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ON THE DETERMINATION OF THE HEAT CONDUCTIVITY FROM THE HEAT FLOW

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Introduction

We study the inverse problem to determine $a(t)$ of the parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} = a(t)\dfrac{\partial^2 u}{\partial x^2} \quad (0 < x < \infty, 0 < t < T), \\ u(x,0) = 0 \quad (0 \le x < \infty), \\ u(0,t) = f(t) \quad (0 \le t < T), \\ -a(t)\dfrac{\partial u}{\partial x}(0,t) = g(t) \quad (0 < t < T), \end{cases} \tag{0.1}$$

so that this (overspecified) system admits a classical solution $u(x,t)$ satisfying, for each $T' < T$,

$$\sup_{0<t<T'} \left\{ |u(x,t)| + \left| \frac{\partial u}{\partial x}(x,t) \right| \right\} = O(e^{x^\alpha}) \quad (x \to \infty). \tag{0.2}$$

with some constant $\alpha < 2$.

This problem was studied by several authors ([1,2,3,5]), and various existence and uniqueness results were established. However they have been accomplished under the assumption that $f(t)$ is a monotonically nondecreasing function. The purpose of the present paper is to investigate the problem without this assumption.

Let us assume that
(I) $a(t)$ is positive and continuous for $0 \le t < T$,
(II) $f(t)$ is continuous for $0 \le t < T$ and $f(0) = 0$.
Then the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = a(t)\dfrac{\partial^2 u}{\partial x^2} \quad (0 < x < \infty, 0 < t < T), \\ u(x,0) = 0 \quad (0 \le x < \infty), \\ u(0,t) = f(t) \quad (0 \le t < T), \end{cases}$$

is uniquely solvable under the assumption (0.2), and the solution $u(x,t)$ can be expressed as

$$u(x,t) = -2\int_0^t \frac{\partial H}{\partial x}\left(x, \int_\tau^t a(\tau)d\tau\right) a(\tau)f(\tau)d\tau,$$

where $H(x,t)$ is the fundamental solution of the heat equation:

$$H(x,t) := \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

Hence, as was shown in [2] (also see [5]), if f is differentiable then the inverse problem mentioned in the beginning is equivalent to finding a positive solution $a(t)$ of the nonlinear integral equation

$$\frac{1}{\sqrt{\pi}} a(t) \int_0^t \frac{f'(\tau)}{\left(\int_\tau^t a(r)dr\right)^{1/2}} d\tau = g(t) \quad (0 < t < T). \tag{0.3}$$

We hereafter focus our attention on the equation (0.3). The main goal here is to show that the equation (0.3) is solvable near $t = 0$ and the continuation of the solution can be made as far as it is bounded above, without the monotonicity of $f(t)$.

Throughout this paper we use the notation

$$C_+(I) := \{a(t) \in C(I) \mid a(t) > 0 \quad (t \in I)\}.$$

In Section 1 we shall establish a uniqueness result. In Section 2 we shall establish a local existence result. In Section 3 we shall discuss the continuation of solution. The main result will be given in Section 4.

1. Uniueness

In this section we shall establish the following uniqueness result:

Theorem 1.1. *Assume that*

(i) $f(t) \in C[0,T) \cap C^1(0,T)$, $\lim_{t\to 0} t^{1-\mu} f'(t) > 0$ *with some* $\mu > 0$*;*

(ii) $g(t) \in C_+(0,T)$.

If $a_1(t), a_2(t) \in C_+[0,T)$ *are solutions of* (0.3) *then* $a_1(t) \equiv a_2(t)$.

Before the proof we shall give some remarks on the assumptions:

Remark 1.2. By the substitution $\tau = t\rho$, (0.3) can be rewritten as

$$t^{\mu-1/2} a(t) \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{\left(\int_\rho^1 a(tr)dr\right)^{1/2}} \frac{d\rho}{\rho^{1-\mu}} = \sqrt{\pi} g(t) \quad (0 < t < T). \tag{1.1}$$

Accordingly the assumption (i) implies that there exists the limit

$$\lim_{t\to 0} t^{1/2-\mu} g(t) > 0 \tag{1.2}$$

In addition to the assumption (ii) we assume that $g(t) \in C[0,T)$. Then it follows from (1.2) that the condition $\mu \geq 1/2$ is necessary. Moreover if (0.3) has a solution $a(t) \in C_+[0,T)$ then (0.3) holds even at $t = 0$.

