta)w)$ tends to $(\mu(s), \sigma(s)w)$ as $\delta \to 0^+$ uniformly for $s \in R^+$; consequently, the integral μ must be uniformly continuous on R^+. From Ascoli-Arzéla's theorem and the sequential compactness of $H_\sigma(w)$, it follows that for any sequence $\{\tau_n'\} \subset R^+$, there exist a subsequence $\{\tau_n\}$ of $\{\tau_n'\}$, a $\chi \in H_\sigma(w)$ and a function $\nu : R^+ \mapsto \mathcal{Y}$ such that $\lim_{n\to\infty} \sigma(\tau_n)w = \chi$ and $\lim_{n\to\infty} \mu(t+\tau_n) = \nu(t)$ uniformly on any compact interval in R^+. In this case, we write as

$$(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, \chi) \text{ compactly,}$$

for simplicity. Denote by $H_\sigma(\mu, w)$ the set of all (ν, χ) such that $(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, \chi)$ compactly for some sequence $\{\tau_n\} \subset R^+$. In particular, we denote by $\Omega_\sigma(\mu, w)$ the set of all $(\nu, \chi) \in H_\sigma(\mu, w)$ for which one can choose a sequence $\{\tau_n\} \subset R^+$ so that $\lim_{n\to\infty} \tau_n = \infty$ and $(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, \chi)$ compactly. We easily see that ν is an integral of the quasi-process χ on R^+ whenever $(\nu, \chi) \in H_\sigma(\mu, w)$.

For any $x_0 \in \mathcal{X}$ and $\varepsilon > 0$, we set $V_\varepsilon(x_0) = \{x \in \mathcal{X} : d(x, x_0) < \varepsilon\}$. We will give the definition of stabilities for the integral μ of the quasi-process w.

Definition The integral $\mu : R^+ \mapsto \mathcal{Y}$ of the quasi-process w is said to be:

(i) $\mathcal{Y}$-uniformly stable ($\mathcal{Y}$-US) (resp. $\mathcal{Y}$-uniformly stable in $\Omega_\sigma(w)$) if for any $\varepsilon > 0$, there exists a $\delta := \delta(\varepsilon) > 0$ such that $w(t, s, \mathcal{Y} \cap V_\delta(\mu(s))) \subset V_\varepsilon(\mu(t+s))$ for $(t, s) \in R^+ \times R^+$ (resp. $\chi(t, s, \mathcal{Y} \cap V_\delta(\nu(s))) \subset V_\varepsilon(\nu(t+s))$ for $(\nu, \chi) \in \Omega_\sigma(\mu, w)$ and $(t, s) \in R^+ \times R^+$);

(ii) $\mathcal{Y}$-uniformly asymptotically stable ($\mathcal{Y}$-UAS) (resp. $\mathcal{Y}$-uniformly asymptotically stable in $\Omega_\sigma(w)$), if it is $\mathcal{Y}$-US (resp. $\mathcal{Y}$-US in $\Omega_\sigma(w)$) and there exists a $\delta_0 > 0$ with the property that for any $\varepsilon > 0$, there is a $t_0 > 0$ such that $w(t, s, \mathcal{Y} \cap V_{\delta_0}(\mu(s))) \subset V_\varepsilon(\mu(t+s))$ for $t \geq t_0, \ s \in R^+$ (resp. $\chi(t, s, \mathcal{Y} \cap V_{\delta_0}(\nu(s))) \subset V_\varepsilon(\nu(t+s))$ for $(\nu, \chi) \in \Omega_\sigma(\mu, w)$ and $t \geq t_0, \ s \in R^+$);

(iii) $\mathcal{Y}$-attractive (resp. $\mathcal{Y}$-attractive in $\Omega_\sigma(w)$) if there is a $\delta_0 > 0$ such that for $y \in \mathcal{Y} \cap V_{\delta_0}(\mu(0))$ (resp. $y \in \mathcal{Y} \cap V_{\delta_0}(\nu(0))$ and $(\nu, \chi) \in \Omega_\sigma(\mu, w)$), $d(w(t, 0, y), \mu(t)) \to 0$ (resp. $d(\chi(t, 0, y), \nu(t)) \to 0$) as $t \to \infty$;

(iv) $\mathcal{Y}$-weakly uniformly asymptotically stable ($\mathcal{Y}$-WUAS) in $\Omega_\sigma(w)$ if it is $\mathcal{Y}$-US in $\Omega_\sigma(w)$ and $\mathcal{Y}$-attaractive in $\Omega_\sigma(w)$.

We assume the following property on $\mathcal{Y}$, w and μ:

(p6) There is a $\delta_1 > 0$ such that for any $s \in R^+$ and $t_0 > 0$, $w(t_0, s, y) \in \mathcal{Y}$ whenever $y \in \mathcal{Y}$ and $w(t, s, y) \in V_{\delta_1}(\mu(t+s))$ for all $t \in (0, t_0]$.

If $\mathcal{Y} = \mathcal{X}$, then (p6) is clearly satisfied. In the next section, we will give a nontivial example for which (p6) is satisfied.

Theorem 1 *Let $\mathcal{Y}$ be a closed set in a metric space $\mathcal{X}$ and suppose that w is a $\mathcal{Y}$-quasi-process on $\mathcal{X}$ for which $H_\sigma(w)$ is sequentially compact and that the skew product flow $\pi(t) : \mathcal{X} \times H_\sigma(w) \mapsto \mathcal{X} \times H_\sigma(w)$ of the quasi-process w is $\mathcal{Y}$-strongly asymptotically smooth. Also, suppose that $\mu : R^+ \mapsto \mathcal{Y}$ is an integral of w on R^+ such that $O^+(\mu)$ is a relative compact subset of $\mathcal{Y}$ and that* (p4) *and* (p6) *are satisfied. Then the following statements are equivalent:*

(i) *The integral μ is $\mathcal{Y}$-UAS.*

(ii) *The integral μ is $\mathcal{Y}$-US and $\mathcal{Y}$-attractive in $\Omega_\sigma(w)$.*

(iii) *The integral μ is $\mathcal{Y}$-UAS in $\Omega_\sigma(w)$.*

(iv) *The integral μ is $\mathcal{Y}$-WUAS in $\Omega_\sigma(w)$.*

Proof. In the case of $\mathcal{Y} = \mathcal{X}, \mu \equiv 0$ and $\mathcal{X}$ is complete, the equivalence (i) $\Leftrightarrow$ (ii) and the implication (i)$\Rightarrow$ (iii) are direct consequence of [3, Theorem 3.7.4 and Lemma 3.7.3], because the $\mathcal{Y}$-strong asymptotic smoothness of $\pi(t)$ implies the asymptotic smothness of $\pi(t)$. Following the argument employed in [3, Theorem 3.7.4 and Lemma 3.7.3], we can see that the equivalence and the implication mentioned above hold true even if $\mathcal{Y} \neq \mathcal{X}$ or $\mu \not\equiv 0$ or $\mathcal{X}$ is not necessarily complete; we omit the details. In order to establish the theorem, it suffices to show that (iv) yields the $\mathcal{Y}$-US of the integral μ. To do this by a contradiction, we assume that the integral μ is $\mathcal{Y}$-WUAS in $\Omega_\sigma(w)$, but not $\mathcal{Y}$-US. Then there exist an $\varepsilon > 0$, $\varepsilon < \min\{\delta_0, \delta_1\}$, and a sequence $\{(t_n, s_n, y_n)\}$ in $R^+ \times R^+ \times \mathcal{Y}$ such that $d(y_n, \mu(s_n)) \to 0$ as $n \to \infty$, $d(w(t_n, s_n, y_n), \mu(t_n + s_n)) = \varepsilon$ and $d(w(t, s_n, y_n), \mu(t + s_n)) < \varepsilon$ for $0 \le t < t_n$, where δ_1 is the one ensured in (p6) and δ_0 is the one given for the $\mathcal{Y}$-attractivity in $\Omega_\sigma(w)$ of the integral μ. Take a positive constant γ, $\gamma < \varepsilon$, so that $\chi(t, s, \mathcal{Y} \cap V_\gamma(\nu(s))) \subset V_\varepsilon(\nu(t + s))$ for $(t, s) \in R^+ \times R^+$ and $(\nu, \chi) \in \Omega_\sigma(\mu, w)$, which is possible by the $\mathcal{Y}$-US in $\Omega_\sigma(w)$ of the integral μ. Since $d(y_n, \mu(s_n)) \to 0$ as $n \to \infty$, there exists a sequence $\{\tau_n\}$, $0 < \tau_n < t_n$, such that $d(w(\tau_n, s_n, y_n), \mu(\tau_n + s_n)) = \gamma/2$ and $d(w(t, s_n, y_n), \mu(t + s_n)) \ge \gamma/2$ for all $t \in [\tau_n, t_n]$.