We now give the proof of Theorem 1.1. Let $T_1 \in (0,T)$ be fixed. By (1.1) we obtain for $0 < t \leq T_1$,

$$a_2(t)\int_0^1 \frac{(t\rho)^{1-\mu}f'(t\rho)}{\left(\int_\rho^1 a_2(tr)dr\right)^{1/2}}\frac{d\rho}{\rho^{1-\mu}} = a_1(t)\int_0^1 \frac{(t\rho)^{1-\mu}f'(t\rho)}{\left(\int_\rho^1 a_1(tr)dr\right)^{1/2}}\frac{d\rho}{\rho^{1-\mu}}. \tag{1.3}$$

By taking the limit as $t \to 0$, this yields

$$a_2(0) = a_1(0). \tag{1.4}$$

We put

$$b(t) := a_2(t) - a_1(t), \quad p(t) := \int_0^1 \frac{(t\rho)^{1-\mu}f'(t\rho)}{\left(\int_\rho^1 a_2(tr)dr\right)^{1/2}}\frac{d\rho}{\rho^{1-\mu}}.$$

Then, from (1.3), we have

$$\begin{aligned}
b(t)p(t) &= (a_2(t) - a_1(t))p(t)\\
&= a_1(t)\int_0^1 \left\{\frac{1}{\left(\int_\rho^1 a_1(tr)dr\right)^{1/2}} - \frac{1}{\left(\int_\rho^1 a_2(tr)dr\right)^{1/2}}\right\}(\rho t)^{1-\mu}f'(t\rho)\frac{d\rho}{\rho^{1-\mu}}\\
&= a_1(t)\int_0^1 \frac{\int_\rho^1 b(tr)dr}{\prod_{j=1}^2\left(\int_\rho^1 a_j(tr)dr\right)^{1/2}\left[\sum_{j=1}^2\left(\int_\rho^1 a_j(tr)dr\right)^{1/2}\right]}(\rho t)^{1-\mu}f'(t\rho)\frac{d\rho}{\rho^{1-\mu}}\\
&= a_1(t)\int_0^1 b(t\sigma)d\sigma\int_0^\sigma \frac{(t\rho)^{1-\mu}f'(t\rho)}{\prod_{j=1}^2\left(\int_\rho^1 a_j(tr)dr\right)^{1/2}\left[\sum_{j=1}^2\left(\int_\rho^1 a_j(tr)dr\right)^{1/2}\right]}\frac{d\rho}{\rho^{1-\mu}},
\end{aligned}$$

where we have used interchange of the order of integration. Therefore, by setting

$$\Phi(t,\sigma) := \frac{a_1(t)}{p(t)}\int_0^\sigma \frac{(t\rho)^{1-\mu}f'(t\rho)}{\prod_{j=1}^2\left(\int_\rho^1 a_j(tr)dr\right)^{1/2}\left[\sum_{j=1}^2\left(\int_\rho^1 a_j(tr)dr\right)^{1/2}\right]}\frac{d\rho}{\rho^{1-\mu}},$$

we arrive at

$$b(t) = \int_0^1 \Phi(t,\sigma)b(t\sigma)d\sigma \quad (0 \leq t \leq T'). \tag{1.5}$$

In view of (1.1), $p(t) = \sqrt{\pi}t^{1/2-\mu}a_2(t)^{-1}g(t)$. Hence, by the assumption (ii), $p(t)$ is positive for $0 < t \leq T'$. But, in view of the definition of $p(t)$ and the assumption (i), $p(t)$ is a continuous function on the interval $[0, T_1]$ with $p(0) > 0$. So $\min_{0\leq t\leq T'} p(t) =: c > 0$. This shows that

$$|\Phi(t,\sigma)| \leq M_1\int_0^\sigma \frac{1}{(1-\rho)^{3/2}}\frac{d\rho}{\rho^{1-\mu}} \leq \frac{M}{(1-\sigma)^{1/2}}. \tag{1.6}$$

Moreover, from (1.4), we get

$$\begin{aligned}\Phi(\sigma) := \lim_{t\to 0}\Phi(t,\sigma) &= \frac{1}{2}\frac{1}{\int_0^1 \frac{d\rho}{(1-\rho)^{1/2}\rho^{1-\mu}}}\int_0^{\sigma}\frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}} \\ &= \frac{1}{2}\frac{1}{B(\mu,1/2)}\int_0^{\sigma}\frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}} > 0,\end{aligned} \tag{1.7}$$

where $B(\cdot,\cdot)$ denote the beta function. Note that this convergence is uniform with respect to σ in the following sense:

$$\lim_{t\to 0}\sup_{0\le\sigma<1}(1-\sigma)^{1/2}\,|\Phi(t,\sigma)-\Phi(\sigma)| = 0. \tag{1.8}$$

We now define

$$J_\Phi z(t) := \int_0^1 \Phi(\sigma)z(t\sigma)d\sigma \quad (0\le t\le\Lambda)$$

for all $z(t)$ in the Banach space $C[0,\Lambda]$ of all continuous functions on $[0,\Lambda]$ (with norm $||\cdot||_\Lambda$ given $||z||_\Lambda := \max_{0\le t\le\Lambda}|z(t)|$). Then J_Φ is a bounded linear operator from $C[0,\Lambda]$ to itself, and the operator norm $||J_\Phi||_\Lambda$ of $J_\Phi : C[0,\Lambda]\to C[0,\Lambda]$ is computed as

$$\begin{aligned}||J_\Phi||_\Lambda = \int_0^1\Phi(\sigma)d\sigma &= \frac{1}{2}\frac{1}{B(\mu,1/2)}\int_0^1 d\sigma\int_0^{\sigma}\frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}} \\ &= \frac{1}{2}\frac{1}{B(\mu,1/2)}\int_0^{\sigma}\frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}}\int_\rho^1 d\sigma = \frac{1}{2}.\end{aligned}$$

Accordingly, by means of the Neumann series, the operator $I-J_\Phi : C[0,\Lambda]\to C[0,\Lambda]$ has the bounded inverse $(I-J_\Phi)^{-1}$, where I denotes the identity operator in $C[0,\Lambda]$.

Since (1.5) can be written as

$$(I-J_\Phi)b(t) = \int_0^1[\Phi(t,\sigma)-\Phi(\sigma)]b(t\sigma)d\sigma,$$

we obtain for $0<\Lambda\le T_1$,

$$\begin{aligned}||b||_\Lambda &\le ||(I-J_\Phi)^{-1}||_\Lambda \max_{0\le t\le\Lambda}\int_0^1|\Phi(t,\sigma)-\Phi(\sigma)|d\sigma\ ||b||_\Lambda \\ &\le 2\int_0^1\max_{0\le t\le\Lambda}(1-\sigma)^{1/2}|\Phi(t,\sigma)-\Phi(\sigma)|\frac{d\sigma}{(1-\sigma)^{1/2}}\ ||b||_\Lambda.\end{aligned}$$

This, together with (1.8), shows that there exists $\delta>0$ such that $||b||_\delta = 0$, that is, $b(t)=0$ for any $t\in[0,\delta]$.

For $\delta \le t \le T_1$ it follows from (1.5), (1.6) that

$$|b(t)| = \left| \int_0^1 \Phi(t,\sigma) b(t\sigma) d\sigma \right| \le M \int_0^1 \frac{|b(t\sigma)|}{(1-\sigma)^{1/2}} d\sigma$$
$$= \frac{M}{t^{1/2}} \int_\delta^t \frac{|b(\tau)|}{(t-\tau)^{1/2}} d\tau \le \frac{M}{\delta^{1/2}} \int_\delta^t \frac{|b(\tau)|}{(t-\tau)^{1/2}} d\tau.$$

This leads to

$$|b(t)| \le \frac{M^2}{\delta} \int_\delta^t \frac{d\tau}{(t-\tau)^{1/2}} \int_\delta^\tau \frac{|b(s)|}{(\tau-s)^{1/2}} ds = \pi \frac{M^2}{\delta} \int_\delta^t |b(s)| ds \quad (\delta \le t \le T_1).$$

By virtue of Gronwall's inequality this shows that $b(t) = 0$ $(\delta \le t \le T_1)$. The proof of Theorem 1.1 is complete.

We wish to point out that, even under the assumption that $f(t)$ is monotonically nondecreasing, there apear cases in which Theorem 1.1 is of vital importance. For instance, we consider the case $f(t) \equiv t, g(t) = (2/\sqrt{\pi})t^{1/2}$. Then it is clear that $a(t) \equiv 1$ is a solution of (0.3). Since the assumptions in Theorem 1.1 are satisfied we can apply the theorem to conclude that this trivial solution is a unique solution of (0.3).

2. Local existence

In this section we shall establish the following local existence theorem:

Theorem 2.1. *Assume that, with some* $\mu > 0$,

(i) $f(t) \in C[0,T) \cap C^1(0,T)$, $\lim_{t\to 0} t^{1-\mu} f'(t) > 0$;

(ii) $g(t) \in C_+(0,T)$, $\lim_{t\to 0} t^{1/2-\mu} g(t) > 0$.

Then, for sufficiently small $T_0 > 0$, (0.3) *has a solution* $a(t) \in C_+[0,T_0]$.

Since the assumptions (i) and (ii) imply that $f'(t) > 0, g(t) > 0$ near $t = 0$, in the case $1/2 \le \mu$, this result is a direct consequense of [5, Theorem 3]; and also, in the case $1/2 \le \mu < 1$, of [2, Theorem 4]. We give an alternative proof of Theorem 2.1, however, in order to make the present paper readable, and in order to make the sprit in the paper transparent.

Proof of Theorem 2.1. Let $f(t), g(t)$ be a function satisfying (i), (ii) and put

$$P := \lim_{t\to 0} t^{1-\mu} f'(t); \quad Q := \lim_{t\to 0} t^{1/2-\mu} g(t),$$

Moreover we define a function $g_0(t)$ by

$$g_0(t) := \frac{Q/P}{B(\mu,1/2)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{1/2}} d\tau = \frac{Q/P}{B(\mu,1/2)} t^{\mu-1/2} \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{1/2}\rho^{1-\mu}} d\rho, \tag{2.1}$$

and consider a mapping defined by

$$F(a(t)) = t^{1/2-\mu} a(t) \int_0^t \frac{f'(\tau)}{\left(\int_\tau^t a(r) dr\right)^{1/2}} d\tau - \sqrt{\pi} t^{1/2-\mu} g_0(t).$$