We assert that $\tau_n \to \infty$ as $n \to \infty$. Suppose that the assertion is false. Then, without loss of generality, we may assume that $\tau_n \to \tau_0$ and $(\mu^{s_n}, \sigma(s_n)w) \to (\tilde{\mu}, \tilde{w})$ as $n \to \infty$, for some $\tau_0 < \infty$ and $(\tilde{\mu}, \tilde{w}) \in H_\sigma(\mu, w)$. From (p5) it follows that $\pi(\tau_n)(y_n, \sigma(s_n)w)$ tends to $\pi(\tau_0)(\tilde{\mu}(0), \tilde{w})$ in $\mathcal{Y} \times H_\sigma(w)$ as $n \to \infty$, which implies that $(\sigma(s_n)w)(\tau_n, 0, y_n) = w(\tau_n, s_n, y_n)$ tends to $\tilde{w}(\tau_0, 0, \tilde{\mu}(0)) = \tilde{\mu}(\tau_0)$ as $n \to \infty$. On the other hand, since $d(w(\tau_n, s_n, y_n), \mu(\tau_n + s_n)) = \gamma/2$, we must get $d(\tilde{w}(\tau_0, 0, \tilde{\mu}(0)), \tilde{\mu}(\tau_0)) = \gamma/2$, a contradiction.

Now we may assume that $(\mu^{\tau_n + s_n}, \sigma(\tau_n + s_n)w) \to (\nu, \chi)$ as $n \to \infty$, for some $(\nu, \chi) \in \Omega_\sigma(\mu, w)$. Notice that $\pi(t)(y_n, \sigma(s_n)w) = (w(t, s_n, y_n), \sigma(t + s_n)w) \in V_{\delta_1}(\mu(t + s_n)) \times H_\sigma(w)$ for $t \in [0, \tau_n]$. By virtue of (p6), we get $w(t, s_n, y_n) \in \mathcal{Y}$ for all $t \in [0, \tau_n]$. Since $\pi(t)$ is $\mathcal{Y}$-strongly asymptotically smooth, taking a subsequence if necessary, we can assume that $w(\tau_n, s_n, y_n) \to \tilde{y}$ for some $\tilde{y} \in \mathcal{Y}$ as $n \to \infty$. Note that $\tilde{y} \in V_\gamma(\nu(0))$. We first consider the case where the sequence $\{t_n - \tau_n\}$ has a convergent subsequence. Without loss of generality, we can assume that $t_n - \tau_n \to \tilde{t}$ as $n \to \infty$, for some $\tilde{t} < \infty$. Then (p5) implies that $\pi(t_n - \tau_n)(w(\tau_n, s_n, y_n), \sigma(\tau_n + s_n)w) = (w(t_n, s_n, y_n), \sigma(t_n + s_n)w)$ tends to $\pi(\tilde{t})(\tilde{y}, \chi)$ in $\mathcal{Y} \times H_\sigma(w)$ as $n \to \infty$. Letting $n \to \infty$ in the relation $d((\sigma(\tau_n + s_n)w)(t_n - \tau_n, 0, w(\tau_n, s_n, y_n)), \mu(t_n + s_n)) = d(w(t_n - \tau_n, s_n + \tau_n, w(\tau_n, s_n, y_n)), \mu(t_n + s_n)) = d(w(t_n, s_n, y_n), \mu(t_n + s_n)) = \varepsilon$, we get $d(\chi(\tilde{t}, 0, \tilde{y}), \nu(\tilde{t})) = \varepsilon$. This is a contradiction, because of $\chi(\tilde{t}, 0, \tilde{y}) \in \chi(\tilde{t}, 0, \mathcal{Y} \cap V_\gamma(\nu(0))) \subset V_\varepsilon(\nu(\tilde{t}))$. Thus we must have $\lim_{n\to\infty}(t_n - \tau_n) = \infty$. Now, letting $n \to \infty$ in the relation $d((\sigma(s_n + \tau_n)w)(t, 0, w(\tau_n, s_n, y_n)), \mu(t + \tau_n + s_n)) = d(w(t + \tau_n, s_n, y_n), \mu(t + \tau_n + s_n)) \le \varepsilon$ for $t \in [0, t_n - \tau_n]$, we get $d(\chi(t, 0, \tilde{y}), \nu(t)) \le \varepsilon < \delta_0$ for all $t \ge 0$. Then, from the $\mathcal{Y}$-attractivity in $\Omega_\sigma(w)$ of the integral μ, it follows that $d(\chi(t, 0, \tilde{y}), \nu(t)) \to 0$ as $t \to \infty$.

On the other hand, since $d(w(t+\tau_n, s_n, y_n), \mu(t+\tau_n+s_n)) \geq \gamma/2$ for $t \in [0, t_n - \tau_n]$, we must get $d(\chi(t,0,\tilde{y}), \nu(t)) \geq \gamma/2$ for all $t \geq 0$; hence $d(\chi(t,0,\tilde{y}), \nu(t)) \not\to 0$ as $t \to \infty$, a contradiction.

3. Quasi-processes generated by abstract functional differential equations

In this section, we shall treat abstract functional differential equations on a fading memory space (resp. uniform fading memory space) and show that quasi-processes (resp. processes) are naturally generated by functional differential equations.

We first explain some notation and convention employed throughout this section. Let X be a Banach space with norm $|\cdot|_X$. For any interval $J \subset R := (-\infty, \infty)$, we denote by $\mathrm{BC}(J;X)$ the space of all bounded and continuous functions mapping J into X. Clearly $\mathrm{BC}(J;X)$ is a Banach space with the norm $|\cdot|_{\mathrm{BC}(J;X)}$ defined by $|\phi|_{\mathrm{BC}(J;X)} = \sup\{|\phi(t)|_X : t \in J\}$. If $J = R^- := (-\infty, 0]$, then we simply write $\mathrm{BC}(J;X)$ and $|\cdot|_{\mathrm{BC}(J;X)}$ as BC and $|\cdot|_{\mathrm{BC}}$, respectively. For any function $u : (-\infty, a) \mapsto X$ and $t < a$, we define a function $u_t : R^- \mapsto X$ by $u_t(s) = u(t+s)$ for $s \in R^-$. Let $\mathcal{B} = \mathcal{B}(R^-; X)$ be a real Banach space of functions mapping R^- into X with a norm $|\cdot|_{\mathcal{B}}$. The space $\mathcal{B}$ is assumed to have the following properties:

(A1) There exist a positive constant N and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on R^+ with the property that if $u : (-\infty, a) \mapsto X$ is continuous on $[\sigma, a)$ with $u_\sigma \in \mathcal{B}$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,

(i) $u_t \in \mathcal{B}$,

(ii) u_t is continuous in t (w.r.t. $|\cdot|_{\mathcal{B}}$),

(iii) $N|u(t)|_X \leq |u_t|_{\mathcal{B}} \leq K(t-\sigma)\sup_{\sigma \leq s \leq t}|u(s)|_X + M(t-\sigma)|u_\sigma|_{\mathcal{B}}$.

(A2) If $\{\phi^n\}$ is a sequence in $\mathcal{B} \cap \mathrm{BC}$ converging to a function ϕ uniformly on any compact intertval in R^- and $\sup_n |\phi^n|_{\mathrm{BC}} < \infty$, then $\phi \in \mathcal{B}$ and $|\phi^n - \phi|_{\mathcal{B}} \to 0$ as $n \to \infty$.

It is known [7, Proposition 7.1.1] that the space $\mathcal{B}$ contains BC and that there is a constant $\ell > 0$ such that

$$|\phi|_{\mathcal{B}} \leq \ell |\phi|_{\mathrm{BC}}, \quad \phi \in \mathrm{BC}. \tag{1}$$

Set $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$ and define an operator $S_0(t) : \mathcal{B}_0 \mapsto \mathcal{B}_0$ by

$$[S_0(t)\phi](s) = \begin{cases} \phi(t+s) & \text{if } t+s \leq 0, \\ 0 & \text{if } t+s > 0 \end{cases}$$

for each $t \geq 0$. In virtue of (A1), one can see that the family $\{S_0(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{B}_0$. We consider the following properties:

(A3) $\lim_{t\to\infty} |S_0(t)\phi|_{\mathcal{B}} = 0, \quad \phi \in \mathcal{B}_0.$

(A3') $\lim_{t\to\infty} \|S_0(t)\| = 0.$

Here and hereafter, we denote by $\|\cdot\|$ the operator norm of linear bounded operators. The space $\mathcal{B}$ is called *a fading memory space* (resp. *a uniform fading memory space*), if it satisfies (A3) (resp. (A3')) in addition to (A1) and (A2). It is obvious that $\mathcal{B}$ is a fading memory space whenever it is a uniform fading memory space. It is known [7, Proposition 7.1.5] that the functions $K(\cdot)$ and $M(\cdot)$ in (A1) can be chosen as $K(t) \equiv \ell$ and $M(t) \equiv (1 + (\ell/N))\|S_0(t)\|$. Note that (A3) implies $\sup_{t\geq 0} \|S_0(t)\| < \infty$ by the Banach-Steinhaus theorem. Therefore, whenever $\mathcal{B}$ is a fading memory space, we can assume that the functions $K(\cdot)$ and $M(\cdot)$ in (A1) satisfy $K(\cdot) \equiv K$ and $M(\cdot) \equiv M$, constants.

We provide a typical example of fading memory spaces. Let $g : R^- \mapsto [1, \infty)$ be any continuous, nonincreasing function such that $g(0) = 1$ and $g(s) \to \infty$ as $s \to -\infty$. We set

$$C_g^0 := C_g^0(X) = \{\phi : R^- \mapsto X \text{ is continuous with } \lim_{s\to-\infty} |\phi(s)|_X / g(s) = 0\}.$$

Then the space C_g^0 equipped with the norm

$$|\phi|_g = \sup_{s\leq 0} \frac{|\phi(s)|_X}{g(s)}, \quad \phi \in C_g^0,$$

is a separable Banach space and it satisfies (A1)–(A3). Moreover, one can see that (A3') holds if and only if $\sup\{g(s+t)/g(s) : s \leq -t\} \to 0$ as $t \to \infty$. Therefore, if $g(s) \equiv e^{-s}$, then the space C_g^0 is a uniform fading memory space. On the other hand, if $g(s) = 1+|s|^k$ for some $k > 0$, then the space C_g^0 is a fading memory space, but not a uniform fading memory space.