It is easy to see that the constant function

$$a_0(t) := \left(\sqrt{\pi}\frac{Q/P}{B(\mu,1/2)}\right)^2$$

satisfies $F(a(t)) = 0$, and that, for each $T_1 < T$, F is a C^1-mapping of an open neighbourhood of a_0 in $C[0,T_1]$ to $C[0,T_1]$. The Fréchet derivative $F_a(a_0)$ at a_0 is computed as,

$$\begin{aligned} F_a(a_0)a(t) &= At^{1/2-\mu}\left\{\int_0^t \frac{f'(\tau)}{(t-\tau)^{1/2}}d\tau - \frac{1}{2}\int_0^t \frac{f'(\tau)}{(t-\tau)^{3/2}}d\tau \int_\tau^t a(r)dr\right\} \\ &= At^{1/2-\mu}\left\{\int_0^t \frac{f'(\tau)}{(t-\tau)^{1/2}}d\tau - \frac{1}{2}\int_0^t a(r)dr \int_0^r \frac{f'(\tau)}{(t-\tau)^{3/2}}d\tau\right\}, \\ &= A\left\{\omega(t)a(t) - \frac{1}{2}\int_0^1 a(t\sigma)d\sigma \int_0^\sigma \frac{(t\rho)^{1-\mu}f'(t\rho)}{(1-\rho)^{3/2}\rho^{1-\mu}}d\rho\right\}, \end{aligned}$$

for each $a(t) \in C[0,T_1]$. Here we set

$$A := \left(\sqrt{\pi}\frac{Q/P}{B(\mu,1/2)}\right)^{-1}, \quad \omega(t) := \int_0^1 \frac{(t\rho)^{1-\mu}f'(t\rho)}{(1-\rho)^{1/2}\rho^{1-\mu}}d\rho$$

Let $h(t) \in C[0,T_1]$ and consider the equation

$$F_a(a_0)a(t) = h(t), \ (0 \le t \le T_1). \tag{2.2}$$

By assumption, the function $\omega(t)$ is positive for sufficiently small t. Hence, if T_1 is sufficiently small then the equation (2.2) is equivalent to

$$a(t) - \int_0^1 \Omega(t,\sigma)a(t\sigma)d\sigma = \tilde{h}(t), \ (0 \le t \le T_1), \tag{2.3}$$

where we put

$$\Omega(t,\sigma) := \frac{1}{2\omega(t)}\int_0^\sigma \frac{(t\rho)^{1-\mu}f'(t\rho)}{(1-\rho)^{1/2}\rho^{1-\mu}}d\rho, \quad \tilde{h}(t) := (A\omega(t))^{-1}h(t).$$

By interchange of the order of integration we have

$$\lim_{t\to 0}\int_0^1 |\Omega(t,\sigma)|d\sigma = \frac{1}{2B(1/2,\mu)}\int_0^1 d\sigma \int_0^\sigma \frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}} = 1/2.$$

Therefore, by means of the Neumann series, the equation (2.3) is uniquely solvable in the space $C[0,T_1]$, provided that T_1 is sufficiently small. This shows that $F_a(a_0) : C[0,T_1] \to C[0,T_1]$ has a bounded linear inverse. Hence, by the implicit function theorem (see e.g. [4, Theorem 1.20]), we conclude that there exists $\delta > 0$ such that the equation $F(a(t)) = h(t)$ has a solution $a(t)$ in $C[0,T_1]$ if $\max_{0\le t\le T_1} |h(t)| < \delta$.

We now set

$$k(t) := \sqrt{\pi} t^{1/2-\mu} g(t) - \sqrt{\pi} t^{1/2-\mu} g_0(t).$$

By the definition (2.1) it follows that $\lim_{t \to 0} k(t) = 0$. Noting that δ may depend on T_1 we introduce a function $\tilde{k}(t)$ so that $\tilde{k}(t) = k(t)$ near 0 : in $[0, T_1']$, say; and so that $\max_{0 \le t \le T_1} |\tilde{k}(t)| < \delta$. Then $F(a)(t) = \tilde{k}(t)$ has a solution $a(t)$ in $C[0, T_1]$. Then $a(t)$ satisfies (0.3) for $0 \le t \le T_2'$. This completes the proof of Theorem 2.1.

3. Continuation

In this section we shall establish the following continuation theorem:

Theorem 3.1. *Assume that*
(i) $f(t) \in C[0,T) \cap C^1(0,T)$;
(ii) $g(t) \in C_+(0,T)$.
Let $0 < T_1 < T$ *and there exists a solution* $a(t) \in C_+[0, T_1]$ *of* (0.3). *Then the solution* $a(t)$ *can be continued to the right of* T_1.

The main idea of the proof of Theorem 3.1 is the use of the implicit function theorem in an appropriate function space setting. Let T_2 be fixed so that $T_1 < T_2 < T$ and define a constant function $a_0(t)$ in the interval $[T_1, T_2]$ by $a_0(t) \equiv a(T_0)$ and $\tilde{a}(t)$ in the interval $[0, T_2]$ by

$$\tilde{a}(t) := \begin{cases} a(t) & (0 \le t \le T_1), \\ a_0(t) & (T_1 \le t \le T_2). \end{cases}$$

Moreover we define a function $g_0(t)$ in $[T_1, T_2]$ by

$$g_0(t) := \frac{1}{\sqrt{\pi}} a_0(t) \int_0^t \frac{f'(\tau)}{\left(\int_\tau^t \tilde{a}(r) dr\right)^{1/2}} d\tau. \tag{3.1}$$

Let X be a function space defined by

$$X := \{ b(t) \in C[T_1, T_2] \mid b(T_1) = 0 \}$$

with the maximal norm, and consider the mapping

$$F(b)(t) := (a_0(t) + b(t)) \int_0^t \frac{f'(\tau)}{\left(\int_\tau^t \tilde{a}(r) + \tilde{b}(r) dr\right)^{1/2}} d\tau - \sqrt{\pi} g_0(t) \quad (T_0 \le t \le T_1),$$

where

$$\tilde{b}(t) := \begin{cases} 0 & (0 \le t \le T_1), \\ b(t) & (T_1 \le t \le T_2). \end{cases}$$

Clearly $F(0) = 0$. Moreover we have:

Lemma 3.2. *F is a C^1-mapping of an open neighbourhood of 0 in X to X. The Fréchet derivative $F_b(0)$ at 0 is written as, for $b \in X$,*

$$F_b(0)b(t) = \sqrt{\pi}\frac{g_0(t)}{a_0(t)}b(t) - \frac{1}{2}a_0(t)\int_{T_1}^{t} b(s)ds \int_0^s \frac{f'(\tau)}{\left(\int_\tau^t \tilde{a}(r)dr\right)^{3/2}}d\tau. \tag{3.2}$$

Proof of Lemma 3.2. It is easy to see that $F(b)$ is a continuous mapping of an open neighbourhood of 0 in X to X. The Fréchet derivative $F_b(b_0)$ at b_0 is computed as,

$$\begin{aligned} F_b(b_0)b(t) =& b(t)\int_0^t \frac{f'(\tau)}{\left(\int_\tau^t \tilde{a}(r) + \tilde{b}_0(r)dr\right)^{1/2}}d\tau \\ &- \frac{1}{2}(a_0(t) + b_0(t))\int_{T_1}^{t} b(s)ds \int_0^s \frac{f'(\tau)}{\left(\int_\tau^t \tilde{a}(r) + \tilde{b}_0(r)dr\right)^{3/2}}d\tau, \end{aligned}$$

for $b(t) \in X$. As is easily seen, $F_b(b_0)$ is continuous in b_0 in the sence of operator norm. In the case $b_0 = 0$ we have

$$F_b(0)b(t) = b(t)\int_0^t \frac{f'(\tau)}{\left(\int_\tau^t \tilde{a}(r)dr\right)^{1/2}}d\tau - \frac{1}{2}a_0(t)\int_{T_1}^{t} b(s)ds \int_0^s \frac{f'(\tau)}{\left(\int_\tau^t \tilde{a}(r)dr\right)^{3/2}}d\tau,$$

which, together with (3.1), yields (3.2). The proof of Lemma 3.2 is complete.

We now let $\phi(t) \in X$ and consider the equation

$$F_b(0)b(t) = \phi(t) \quad (T_1 \le t \le T_2). \tag{3.3}$$

If T_2 is sufficiently near T_1 then $g_0(t) > 0$ for $T_1 \le t \le T_2$. Therefore (3.3) is equivalent to

$$b(t) - \int_{T_1}^{t} L(t,s)b(s)ds = \tilde{\phi}(t) \quad (T_1 \le t \le T_2), \tag{3.4}$$

where we set

$$L(t,s) := -\frac{1}{2\sqrt{\pi}}\frac{a_0(t)^2}{g_0(t)}\int_0^s \frac{f'(\tau)}{\left(\int_\tau^t \tilde{a}(r)dr\right)^{3/2}}d\tau \quad (T_1 \le s \le t \le T_2),$$

$$\tilde{\phi}(t) := \frac{a_0(t)}{\sqrt{\pi}g_0(t)}\phi(t) \quad (T_1 \le t \le T_2).$$

Since $\tilde{a}(r) > 0$ for $0 \le t \le T_2$ there exists a constant M such that $|L(t,s)| \le M(t-s)^{-1/2}$. So, by a standard solving metod (see e.g [6, §39]) of the Volterra equation of the second kind, it follows that (3.4) has a unique solution $b(t)$ in X for each $\tilde{\phi}(t) \in X$, and that the correspondence $\tilde{\phi}(t) \mapsto b(t)$ is a bounded linear operator in X. This shows that $F_b(0) : X \to X$ has a bounded linear inverse.

Hence, by the implicit function theorem (see e.g. [4, Theorem 1.20]), we conclude that there exists $\delta > 0$ such that the equation $F(b)(t) = \sqrt{\pi}(g(t) - g_0(t))$ has a solution $b(t)$ in X if $\max_{T_1 \le t \le T_2} \sqrt{\pi}|g(t) - g_0(t)| < \delta$. Noting that δ may depend on T_2 we introduce a function $\tilde{g}(t)$ so that $\tilde{g}(t) = g(t)$ near T_1 : in $[T_1, T_2']$, say; and so that $\max_{T_1 \le t \le T_2} \sqrt{\pi}|\tilde{g}(t) - g_0(t)| < \delta$. Then $F(b)(t) = \sqrt{\pi}(\tilde{g}(t) - g_0(t))$ has a solution $b(t)$ in X. Using the solution $b(t)$ we set $a(t) := a_0(t) + b(t)$. Then $a(t)$ satisfies (0.3) for $T_1 \le t \le T_2'$. This completes the proof of Theorem 3.1.