Throughout the remainder of this paper, we assume that $\mathcal{B}$ is a fading memory space or a uniform fading memory space which is separable and let $C(R^+ \times \mathcal{B}; X)$ be the space of all continuous functions on $R^+ \times \mathcal{B}$ with values in a Banach space X, which is equipped with the compact open topology.

Now we consider the following functional differential equation

$$\frac{du}{dt} = Au(t) + F(t, u_t), \tag{2}$$

where A is the infinitesimal generator of a compact semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X and $F(t,\phi) \in C(R^+ \times \mathcal{B}; X)$. In what follows, we shall show that (2) generates a quasi-process on an appropriate space under some conditions and deduce equivalence relationships between some stability properties of (2) and those of its limiting equations as an application of Theorem 1.

We assume the following conditions on F:

(H1) $F(t,\phi)$ is uniformly continuous on $R^+ \times \mathcal{K}$ for any compact set $\mathcal{K}$ in $\mathcal{B}$, and $\{F(t,\phi) \mid t \in R^+\}$ is a relative compact subset of X for each $\phi \in \mathcal{B}$.

(H2) For any $H > 0$, there is an $L(H) > 0$ such that $|F(t,\phi)|_X \leq L(H)$ for all $t \in R^+$ and $\phi \in \mathcal{B}$ such that $|\phi|_{\mathcal{B}} \leq H$.

For $\tau \in R^+$, we denote the τ-translation F^τ of $F(t,\phi)$ by

$$F^\tau(t,\phi) = F(t+\tau,\phi), \quad (t,\phi) \in R^+ \times \mathcal{B}.$$

Clearly, F^τ is in $C(R^+ \times \mathcal{B}; X)$, too. Set

$$H(F) = \overline{\{F^\tau; \tau \in R^+\}},$$

where $\overline{\{F^\tau; \tau \in R^+\}}$ denotes the closure of $\{F^\tau; \tau \in R^+\}$ in $C(R \times \mathcal{B}; X)$. The subspace $H(F)$ of $C(R^+ \times \mathcal{B}; X)$ is called the hull of F. It is known [7, Proposition 8.1.3] that $H(F)$ is metrizable. Clearly, the hull $H(F)$ is invariant with respect to the τ-translation; that is, $G^\tau \in H(F)$ whenever $G \in H(F)$ and $\tau \in R^+$: Moreover, from (H1) and [7, Theorem 7.1.4] it follows that $H(F)$ is a compact set in $C(R^+ \times \mathcal{B}; X)$. If $G \in H(F)$, one can choose a sequence $\{\tau_n\} \subset R^+$ so that F^{τ_n} tends to G in $C(R^+ \times \mathcal{B}; X)$, that is, $F(t+\tau_n,\phi) \to G(t,\phi)$ as $n \to \infty$ uniformly on any compact subset of $R^+ \times \mathcal{B}$. It is easy to see that each $G \in H(F)$ satisfies (H1) and (H2) (with the common $L(\cdot)$) when F does. We denote by $\Omega(F)$ the set of all elements G in $H(F)$ for which one can choose a sequence $\{\tau_n\} \subset R^+$ so that $\{\tau_n\} \to \infty$ as $n \to \infty$ and $F^{\tau_n} \to G$ in $C(R \times \mathcal{B}; X)$. If $G \in H(F)$, the system

$$\frac{du}{dt} = Au(t) + G(t,u_t) \qquad t \in R^+, \tag{3}$$

is called an equation in the hull of System (2). In particular, if $G \in \Omega(F)$, then it is called a limiting equation of (2).

Under the conditions (H1) and (H2), it is known that for any $(\sigma,\phi) \in R^+ \times \mathcal{B}$, there exists a function $u \in C((-\infty,t_1); X)$ such that $u_\sigma = \phi$ and the following relation holds:

$$u(t) = T(t-\sigma)\phi(0) + \int_\sigma^t T(t-s)G(s,u_s)ds, \quad \sigma \leq t \leq t_1,$$

(cf. [5, Theorem 1]). The function u is called the (mild) solution of (3) through (σ, ϕ) defined on $[\sigma, t_1]$ and denoted by $u(t) := u(t, \sigma, \phi, G)$. In the above, t_1 can be taken as $t_1 = \infty$ if $\sup_{t \le t_1} |u(t)|_X < \infty$ (cf. [5, Corollary 2]). In the following, we always assume the following condition, too:

(H3) For any $G \in H(F)$ and $(\sigma, \phi) \in R^+ \times \mathcal{B}$, Equation (3) has a *unique* solution through (σ, ϕ) which exists for *all* $t \ge \sigma$.

Consider a mapping $\Phi : R^+ \times R^+ \times \mathcal{B} \times H(F) \mapsto \mathcal{B}$ defined by

$$\Phi(t, s, \phi, G) = u_{t+s}(s, \phi, G) \in \mathcal{B}, \qquad (t, s, \phi, G) \in R^+ \times R^+ \times \mathcal{B} \times H(F).$$

Proposition 1 *Assume that $\mathcal{B}$ is a fading memory space and that conditions* (H1)–(H3) *are hold. Then the mapping Φ is continuous.*

Proof. Assume that the mapping Φ is not continuous. Then there exist an $\varepsilon > 0$, $(t_0, s_0, \phi^0, G) \in R^+ \times R^+ \times \mathcal{B} \times H(F)$ and sequences $\{t_k\} \subset R^+, \{s_k\} \subset R^+, \{\phi^k\} \subset \mathcal{B}$ and $\{G_k\} \subset H(F)$ such that $(t_k, s_k, \phi^k, G_k) \to (t_0, s_0, \phi^0, G)$ and $|\Phi(t_k, s_k, \phi^k, G_k) - \Phi(t_0, s_0, \phi^0, G)|_{\mathcal{B}} \ge 3\varepsilon$ for $k \in \mathbf{N}$ ($\mathbf{N}$ denotes the set of all positive integers). Since $\Phi(t, s_0, \phi^0, G) \in \mathcal{B}$ is continuous in $t \in R^+$ by (A1–ii), we may assume that $|x^k_{t_k} - x_{t_k}|_{\mathcal{B}} \ge 2\varepsilon$ for all $k \in \mathbf{N}$, where $x(t) = u(t + s_0, s_0, \phi^0, G)$ and $x^k(t) = u(t + s_k, s_k, \phi^k, G_k)$ for $k \in \mathbf{N}$. There exist $\gamma > 0$, σ_k and τ_k, $0 < \sigma_k < \tau_k \le t_k$, such that $\gamma < \min\{\varepsilon/(1 + M), N\varepsilon/K\}$,

$$|x^k_{\sigma_k} - x_{\sigma_k}|_{\mathcal{B}} = \gamma, \qquad |x^k_{\tau_k} - x_{\tau_k}|_{\mathcal{B}} = 2\varepsilon,$$
$$|x^k_t - x_t|_{\mathcal{B}} < \gamma \quad (0 \le t < \sigma_k)$$

and

$$|x^k_t - x_t|_{\mathcal{B}} < 2\varepsilon \quad (0 \le t < \tau_k)$$

for $k \in \mathbf{N}$; here the functions $M(\cdot)$ and $K(\cdot)$ in (A1) may be chosen as positive constants M and K, respectively, because $\mathcal{B}$ is a fading memory space. By choosing a subsequence if necessarily, we may assume that $\sigma_k \to \sigma_0 \in [0, t_0]$. We claim that

$$\sigma_0 > 0. \tag{4}$$

Indeed, if (4) is false, then we have, for any $0 \le t \le \min\{\sigma_k, 1\}$,

$$\begin{aligned} |x^k(t) - x(t)|_X &= |T(t)x^k(0) + \int_0^t T(t-\tau)G_k(s_k + \tau, x^k_\tau)d\tau \\ &\quad -T(t)x(0) - \int_0^t T(t-\tau)G(s_0 + \tau, x_\tau)d\tau|_X \\ &\le C_1\{(1/N)|\phi^k - \phi^0|_{\mathcal{B}} + 2\int_0^t L(H)d\tau\}, \end{aligned}$$

where $H = \sup\{|x_t|_{\mathcal{B}}, |x_t^k|_{\mathcal{B}} : 0 \le t \le \tau_k, k \in \mathbf{N}\}$ and $C_1 = \sup_{0 \le s \le 1} ||T(s)||$; hence

$$\begin{aligned} \gamma &= |x_{\sigma_k}^k - x_{\sigma_k}|_{\mathcal{B}} \\ &\le K \sup_{0 \le t \le \sigma_k} |x^k(t) - x(t)|_X + M|\phi^k - \phi^0|_{\mathcal{B}} \\ &\le KC_1\{(1/N)|\phi^k - \phi^0|_{\mathcal{B}} + 2\sigma_k L(H)\} + M|\phi^k - \phi^0|_{\mathcal{B}} \to 0 \end{aligned}$$

as $k \to \infty$, a contradiction.

Next we prove that the set $O := \{x^k(t) : 0 \le t \le \tau_k, \ k \in \mathbf{N}\}$ is relatively compact in X. To do this, we consider the sets $O_\eta = \{x^k(t) : \eta \le t \le \tau_k, \ k \in \mathbf{N}\}$ and $\tilde{O}_\eta = \{x^k(t) : 0 \le t \le \eta, \ k \in \mathbf{N}\}$ for any $\eta > 0$ such that $\eta < \inf_k \tau_k$. Then $\alpha(O) = \max\{\alpha(O_\eta), \ \alpha(\tilde{O}_\eta)\}$, where $\alpha(\cdot)$ is Kuratowski's measure of noncompactness of sets in X. For the details of the properties of $\alpha(\cdot)$, see [10, Section 1.4]. Let $0 < \nu < \min\{1, \eta\}$. Since $x^k(t)$ is a mild solution of $(d/dt)u = Au(t) + G_k(t + s_k, u_t)$ through $(0, \phi^k)$, we get

$$\begin{aligned} x^k(t) &= T(t)\phi^k(0) + \int_0^t T(t-s)h^k(s)ds \\ &= T(\eta)[T(t-\eta)\phi^k(0) + \int_0^{t-\eta} T(t-\eta-s)h^k(s)ds] + \int_{t-\eta}^t T(t-s)h^k(s)ds \\ &= T(\eta)x^k(t-\eta) + T(\nu)\int_{t-\eta}^{t-\nu} T(t-s-\nu)h^k(s)ds + \int_{t-\nu}^t T(t-s)h^k(s)ds \end{aligned}$$

for $t \ge \eta$, where $h^k(t) = G_k(t+s_k, x_t^k)$. The set $\{\int_{t-\eta}^{t-\nu} T(t-s-\nu)h^k(s)ds : \eta \le t \le \tau_k, \ k \in \mathbf{N}\}$ is bounded in X, and hence $T(\nu)\{\int_{t-\eta}^{t-\nu} T(t-s-\nu)h^k(s)ds : \eta \le t \le \tau_k, \ k \in \mathbf{N}\}$ is relatively compact in X by the compactness of the semigroup $\{T(t)\}_{t \ge 0}$. Similarly, one can get the relative compactness of the set $T(\eta)\{x^k(t-\eta) : \eta \le t \le \tau_k, \ k \in \mathbf{N}\}$. Since

$$\begin{aligned} \alpha(O_\eta) &= \alpha(\{\int_{t-\nu}^t T(t-s)h^k(s)ds : \eta \le t \le \tau_k, \ k \in \mathbf{N}\}) \\ &\le C_1 L(H)\nu, \end{aligned}$$

letting $\nu \to 0$ in the above, we get $\alpha(O_\eta) = 0$. Hence

$$\begin{aligned} \alpha(O) = \alpha(\tilde{O}_\eta) &= \alpha(\{T(t)\phi^k(0) + \int_0^t T(t-s)h^k(s)ds : 0 \le t \le \eta, \ k \in \mathbf{N}\}) \\ &= \alpha(\{\int_0^t T(t-s)h^k(s)ds : 0 \le t \le \eta, \ k \in \mathbf{N}\}) \\ &\le C_1 L(H)\eta \end{aligned}$$

for all $0 < \eta < \inf_k \tau_k$, which shows $\alpha(O) = 0$; consequently, O must be relatively compact in X.

Since the set $\{x^k(\sigma_k), \ x(\sigma_k) : k \in \mathbf{N}\}$ is relatively compact in X, $|[T(t-\sigma_k) - I](x^k(\sigma_k) - x(\sigma_k))|_X \to 0$ as $|t - \sigma_k| \to 0$. Therefore, repeating almost the same arguments as in the proof of (4), we obtain $\inf\{\tau_k - \sigma_k : k \in \mathbf{N}\} =: 2a > 0$, because of the inequality

$|x^k_{\tau_k} - x_{\tau_k}|_{\mathcal{B}} \leq K \sup_{\sigma_k \leq t \leq \tau_k} |x^k(t) - x(t)|_X + M|x^k_{\sigma_k} - x_{\sigma_k}|_{\mathcal{B}}$ or $\varepsilon < K \sup_{\sigma_k \leq t \leq \tau_k} |x^k(t) - x(t)|_X$. Noting that $|x^k(t) - x^k(s)|_X \leq \sup\{|T(t-s)z - z|_X : z \in O\} + C_1 L(H)|t-s|$ when $0 \leq s \leq t \leq \sigma_0 + a$ and $t \leq s+1$, we have that $x^k(t)$ is equicontinuous on $[0, \sigma_0 + a]$. Hence, one may assume that $x^k(t)$ converges to some continuous function $y(t)$ uniformly on $[0, t_0 + a]$ as $k \to \infty$. Putting $y_0 = \phi^0$, we have $x^k_t \to y_t$ in $\mathcal{B}$ uniformly on $[0, \sigma_0 + a]$, because of $x^k_0 = \phi^k \to \phi^0$ in $\mathcal{B}$. Letting $k \to \infty$ in the relation

$$x^k(t) = T(t)\phi^k(0) + \int_0^t T(t-\tau)G_k(s_k + \tau, x^k_\tau)d\tau$$

for $t \in [0, \sigma_k + a]$, we have

$$y(t) = T(t)y(0) + \int_0^t T(t-\tau)G(s_0 + \tau, y_\tau)d\tau$$

for $t \in [0, \sigma_0 + (a/2)]$; hence $y(t - s_0) \equiv u(t, s_0, \phi^0, G) = x(t - s_0)$ on $[s_0, s_0 + \sigma_0 + (a/2)]$ by (H3). Consequently $|y_{\sigma_0} - x_{\sigma_0}|_{\mathcal{B}} = 0$. This is a contradiction, because we must get $|y_{\sigma_0} - x_{\sigma_0}|_{\mathcal{B}} = \gamma$ by letting $k \to \infty$ in $|x^k_{\sigma_k} - x_{\sigma_k}|_{\mathcal{B}} = \gamma$. This completes the proof of the proposition.

Now we take $\mathcal{X} = \mathcal{Y} = \mathcal{B}$ and consider a function $w^G_{\mathcal{B}} : R^+ \times R^+ \times \mathcal{B} \mapsto \mathcal{B}$ defined by

$$w^G_{\mathcal{B}}(t, s, \phi) = u_{t+s}(s, \phi, G), \qquad (t, s, \phi) \in R^+ \times R^+ \times \mathcal{B}.$$

By virtue of (H3), we see that the mapping $w^G_{\mathcal{B}}$ satisfies (p1) and (p2) with $\mathcal{X} = \mathcal{B}$. Moreover from Proposition 1 it follows that $w^G_{\mathcal{B}}$ satisfies (p3). Thus the mapping $w^G_{\mathcal{B}}$ is a $\mathcal{B}$-quasi-process on $\mathcal{B}$. In fact, $w^G_{\mathcal{B}}$ is precisely a process on $\mathcal{B}$ in a sense of [1–3]. We call $w^G_{\mathcal{B}}$ a process on $\mathcal{B}$ generated by (3).

Now we consider the process $w^F_{\mathcal{B}}$ on $\mathcal{B}$ generated by (2). (H3) yields the relation $u(t + \tau, s + \tau, \phi, F) = u(t, s, \phi, F^\tau)$ for $t \in R$ and $(s, \tau, \phi) \in R^+ \times R^+ \times \mathcal{B}$. Hence $(\sigma(\tau)w^F_{\mathcal{B}})(t, s, \phi) = w^F_{\mathcal{B}}(t, \tau + s, \phi) = u_{t+\tau+s}(\tau + s, \phi, F) = u_{t+s}(s, \phi, F^\tau) = w^{F^\tau}_{\mathcal{B}}(t, s, \phi)$ for $(t, s, \phi) \in R^+ \times R^+ \times \mathcal{B}$; in other words,

$$\sigma(\tau)w^F_{\mathcal{B}} \equiv w^{F^\tau}_{\mathcal{B}}.$$

Therefore $H_\sigma(w^F_{\mathcal{B}}) = \{w^G_{\mathcal{B}} : G \in H(F)\}$ and $\Omega_\sigma(w^F_{\mathcal{B}}) = \{w^G_{\mathcal{B}} : G \in \Omega(F)\}$. In particular, the process $w^F_{\mathcal{B}}$ satisfies the condition (p4). Moreover, we see that $H_\sigma(w^F_{\mathcal{B}})$ is sequentially compact and that a mapping $\pi_{\mathcal{B}}(t) : \mathcal{B} \times H_\sigma(w^F_{\mathcal{B}}) \mapsto \mathcal{B} \times H_\sigma(w^F_{\mathcal{B}})$ defined by

$$\pi_{\mathcal{B}}(t)(\phi, w^G_{\mathcal{B}}) = (u_t(0, \phi, G), \sigma(t)w^G_{\mathcal{B}}), \qquad (\phi, w^G_{\mathcal{B}}) \in \mathcal{B} \times H_\sigma(w^F_{\mathcal{B}}),$$

satisfies the condition (p5), and hence $\pi_{\mathcal{B}}(t)$ is the skew product flow of $w^F_{\mathcal{B}}$.