4. Alternative theorem

In this section we shall establish the following:

Theorem 4.1. *Assume that, with some* $\mu > 0$,

(i) $f(t) \in C[0,T) \cap C^1(0,T)$, $\lim_{t \to 0} t^{1-\mu} f'(t) > 0$;

(ii) $g(t) \in C_+(0,T)$, $\lim_{t \to 0} t^{1/2-\mu} g(t) > 0$.

Then a solution $a(t) \in C_+[0,T_1)$ *of* (0.3) *that does not become infinite as* $t \to T_1$ *can be continued to the right of* T_1.

An obvious consequence of Theorem 4.1 is the following:

Corollary 4.2. *Assume* (i) *and* (ii). *If a solution* $a(t) \in C_+[0,T_*)$ *of* (0.3) *can not be continued any further, then* $\lim_{t \to T_*} a(t) = +\infty$.

We base the proof of Theorem 4.1 on the following *a priori* property of solutions of (0.3):

Lemma 4.3. *Under the same assumption as in Theorem* 4.1, *a solution* $a(t) \in C_+[0,T_1)$ *of* (0.3) *for some* $T_1 < T$ *satisfies* $\inf_{0 \le t < T_1} a(t) > 0$.

Proof. Let $T_1' < T_1$. From (1.1) we have for $0 \le t \le T_1'$,

$$0 < \sqrt{\pi} \min_{0 \le t \le T_1} (t^{1/2-\mu} g(t)) \le \left| a(t) \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{\left(\int_\rho^1 a(tr)dr\right)^{1/2}} \frac{d\rho}{\rho^{1-\mu}} \right|$$

$$\le a(t) \frac{\max_{0 \le t \le T_1} |t^{1-\mu} f'(t)|}{\left(\min_{0 \le t \le T_1'} a(t)\right)^{1/2}} \int_0^1 \frac{d\rho}{(1-\rho)^{1/2} \rho^{1-\mu}},$$

which yields

$$\frac{\sqrt{\pi}}{B(1/2,\mu)} \frac{\min_{0 \le t \le T_1} (t^{1/2-\mu} g(t))}{\max_{0 \le t \le T_1} |t^{1-\mu} f'(t)|} \le \left(\min_{0 \le t \le T_1'} a(t)\right)^{1/2}$$

Noting that the left side is a constant independent of T_1', we complete the proof.

Lemma 4.3 leads to the following *alternative* for a solution of (0.3):

Lemma 4.4. *Assume* (i) *and* (ii) *in Theorem* 4.1, *and let* $a(t) \in C_+[0, T_1)$ *be a solution of* (0.3) *for some* $T_1 < T$. *Then,* either $a(t)$ *tends to a finite, positive value as* $t \to T_1$: $0 < \lim_{t \to T_1} a(t) < \infty$; or $a(t)$ *tends to infinity as* $t \to T_1$: $\lim_{t \to T_1} a(t) = +\infty$.

Proof. We proceed in two steps.

Step 1. We shall show that if $\liminf_{t \to \infty} a(t) < \infty$ then $\sup_{0 \le t < T_1} a(t) < \infty$. By the assumption there exists a sequence $\{t_k\}_{k=1}^{\infty} \to T_1$ as $k \to \infty$, such that

$$\sup_k a(t_k) \le M_1 < \infty, \tag{4.1}$$

with some constant M_1 independent of k. The equation (0.3) can be rewritten as

$$\begin{aligned}
\sqrt{\pi} g(t) = & a(t) \int_0^{t_k} \frac{f'(\tau)}{\left(\int_\tau^{t_k} a(r)dr\right)^{1/2}} d\tau \\
& + a(t) \int_0^{t_k} \left(\frac{1}{\left(\int_\tau^{t} a(r)dr\right)^{1/2}} - \frac{1}{\left(\int_\tau^{t_k} a(r)dr\right)^{1/2}} \right) f'(\tau) d\tau \\
& + a(t) \int_{t_k}^{t} \frac{f'(\tau)}{\left(\int_\tau^{t} a(r)dr\right)^{1/2}} d\tau.
\end{aligned}$$

Hence we have

$$\sqrt{\pi} g(t) = \sqrt{\pi} \frac{g(t_k)}{a(t_k)} a(t) + I_1(t, t_k) + I_2(t, t_k), \tag{4.2}$$

where

$$\begin{aligned}
I_1(t, t_k) := & -a(t) \int_{t_k}^{t} a(r)dr \times \\
& \times \int_0^{t_k} \frac{f'(\tau)}{\left(\int_\tau^{t} a(r)dr\right)^{1/2} \left(\int_\tau^{t_k} a(r)dr\right)^{1/2} \left\{ \left(\int_\tau^{t} a(r)dr\right)^{1/2} + \left(\int_\tau^{t_k} a(r)dr\right)^{1/2} \right\}} d\tau, \\
I_2(t, t_k) := & \, a(t) \int_{t_k}^{t} \frac{f'(\tau)}{\left(\int_\tau^{t} a(r)dr\right)^{1/2}} d\tau.
\end{aligned}$$