Lemma 1 *The skew product flow $\pi_{\mathcal{B}}(t)$ is $\mathcal{B}$-strongly asymptotically smooth whenever $\mathcal{B}$ is a uniform fading memory space.*

Proof. Let B be any bounded closed set in $\mathcal{B} \times H_\sigma(w_{\mathcal{B}}^F)$. Then there is an $H > 0$ such that $B \subset \mathcal{B}_H \times H_\sigma(w_{\mathcal{B}}^F)$, where $\mathcal{B}_H = \{\phi \in \mathcal{B} : |\phi|_{\mathcal{B}} \le H\}$. Let $L := L(H)$ be a constant in (H2), and set

$$Q_1 = \{\int_0^1 T(\tau)h(\tau)d\tau : \ h \in BC([0,1];X) \text{ with } |h|_{\mathrm{BC}([0,1];X)} \le L\}.$$

By the same reason as the one for the set O in the proof of Proposition 1, we see that the set Q_1 is relativly compact in X. Then there exists a compact set O_B in X satisfying

$$T(1)X_{H/N} + Q_1 \subset O_B, \tag{5}$$

where N is the constant in (A1), and $X_{H/N} = \{x \in X : |x|_X \le H/N\}$. Denote by J_B the set of all elements ϕ in BC with the property that $\phi(\theta) \in O_B$ for $\theta \in R^-$ and

$$|\phi(t) - \phi(s)|_X \le \sup\{|T(t-s)z - z|_X : z \in O_B\} + C_1 L|t-s|$$

for all s, t satisfying $s - 1 \le t - 1 \le s \le 0$, where $C_1 = \sup_{0 \le s \le 1} ||T(s)||$. From (A2) and Ascoli-Arzéela's theorem, we see that J_B is a compact set in $\mathcal{B}$.

Now, let $\{t_n\} \subset R^+$ and $\{(\phi^n, w_{\mathcal{B}}^{G_n})\} \subset B$ be sequences with the property that $lim_{n\to\infty} t_n = \infty$ and $\pi_{\mathcal{B}}(t)(\phi^n, w_{\mathcal{B}}^{G_n}) = (u_t(0, \phi^n, G_n), w_{\mathcal{B}}^{G_n^t}) \in B$ for all $t \in [0, t_n]$. We shall show that $\{\pi_{\mathcal{B}}(t_n)(\phi^n, w_{\mathcal{B}}^{G_n})\}$ has a subsequence which approaches to the compact set $J_B \times H_\sigma(w_{\mathcal{B}}^F)$. We may assume that $t_n > 2$ for $n = 1, 2, \cdots$. Set $x^n(t) = u(t, 0, \phi^n, G_n), n = 1, 2, \cdots$. Since $|x_t^n|_{\mathcal{B}} \le H$ for $t \in [0, t_n]$, we get

$$\begin{aligned} x^n(t) &= T(1)x^n(t-1) + \int_{t-1}^t T(t-s)G_n(s, x_s^n)ds \\ &= T(1)x^n(t-1) + \int_0^1 T(\tau)h^{n,t}(\tau)d\tau \end{aligned}$$

for any $t \in [1, t_n]$, where $h^{n,t}(\tau) = G_n(t-\tau, x_{t-\tau}^n)$. Note that $h^{n,t} \in \mathrm{BC}([0,1];X)$ with $|h^{n,t}|_{\mathrm{BC}([0,1];X)} \le L$ and that $|x^n(t-1)|_X \le (1/N)|x_{t-1}^n|_{\mathcal{B}} \le H/N$. It follows from (5) that the set $\{x^n(t) : 1 \le t \le t_n, \ n = 1, 2, \cdots\} \subset O_B$. Moreover, if $1 \le s \le t \le t_n$ and $|t - s| \le 1$, then

$$\begin{aligned} |x^n(t) - x^n(s)|_X &\le |T(t-s)x^n(s) - x^n(s)|_X + |\int_s^t T(t-\tau)G_n(\tau, x_\tau^n)d\tau|_X \\ &\le \sup\{|T(t-s)z - z|_X : z \in O_B\} + C_1 L|t-s|. \end{aligned}$$

Therefore, if we consider a function y^n defined by $y^n(t) = x^n(t)$ if $1 \le t \le t_n$, and $y^n(t) = x^n(1)$ if $t \le 1$, then $y_{t_n}^n \in J_B$. Observe that

$$u_{t_n}(0, \phi_n, G_n) = y_{t_n}^n + S_0(t_n - 1)[x_1^n - x^n(1)\xi],$$

where $\xi(\theta) = 1$ for $\theta \leq 0$. Since $|x_1^n - x^n(1)\xi|_{\mathcal{B}} \leq H + \ell H/N$ by (1), we get $|u_{t_n}(0, \phi^n, G_n) - y_{t_n}^n|_{\mathcal{B}} \leq ||S_0(t_n - 1)||(1 + \ell/N)H \to 0$ as $n \to \infty$, because $\mathcal{B}$ is a uniform fading memory space. Thus $\{\pi_{\mathcal{B}}(t_n)(\phi^n, w_{\mathcal{B}}^{G_n})\}$ approaches to the compact set $J_B \times H_\sigma(w_{\mathcal{B}}^F)$.

Suppose the following condition:

(H4) Equation (2) has a bounded solution $\bar{u}(t)$ defined on R^+ such that $\bar{u}_0 \in$ BC.

By virtue of [8, Lemma 2] and the proof of Proposition 1, we see that the set $\{\bar{u}_t : t \in R^+\}$ is relatively compact in $\mathcal{B}$ and that $\bar{u}_t \in \mathcal{B}$ is uniformly continuous in $t \in R^+$. Therefore, for any sequence $\{\tau_n'\} \subset R^+$ one can choose a subsequence $\{\tau_n\} \subset \{\tau_n'\}$, $\bar{v} \in C(R; X)$ and $G \in H(F)$ such that $\lim_{n\to\infty} F^{\tau_n} = G$ in $C(R^+ \times \mathcal{B}; X)$ and $\lim_{n\to\infty} |\bar{u}_{t+\tau_n} - \bar{v}_t|_{\mathcal{B}} = 0$ uniformly on any compact interval in R^+. In this case, we write as

$$(\bar{u}^{\tau_n}, F^{\tau_n}) \to (\bar{v}, G) \text{ compactly,}$$

for simplicity. Denote by $H(\bar{u}, F)$ the set of all $(\bar{v}, G) \in C(R; X) \times H(F)$ such that $(\bar{u}^{\tau_n}, F^{\tau_n}) \to (\bar{v}, G)$ compactly for some sequence $\{\tau_n\} \subset R^+$. In particular, we denote by $\Omega(\bar{u}, F)$ the set of all elements $(\bar{v}, G)$ in $H(\bar{u}, F)$ for which one can choose a sequence $\{\tau_n\} \subset R^+$ so that $\lim_{n\to\infty} \tau_n = \infty$ and $(\bar{u}^{\tau_n}, F^{\tau_n}) \to (\bar{v}, G)$ compactly. We can easily see that $\bar{v}$ is a solution of (3) whenever $(\bar{v}, G) \in H(\bar{u}, F)$.

For any function $\xi : R \mapsto X$ such that $\xi_0 \in \mathcal{B}$ and ξ is continuous on R^+, we define a continuous function $\mu_{\mathcal{B}}^\xi : R^+ \mapsto \mathcal{B}$ by

$$\mu_{\mathcal{B}}^\xi(t) = \xi_t, \qquad t \in R^+.$$

It is clear that $\mu_{\mathcal{B}}^{\bar{u}}$ is an integral of the process $w_{\mathcal{B}}^F$ on R^+. Also, we get $H_\sigma(\mu_{\mathcal{B}}^{\bar{u}}, w_{\mathcal{B}}^F) = \{(\mu_{\mathcal{B}}^{\bar{v}}, w_{\mathcal{B}}^G) : (\bar{v}, G) \in H(\bar{u}, F)\}$ and $\Omega_\sigma(\mu_{\mathcal{B}}^{\bar{u}}, w_{\mathcal{B}}^F) = \{(\mu_{\mathcal{B}}^{\bar{v}}, w_{\mathcal{B}}^G) : (\bar{v}, G) \in \Omega(\bar{u}, F)\}$.

The $\mathcal{B}$-stabilities for the solution $\bar{u}(t)$ of (2) is defined via those of the integral $\mu_{\mathcal{B}}^{\bar{u}}$ of the process $w_{\mathcal{B}}^F$ in Definition with $\mathcal{X} = \mathcal{Y} = \mathcal{B}$. For example, the solution $\bar{u}(t)$ of (2) is $\mathcal{B}$-uniformly stable in $\Omega(F)$($\mathcal{B}$-US in $\Omega(F)$), if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|u_t(s, \phi, G) - \bar{v}_t|_{\mathcal{B}} < \varepsilon$ for $t \geq s \geq 0$ whenever $(\bar{v}, G) \in \Omega(\bar{u}, F)$ and $|\phi - \bar{v}_s|_{\mathcal{B}} < \delta(\varepsilon)$. The other $\mathcal{B}$-stabilities for $\bar{u}(t)$ are given in a similar way; we omit the details.

Combining the above observation with Theorem 1 and Lemma 1, we get the following result on $\mathcal{B}$-stabilities (cf. [4, 6, 11]). We emphasize that the additional condition that $\mathcal{B}$ is a uniform fading memory space cannot be removed because a fading memory space $\mathcal{B}$ must be a uniform fading memory space whenever there is a functional differential equation on $\mathcal{B}$ which has a $\mathcal{B}$-UAS solution ([7, Theorem 7.2.6]).