By subtracting $g(t_k)$ from (4.2) we get

$$\sqrt{\pi}(a(t) - a(t_k)) = \sqrt{\pi} \frac{a(t_k)}{g(t_k)} (g(t) - g(t_k)) - \frac{a(t_k)}{g(t_k)} I_1(t, t_k) - \frac{a(t_k)}{g(t_k)} I_2(t, t_k) \tag{4.3}$$

for $t \ge t_k$. By setting

$$\begin{aligned}
& b_k(t) := a(t) - a(t_k), \ \varphi(t, t_k) := \\
& = \int_0^{t_k} \frac{f'(\tau)}{\left(\int_\tau^{t} a(r)dr\right)^{1/2} \left(\int_\tau^{t_k} a(r)dr\right)^{1/2} \left\{ \left(\int_\tau^{t} a(r)dr\right)^{1/2} + \left(\int_\tau^{t_k} a(r)dr\right)^{1/2} \right\}} d\tau. \\
& \psi(t, t_k) := \int_{t_k}^{t} \frac{f'(\tau)}{\left(\int_\tau^{t} a(r)dr\right)^{1/2}} d\tau,
\end{aligned}$$

we obtain

$$\begin{aligned} I_1(t,t_k) &= -(b_k(t)+a(t_k))\int_{t_k}^{t}(b_k(r)+a(t_k))dr\,\varphi(t,t_k) \\ &= -\varphi(t,t_k)b_k(t)\int_{t_k}^{t} b_k(r)dr - a(t_k)\,\varphi(t,t_k)\int_{t_k}^{t} b_k(r)dr \\ &\quad - b_k(t)(t-t_k)a(t_k)\varphi(t,t_k) - a(t_k)^2(t-t_k)\varphi(t,t_k), \\ I_2(t,t_k) &= b_k(t)\psi(t,t_k) + a(t_k)\psi(t,t_k). \end{aligned}$$

Substituting this in (4.3) shows that

$$\begin{aligned} &\left[\sqrt{\pi} - \frac{a(t_k)^2}{g(t_k)}(t-t_k)\varphi(t,t_k) + \frac{a(t_k)}{g(t_k)}\psi(t,t_k)\right] b_k(t) \\ &= A(t) + \frac{a(t_k)^2}{g(t_k)}\varphi(t,t_k)\int_{t_k}^{t} b_k(r)dr + \frac{a(t_k)}{g(t_k)} b_k(t)\varphi(t,t_k)\int_{t_k}^{t} b_k(r)dr, \end{aligned}$$

where we put

$$A(t) := \sqrt{\pi}\frac{a(t_k)}{g(t_k)}(g(t)-g(t_k)) + \frac{a(t_k)^3}{g(t_k)}(t-t_k)\varphi(t,t_k) - \frac{a(t_k)^2}{g(t_k)}\psi(t,t_k).$$

We now set $m_a := \inf_{0\le t<T_1} a(t), M_f := \max_{t_1\le t\le T_1} |f'(t)|$. Note that $m_a > 0$ by Lemma 4.3. It follows that for $t_k \le t < T_1$

$$\begin{aligned} |\varphi(t,t_k)| &\le \frac{M_f}{m_a^{3/2}}\int_0^{t_k}\frac{d\tau}{(t-\tau)(t_k-\tau)^{1/2}} \le \frac{M_2}{(t-t_k)^{1/2}} \\ |\psi(t,t_k)| &\le \frac{M_f}{m_a^{1/2}}\int_{t_k}^{t}\frac{d\tau}{(t-\tau)^{1/2}} \le M_2(t-t_k)^{1/2} \end{aligned} \tag{4.4}$$

with a constant M_2 independent of k. This, together with (4.2), shows that

$$\left|-\frac{a(t_k)^2}{g(t_k)}(t-t_k)\varphi(t,t_k) + \frac{a(t_k)}{g(t_k)}\psi(t,t_k)\right| \le \sqrt{\pi}-1 \quad (k \ge N_1),$$

if we take N_1 sufficiently large. Accordingly, from (4.1) and (4.4), we have

$$|b_k(t)| \le |A(t)| + \frac{M_3 + M_4|b_k(t)|}{(t-t_k)^{1/2}}\int_{t_k}^{t}|b_k(r)|dr \quad (k \ge N_1).$$