Theorem 2 *Let $\mathcal{B}$ be a uniform fading memory space which is separable, and suppose that the conditions* (H1)–(H4) *are satisfied. Then the following statements are equivalent:*

(i) *The solution $\bar{u}(t)$ of* (2) *is $\mathcal{B}$-UAS.*

(ii) *The solution $\bar{u}(t)$ of* (2) *is $\mathcal{B}$-US and $\mathcal{B}$-attractive in $\Omega(F)$.*

(iii) *The solution $\bar{u}(t)$ of* (2) *is $\mathcal{B}$-UAS in $\Omega(F)$.*

(iv) *The solution $\bar{u}(t)$ of* (2) *is $\mathcal{B}$-WUAS in $\Omega(F)$.*

Next we shall construct a quasi-process with $\mathcal{X} = \mathrm{BC}_\rho$ associated with (3); here and hereafter, BC_ρ denotes the space BC which is equipped with the metric ρ defined by

$$\rho(\phi,\psi) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|\phi-\psi|_{\mathrm{BC}([-n,0];X)}}{1+|\phi-\psi|_{\mathrm{BC}([-n,0];X)}}, \qquad \phi,\ \psi \in \mathrm{BC}_\rho.$$

It is well known that the topology induced by the metric ρ is equivalent to the compact open topology in BC.

We first provide an example which shows that a process on BC_ρ cannot be always constructed for functional differential equations with infinite delay.

Example. Consider a scalar delay equation

$$\dot{x}(t) = \sum_{n=1}^{\infty}(1/n^3)x(t-n), \tag{6}$$

which is a special case of (2) with $\mathcal{B} = C_g^0(R)$ $(g(s) = s+1)$, $A = 0$ and $F(t,\phi) = \sum_{n=1}^{\infty}(1/n^3)\phi(-n)$. It is clear that the conditions (H1)–(H3) are satisfied for this equation. Consider a sequence $\{\phi^k\} \subset \mathrm{BC}$ defined by $\phi^k(\theta) = 0$ if $-k \le \theta \le 0$, k^4 if $\theta \le -k-1$ and linear if $-k-1 \le \theta \le -k$. Clearly $\phi^k \to 0$ in BC_ρ. Let denote by $x(t,s,\phi)$ the solution of (6) through (s,ϕ). Then

$$\begin{aligned} x(1,0,\phi^k) &= \int_0^1 \sum_{n=1}^{\infty}(1/n^3)\phi^k(s-n)ds \\ &\ge 1/(k+2)^3 \int_0^1 \phi^k(s-k-2)ds \\ &\ge k^4/(k+2)^3 \ge 1 \end{aligned}$$

for $k \ge 10$. Note that $x(t,0,0) \equiv 0$ and $x_1(0,\phi^k) \not\to x_1(0,0)$ in BC_ρ. Hence the associated mapping $w : R^+ \times R^+ \times \mathrm{BC}_\rho \mapsto \mathrm{BC}_\rho$ defined by $w(t,s,\phi) = x_{t+s}(s,\phi)$, $(t,s,\phi) \in R^+ \times R^+ \times \mathrm{BC}_\rho$, is not continuous on $R^+ \times R^+ \times \mathrm{BC}_\rho$.

From the above example, we see that the concept of processes does not fit in with the study of the ρ-stabilities in functional differential equations. In what follows, we shall consider a subset $\mathcal{Y}$ of BC_ρ and construct a $\mathcal{Y}$-quasi-process on BC_ρ associated with (3) to overcome the above difficulty.

Let U be a closed and bounded subset of X whose interior contains the closure of the set $\{\bar{u}(t) : t \in R\}$, where $\bar{u}$ is the one in (H4). Set

$$\mathrm{BC}_\rho^U = \{\phi \in \mathrm{BC}_\rho : \phi(\theta) \in U \text{ for all } \theta \in R^-\}.$$

It is clear that BC_ρ^U is a nonempty closed subset of BC_ρ. With $\mathcal{X} = \mathrm{BC}_\rho$ and $\mathcal{Y} = \mathrm{BC}_\rho^U$, we shall construct the quasi-process associated with (3). Consider a function $w_\rho^G : R^+ \times R^+ \times \mathrm{BC}_\rho \mapsto \mathrm{BC}_\rho$ defined by

$$w_\rho^G(t,s,\phi) = u_{t+s}(s,\phi,G), \qquad (t,s,\phi) \in R^+ \times R^+ \times \mathrm{BC}_\rho,$$

which is the restriction of $w_\mathcal{B}^G$ to $R^+ \times R^+ \times \mathrm{BC}_\rho$.

Lemma 2 *w_ρ^G is a BC_ρ^U-quasi-process on BC_ρ.*

Proof. From (H3) we easily see that w_ρ^G satisfies (p1) and (p2) with $\mathcal{X} = \mathrm{BC}_\rho$. We shall show that w_ρ^G satisfies (p3) with $\mathcal{X} = \mathrm{BC}_\rho$ and $\mathcal{Y} = \mathrm{BC}_\rho^U$. Suppose the condition (p3) is not satisfied for w_ρ^G. Then there exist a point $(\bar{t},\bar{s},\bar{\phi}) \in R^+ \times R^+ \times \mathrm{BC}_\rho$ and sequences $\{t_n\} \subset R^+$, $\{s_n\} \subset R^+$ and $\{\phi^n\} \subset \mathrm{BC}_\rho^U$ such that $(t_n,s_n,\phi^n) \to (\bar{t},\bar{s},\bar{\phi})$ in $R^+ \times R^+ \times \mathrm{BC}_\rho$ as $n \to \infty$ and that $\inf_n \rho(u_{t_n+s_n}(s_n,\phi^n,G), u_{\bar{t}+\bar{s}}(\bar{s},\bar{\phi},G)) > 0$. Then there exists an integer $l > 0$ such that $\inf_n |u_{t_n+s_n}(s_n,\phi^n,G) - u_{\bar{t}+\bar{s}}(\bar{s},\bar{\phi},G)|_{\mathrm{BC}([-l,0];X)} > 0$, and hence there exists a sequence $\{\tau_n\} \subset [-l,0]$ such that

$$\inf_n |u(t_n+s_n+\tau_n, s_n, \phi^n, G) - u(\bar{t}+\bar{s}+\tau_n, \bar{s}, \bar{\phi}, G)|_X > 0. \tag{7}$$

Since $u(t,\bar{s},\bar{\phi},G)$ is continuous in $t \in R$, we get $\inf_n |u(t_n+s_n+\tau_n,s_n,\phi^n,G) - u(t_n+\bar{s}+\tau_n,\bar{s},\bar{\phi},G)|_X > 0$. Therefore it must hold that $t_n + \tau_n \geq 0$ for all sufficiently large n, because of $\rho(\phi^n,\bar{\phi}) \to 0$ as $n \to \infty$. Thus we can assume that $\lim_{n\to\infty} \tau_n = \bar{\tau}$ for some $\bar{\tau} \in [-l,0]$ with $\bar{t} + \bar{\tau} \geq 0$. Since $\lim_{n\to\infty} |\phi^n - \phi|_\mathcal{B} = 0$ by (A2), it follows from Proposition 1 that $\lim_{n\to\infty} |u_{t_n+s_n+\tau_n}(s_n,\phi^n,G) - u_{\bar{t}+\bar{s}+\bar{\tau}}(\bar{s},\bar{\phi},G)|_\mathcal{B} = 0$; which implies that $\lim_{n\to\infty} |u(t_n+s_n+\tau_n,s_n,\phi^n,G) - u(\bar{t}+\bar{s}+\bar{\tau},\bar{s},\bar{\phi},G)|_X = 0$ by (A1-iii). Therefore $\lim_{n\to\infty} |u(t_n+s_n+\tau_n,s_n,\phi^n,G) - u(\bar{t}+\bar{s}+\tau_n,\bar{s},\bar{\phi},G)|_X = 0$, which is a contradiction to (7).

The mapping w_ρ^G constructed above is called the BC_ρ^U-quasi-process on BC_ρ generated by (3).

Now we consider the BC_ρ^U-quasi-process w_ρ^F on BC_ρ generated by (2). By the same calculation as for $w_\mathcal{B}^F$, we see that $\sigma(\tau)w_\rho^F = w_\rho^{F^\tau}$, $H_\sigma(w_\rho^F) = \{w_\rho^G : G \in H(F)\}$ and $\Omega_\sigma(w_\rho^F) = \{w_\rho^G : G \in \Omega(F)\}$. Moreover, we see that $H_\sigma(w_\rho^F)$ is sequentially compact and the BC_ρ^U- quasi-process w_B^F satisfies the condition (p4). For $t \in R^+$, consider a mapping $\pi_\rho(t) : BC_\rho \times H_\sigma(w_\rho^F) \mapsto BC_\rho \times H_\sigma(w_\rho^F)$ defined by

$$\pi_\rho(t)(\phi, w_\rho^G) = (u_t(0,\phi,G), \sigma(t)w_\rho^G), \qquad (\phi, w_\rho^G) \in BC_\rho \times H_\sigma(w_\rho^F).$$

Notice that $\lim_{n\to\infty} |\phi^n - \phi|_{\mathcal{B}} = 0$ whenever $\{\phi^n\} \subset BC_\rho^U$ satisfies $\lim_{n\to\infty} \rho(\phi^n, \phi) = 0$. Therefore, repeating almost the same argument as in the proof of Lemma 2, one can see that $\pi_\rho(t)$ satisfies the condition (p5) with $\mathcal{Y} = BC_\rho^U$, and hence $\pi_\rho(t)$ is the skew product flow of w_ρ^F.

Lemma 3 *The skew product flow $\pi_\rho(t)$ is BC_ρ^U-strongly asymptotically smooth.*

Proof. It suffices to show that for the set $BC_\rho^U \times H_\sigma(w_\rho^F)$ there exists a compact set $J \subset BC_\rho^U \times H_\sigma(w_\sigma^F)$ with the property that $\{\pi_\rho(t_n)(\phi^n, w_\rho^{G_n})\}$ has a subsequence which approaches to J whenever sequences $\{t_n\} \subset R^+$ and $\{(\phi^n, w_\rho^{G_n})\} \subset BC_\rho^U \times H_\sigma(w_\rho^F)$ satisfy $\lim_{n\to\infty} t_n = \infty$ and $\pi_\rho(t)(\phi^n, w_\rho^{G_n}) \subset BC_\rho^U \times H_\sigma(w_\rho^F)$ for all $t \in [0, t_n]$. This can be done by the same arguments as in the proof of Lemma 1. Indeed, since the set U is bounded in X, putting $B = BC_\rho^U \times H_\sigma(w_\rho^F)$ we can construct the set J_B as in the proof of Lemma 1. Then the set $J := (J_B \cap BC_\rho^U) \times H_\sigma(w_\rho^F)$ is a compact set in $BC_\rho^U \times H_\sigma(w_\rho^F)$. By virtue of (A2) and Ascoli-Arzéela's theorem, J has the desired property, because the function $x^n(t)$ in the proof of Lemma 1 satisfies $x^n(t) \in O_B$ and $|x^n(t) - x^n(s)|_X \le \sup\{|T(t-s)z - z|_X : z \in O_B\} + C_1L|t-s|$ for any s, t with $1 \le s \le t \le t_n$ and $|t-s| \le 1$.

For any function $\xi : R \mapsto X$ such that $\xi_0 \in BC_\rho^U$ and ξ is continuous on R^+, we define a continuous function $\mu_\rho^\xi : R^+ \mapsto BC_\rho$ by

$$\mu_\rho^\xi(t) = \xi_t, \qquad t \in R^+.$$

It follows from (H4) that $\mu_\rho^{\bar{u}}$ is an integral of the quasi-process w_ρ^F on R^+. Let $\eta > 0$ be chosen so that the interior of U contains the η-neighborhood of the set $\{\bar{u}(t) : t \in R\}$. Then we easily see that (p6) is satisfied with $\delta_1 := \eta$ as $\mathcal{Y} = BC_\rho^U$, $w = w_\rho^F$ and $\mu = \mu_\rho^{\bar{u}}$ because of the inequality $|u(t+s,s,\phi,F) - \bar{u}(t+s)|_X \le \rho(w_\rho^F(t,s,\phi), \mu_\rho^{\bar{u}}(t))$. Also, we get $H_\sigma(\mu_\rho^{\bar{u}}, w_\rho^F) = \{(\mu_\rho^{\bar{v}}, w_\rho^G) : (\bar{v}, G) \in H(\bar{u}, F)\}$ and $\Omega_\sigma(\mu_\rho^{\bar{u}}, w_\rho^F) = \{(\mu_\rho^{\bar{v}}, w_\rho^G) : (\bar{v}, G) \in \Omega(\bar{u}, F)\}$.

The BC_ρ^U-stabilities of the integral $\mu_\rho^{\bar{u}}$ of the quasi-process w_ρ^F yield the ρ-stabilities with respect to U for the solution $\bar{u}(t)$ of (2). For example, the solution $\bar{u}(t)$ of (2) is ρ-uniformly stable with respect to U in $\Omega(F)$(ρ-US with respect to U in $\Omega(F)$), if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\rho(\bar{u}_t(s,\phi,G), \bar{v}_t) < \varepsilon$ for $t \ge s \ge 0$ whenever $(\bar{v}, G) \in \Omega(\bar{u}, F)$, $\rho(\phi, \bar{v}_s) < \delta(\varepsilon)$ and $\phi(s) \in U$ for all $s \in R^-$. The other ρ-stabilities with respect to U for $\bar{u}(t)$ are given in a similar way; we omit the details.

The following result is a direct consequence of Theorem 1 and Lemmas 2 and 3.

Theorem 3 *Let $\mathcal{B}$ be a fading memory space which is separable and suppose that the conditions* (H1)–(H4) *are satisfied. Also, let U be a closed and bounded subset of X whose interior contains the closure of the set $\{\bar{u}(t) : t \in R\}$. Then the following statements are equivalent:*

(i) *The solution $\bar{u}(t)$ of* (2) *is ρ-UAS with respect to U.*

(ii) *The solution $\bar{u}(t)$ of* (2) *is ρ-US with respect to U and ρ-attractive with respect to U in $\Omega(F)$.*

(iii) *The solution $\bar{u}(t)$ of* (2) *is ρ-UAS with respect to U in $\Omega(F)$.*

(iv) *The solution $\bar{u}(t)$ of* (2) *is ρ-WUAS with respect to U in $\Omega(F)$.*

As an application of Theorem 3 (or Theorem 2), we will investigate stability properties for some solution of the following integrodifferential equation with diffusion

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) - u^3(t,x) \\ &\quad + \int_{-\infty}^{t} k(t,s,x)u(s,x)ds + h(t,x), \quad t>0,\ 0<x<\pi,\end{aligned} \tag{8}$$

under the Neumann boundary condition

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,\pi) = 0, \quad t>0, \tag{9}$$

where the functions $h(t,x)$ and $k(t,s,x)$ are continuous functions satisfying $2 \le h(t,x) \le 7$ and $0 \le k(t,s,x) \le K(t-s)$ for some continuous function $K(\tau)$ with $\int_0^\infty K(\tau)d\tau < 1/4$, and moreover $h(t,x)$ and $k(t,t+s,x)$ are almost periodic in t uniformly for $(s,x) \in R^- \times [0,\pi]$. In [8], it has been shown that (8)–(9) is represented as the functional differential equation (2) satisfying (H1)–(H2) with $X = \mathrm{BC}([0,\pi];R)$, $A\xi = \xi''$ for $\xi \in D(A) := \{\xi \in C^2[0,\pi] : \xi'(0) = \xi'(\pi) = 0\}$ and $F(t,\phi)(x) = h(t,x) - \phi^3(0,x) + \int_{-\infty}^0 k(t,t+s,x)\phi(s,x)ds$ for $\phi \in C_g^0(X)$ (with an appropriate function g such that $\int_0^\infty K(\tau)g(-\tau)d\tau < \infty$), and there exists an almost periodic (mild) solution $\bar{u}(t,x)$ of (8)–(9) such that $1 \le \bar{u}(t,x) \le 2$ on $R \times [0,\pi]$. In the following, we shall see that the solution $\bar{u}(t,x)$ of (8)–(9) is ρ-UAS with respect to the set $U := \{\xi \in \mathrm{BC}([0,\pi];R) : -b \le \xi(x) \le b \text{ on } [0,\pi]\}$, where $b > 2$. Notice that the equations in the hull of (8) are of the form

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) - u^3(t,x) \\ &\quad + \int_{-\infty}^{t} \tilde{k}(t,s,x)u(s,x)ds + \tilde{h}(t,x), \quad t>0,\ 0<x<\pi;\end{aligned} \tag{10}$$

here $\lim_{n\to\infty} k(t+t_n, t+t_n+s) = \tilde{k}(t,t+s,x)$ and $\lim_{n\to\infty} h(t+t_n,x) = \tilde{h}(t,x)$ uniformly for $(t,s,x) \in R \times J \times [0,\pi]$ for any compact set J in R^-, where $\{t_n\}$ is some sequence in R^+. The functions $\tilde{k}(t,s,x)$ and $\tilde{h}(t,x)$ are of the same type as $k(t,s,x)$ and $h(t,x)$. Hence, by employing almost the same manner as in the proof of [8, Lemma 4], one can see that the condition (H3) is satisfied for (8)–(9).