So, for $k \ge N_1$, if $|b_k(t)| \le 1$ then for $t_k \le t < T_1$,

$$\begin{aligned} |b_k(t)| &\le |A(t)| + \frac{M_3+M_4}{(t-t_k)^{1/2}}\int_{t_k}^{t}|b_k(r)|dr \\ &\le |A(t)| + \frac{M_3+M_4}{(t-t_k)^{1/2}}\int_{t_k}^{t}\left\{|A(r)| + \frac{M_3+M_4}{(r-t_k)^{1/2}}\int_{t_k}^{r}|b_k(s)|ds\right\}dr \\ &\le B(t) + (M_3+M_4)^2\int_{t_k}^{t}|b_k(s)|ds, \end{aligned}$$

where

$$B(t) := |A(t)| + \frac{M_3 + M_4}{(t - t_k)^{1/2}} \int_{t_k}^{t} |A(r)| dr.$$

By the definition of $A(t)$ and (4.4), it follows that $\lim_{t \to t_k} B(t) = 0$ uniformly with respect to k. This, together with Gronwall's inequality, shows that $\lim_{t \to t_k} b_k(t) = 0$ uniformly with respect to k. Hence if we take $N(\geq N_1)$ sufficiently large then $|b_k(t)| \leq 1/2$ for $k \geq N$, $t_k \leq t < T_1$, provided that $|b_k(t)| \leq 1$. In other words, for $k \geq N$, $t_k \leq t \leq T_1$, either $|b_k(t)| \geq 1$ or $|b_k(t)| \leq 1/2$. But the former does not occur because $b_k(t)$ is a continuous function in the interval with $b_k(t_k) = 0$. Thereby we conclude that there exists a number N such that, for $k \geq N$, $a(t) \leq a(t_k) + 1/2$ in the interval $t_k \leq t < T_1$. This shows that $\sup_{0 \leq t < T_1} a(t) < \infty$.

Step 2. We shall show that if $\sup_{0 \leq t < T_1} a(t) < \infty$ then $a(t)$ tends to a finite, positive value as $t \to T_1$. Let $T_0 \leq s \leq t < T_1$. Using (4.3) we have

$$\begin{aligned}
\sqrt{\pi}(a(t) - a(s)) &= \sqrt{\pi}\frac{a(s)}{g(s)}(g(t) - g(s)) - \frac{a(s)}{g(s)} I_1(t,s) - \frac{a(s)}{g(s)} I_2(t,s) \\
&= \sqrt{\pi}\frac{a(s)}{g(s)}(g(t) - g(s)) + \frac{a(s)a(t)}{g(s)} \int_s^t a(r) dr\, \varphi(t,s) - \frac{a(s)a(t)}{g(s)} \psi(t,s).
\end{aligned}$$

It follows from this equality, the assupmtion $\sup_{0 \leq t < T_1} a(t) < \infty$, (4.4), and the uniform continuity of $g(t)$ that $a(t)$ is uniformly continuous on $[0, T_1)$. Hence $a(t)$ is extended as a continuous function on $[0, T_1]$. The proof of Lemma 4.4 is complete.

We now give the

Proof of Theorem 4.1. If a solution $a(t) \in C_+[0, T_1)$ of (0.3) does not become infinite as $t \to T_1$, then, by Lemma 4.4, $a(t)$ is extended as a positive solution on $[0, T_1]$. So, by Theorem 3.1, $a(t)$ can be continued to the right of T_1. The proof of Theorem 4.1 is complete.

We treat the case when $f'(t) \geq 0$. The following result is useful.

Lemma 4.5. *In addition to the assumption in Theorem* 4.1 *we assume that* $f'(t) \geq 0$ *for each* $t \in (0, T)$. *Then any solution* $a(t) \in C_+[0, T_1)$ *of* (0.3) *for some* $T_1 < T$ *satisfies* $\sup_{0 \leq t < T_1} a(t) < \infty$.

Proof. Let $T_1' < T_1$. It follows from (1.1) that

$$\begin{aligned}
\sqrt{\pi} \max_{0 \leq t \leq T_1} (t^{1/2-\mu} g(t)) &\geq a(t) \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{\left(\int_\rho^1 a(tr) dr\right)^{1/2}} \frac{d\rho}{\rho^{1-\mu}} \\
&\geq \frac{a(t)}{\left(\max_{0 \leq t \leq T_1'} a(t)\right)^{1/2}} \min_{0 \leq t \leq T_1} \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{1/2} \rho^{1-\mu}} d\rho,
\end{aligned}$$

for $0 \leq t \leq T_1'$. Since the function

$$\int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{1/2}\rho^{1-\mu}} d\rho$$

is a positve, continuous function on $[0, T_1]$, we get

$$\sqrt{\pi}\left(\min_{0\leq t\leq T_1} \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{1/2}\rho^{1-\mu}} d\rho\right)^{-1} \max_{0\leq t\leq T_1} (t^{1/2-\mu} g(t)) \geq \left(\max_{0\leq t\leq T_1'} a(t)\right)^{1/2}.$$

Noting the left side is a constant independent of T_1' we complete the proof.

By virtue of Lemma 4.5 the following is an immediate consequence of Theorem 4.1:

Corollary 4.6. *In addition to the assumptions in Theorem* 4.1 *we assume that* $f'(t) \geq 0$ *for each* $t \in (0, T)$. *Then* (0.3) *has a solution* $a(t) \in C_+[0, T)$.

We wish to point out that Corollary 4.5 is also obtained immediately by [5, Chap 1, Theorem 3]. In the case $1/2 \leq \mu < 1$ this follows also from [2].

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