Proposition 2 *Suppose that the conditions on $h(t,x)$ and $k(t,s,x)$ stated above are satisfied. Then the solution $\bar{u}(t,x)$ of (8)–(9) is ρ-WUAS with respect to the set U in $\Omega(F)$. Consequently, $\bar{u}(t,x)$ is ρ-UAS with respect to U.*

Proof. We first prove that $\bar{u}(t,x)$ is ρ-US with respect to U in $\Omega(F)$. To do this, it is sufficient to show that the solution $v(t,x)$ of (10)–(9) on $[\sigma,\infty)$ such that $v(\sigma+\theta,x)\equiv\phi(\theta,x)$ on $R^-\times[0,\pi]$ satisfies

$$|v(t,x)-\bar{v}(t,x)|<\varepsilon, \qquad t\geq\sigma,\ \ x\in[0,\pi], \tag{11}$$

whenever $|\phi(\theta,x)-\bar{v}(\sigma+\theta,x)|<\varepsilon$ for $(\theta,x)\in[-l,0]\times[0,\pi]$; here $\bar{v}(t,x)$ is the solution of (10)–(9) such that $\lim_{n\to\infty}\bar{u}(t+t_n,x)=\bar{v}(t,x)$ for the same sequence $\{t_n\}$ as for $\tilde{k}$ and $\tilde{h}$, and l is a natural number satisfying $b\int_l^\infty K(\tau)d\tau<\varepsilon/4$. Set $w(t,x)=v(t,x)-\bar{v}(t,x)$. Then $w(t,x)$ is a (mild) solution of

$$\frac{\partial w}{\partial t}(t,x)=\frac{\partial^2 w}{\partial x^2}(t,x)-w(t,x)(v^2(t,x)+v(t,x)\bar{v}(t,x)+\bar{v}^2(t,x))$$

$$+\int_{-\infty}^t\tilde{k}(t,s,x)w(s,x)ds, \quad t>\sigma,\ \ 0<x<\pi,$$

$$\frac{\partial w}{\partial x}(t,0)=\frac{\partial w}{\partial x}(t,\pi)=0, \quad t>\sigma.$$

Assume that (11) is not true. Then there exists a $(t_1,x_1)\in(\sigma,\infty)\times[0,\pi]$ such that $|w(t,x)|<\varepsilon$ on $[\sigma,t_1)\times[0,\pi]$ and $|w(t_1,x_1)|=\varepsilon$. Consider two functions $p(t,x)$ and $q(t,x)$ in $C([\sigma,t_1]\times[0,\pi])$ defined by $p(t,x)=v^2(t,x)+v(t,x)\bar{v}(t,x)+\bar{v}^2(t,x)$ and $q(t,x)=\int_{-\infty}^t\tilde{k}(t,s,x)w(s,x)ds$ and choose sequences $\{p_n(t,x)\}$ and $\{q_n(t,x)\}$ in $C^1([\sigma,t_1]\times[0,\pi])$ such that $\lim_{n\to\infty}p_n(t,x)=p(t,x)$ and $\lim_{n\to\infty}q_n(t,x)=q(t,x)$ uniformly on $[\sigma,t_1]\times[0,\pi]$. Moreover, we choose a sequence $\{\xi_n\}\subset D(A)$ such that $\lim_{n\to\infty}\xi_n(x)=w(\sigma,x)$ uniformly on $[0,\pi]$. Then there exists a (classical) solution $w_n(t,x)$ of the initial-boundary value problem

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x)-p_n(t,x)u(t,x)+q_n(t,x), \quad \sigma<t\leq t_1,\ \ 0<x<\pi,\\
\frac{\partial u}{\partial x}(t,0) &= \frac{\partial u}{\partial x}(t,\pi)=0, \quad \sigma<t\leq t_1,\\
u(\sigma,x) &= \xi_n(x), \quad 0<x<\pi.
\end{aligned}$$

Clearly $\lim_{n\to\infty} w_n(t,x) = w(t,x)$ uniformly on $[\sigma, t_1]\times[0,\pi]$. Since $p(t,x) \geq (3/4)\bar{v}^2(t,x) \geq 3/4$ and $|q(t,x)| \leq \int_0^l K(\theta)|w(t-\theta,x)|d\theta + \int_l^\infty K(\theta)|w(t-\theta,x)|d\theta < \varepsilon/4 + b\int_l^\infty K(\theta)d\theta < \varepsilon/2$ on $[\sigma,t_1]\times[0,\pi]$, one can select $n\in\mathbf{N}$, $\varepsilon' \in (3\varepsilon/4, \varepsilon)$ and $(t_2, x_2) \in (\sigma, t_1)\times[0,\pi]$ so that $p_n(t,x) > 2/3$ and $|q_n(t,x)| < \varepsilon/2$ on $[\sigma,t_1]\times[0,\pi]$ and that $|w_n(t,x)| < \varepsilon'$ on $[\sigma,t_2)\times[0,\pi]$ and $|w_n(t_2,x_2)| = \varepsilon'$, say $w_n(t_2,x_2) = \varepsilon'$. Then the function $W(t,x) := w_n(t,x) - \varepsilon'$ satisfies

$$\begin{aligned}\frac{\partial^2 W}{\partial x^2}(t,x) - \frac{\partial W}{\partial t}(t,x) - p_n(t,x)W(t,x) &= \varepsilon' p_n(t,x) - q_n(t,x)\\ &> 2\varepsilon'/3 - \varepsilon/2 > 0\end{aligned}$$

on $(\sigma,t_1]\times(0,\pi)$. Then we get a contradiction by the strong maximum principle (cf. e.g. [15, Theorems 3.6 and 3.7]). Indeed, if $x_2\in(0,\pi)$, then $W(t,x)\equiv 0$ or $w_n(t,x)\equiv\varepsilon'$ on $[\sigma,t_2]\times[0,\pi]$ by the strong maximum principle, which is a contradiction because of $|w_n(\sigma,x)| < \varepsilon'$ on $[0,\pi]$. We thus obtain $W(t,x) < 0$ on $[\sigma,t_2]\times(0,\pi)$, and $x_2 = 0$ or $x_2 = \pi$; say $x_2 = \pi$. Then we get $(\partial W/\partial x)(t_2,\pi) > 0$ by the strong maximum principle again. This is also a contradiction, because of $(\partial W/\partial x)(t_2,\pi) = (\partial w_n/\partial x)(t_2,\pi) = 0$. Therefore, (11) must hold true.

Next we shall establish the ρ-attractivity with respect to U of the solution $\bar{u}(t,x)$ in $\Omega(F)$. To do this, it is sufficient to show that $\lim_{t\to\infty}\max_{0\leq x\leq\pi}|v(t,x) - \bar{v}(t,x)| = 0$ for any solution $v(t,x)$ of (10)–(9) such that $|v(t,x)-\bar{v}(t,x)| < b$ on $R\times[0,\pi]$. Consider a continuous function $V: R\mapsto R^+$ defined by

$$V(t) = \int_0^\pi |v(t,x) - \bar{v}(t,x)|^2 dx, \quad t\in R.$$

If v and $\bar{v}$ are smooth solutions of (10)–(9), then

$$\begin{aligned}(d/dt)V(t) &= 2\int_0^\pi \{(\partial/\partial t)v(t,x) - (\partial/\partial t)\bar{v}(t,x)\}(v(t,x)-\bar{v}(t,x))dx\\ &= 2\int_0^\pi \{(\partial^2/\partial x^2)(v(t,x)-\bar{v}(t,x)) - (v^3(t,x) - \bar{v}^3(t,x))\\ &\qquad + \int_{-\infty}^t \tilde{k}(t,s,x)(v(s,x)-\bar{v}(s,x))ds\}\{v(t,x)-\bar{v}(t,x)\}dx\\ &\leq -3/2\int_0^\pi |v(t,x)-\bar{v}(t,x)|^2 dx\\ &\qquad +2\int_{-\infty}^t K(t-s)\int_0^\pi |v(s,x)-\bar{v}(s,x)||v(t,x)-\bar{v}(t,x)|dxds\\ &\leq -(3/2)V(t) + 2\sqrt{V(t)}\int_{-\infty}^t K(t-s)\sqrt{V(s)}ds\end{aligned}$$

for $t\geq 0$. When v or $\bar{v}$ is not smooth, approximating v and $\bar{v}$ by smooth solutions of (10)–(9) we see that the Dini-derivative $D^+V(t) = \limsup_{\tau\to t+0}[V(\tau)-V(t)]/(\tau - t)$ exists and it satisfies

$$D^+V(t) \leq -(3/2)V(t) + 2\sqrt{V(t)}\int_{-\infty}^t K(t-s)\sqrt{V(s)}ds, \quad t\in R^+.$$

For any $\eta > 0$, there exists an $m > 0$ such that $2\sqrt{V(t)}\int_{-\infty}^{t-m} K(t-s)\sqrt{V(s)}ds < \eta/2$ for $t \in R^+$. Observe that $D^+V(t) < -(1/2)V(t) + \eta/2$ whenever $2V(t) \geq \sup_{t-m\leq s\leq t} V(s)$. Then, by the standard argument of Razumikhin type (cf., e.g. [16, Theorem 2] or [12, Theorem 2]) we get $\limsup_{t\to\infty} V(t) \leq \eta$. Since η is given arbitrarily, we get $\lim_{t\to\infty} V(t) = 0$; that is, $v(t,\cdot) - \bar{v}(t,\cdot) \to 0$ in $L^2[0,\pi]$ as $t \to \infty$. Notice that v and $\bar{v}$ are bounded on $R^+ \times [0,\pi]$. Then orbits of $v(t,\cdot)$ and $\bar{v}(t,\cdot)$ are relative compact in BC($[0,\pi]; R$); see e.g. [14, p.184]. From these facts, we have that $v(t,\cdot) - \bar{v}(t,\cdot) \to 0$ in BC($[0,\pi]; R$) as $t \to \infty$, as required. Thus the solution $\bar{u}(t,x)$ of (8)–(9) is ρ-WUAS with respect to U in $\Omega(F)$ and it is ρ-UAS with respect to U by Theorem 3. This completes the proof of Proposition 2.

Under the additional assumption that $\int_0^\infty K(s)e^{\gamma s}ds < \infty$ for some constant $\gamma > 0$, (8)–(9) can be formulated as an functional differential equation on a uniform fading memory space $C_g^0(X)$ with $g(\theta) = e^{-\gamma\theta}$. Repeating almost the same argument as in the above, we can check that the solution $\bar{u}(t,x)$ of (8)–(9) is $C_g^0(X)$-WUAS in $\Omega(F)$, and hence $\bar{u}(t,x)$ is $C_g^0(X)$-UAS by Theorem 2; the details are omitted.

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