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STRONG SUBADDITIVITY PROPERTY OF ENTROPY IN FERMION SYSTEMS

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Abstract

We prove that strong subadditibity(SSA) of entropy holds for fermion systems.

1 Main result

Probably,strong subadditibity(SSA) is the one of the most powerful properties of entropy. In this paper, we prove SSA holds for fermion systems. We consider Fermonic systems on an integer lattice $\mathbf{Z}^d$,the set of d-tuples of integers. The creation and annihilation operators will be denoted by a_j^* and a_j $(j \in \mathbf{Z}^d)$. They satisfy the canonical anticommutation relations,

$$\{a_j, a_k\} = 0,$$
$$\{a_j^*, a_k^*\} = 0,$$
$$\{a_j, a_k^*\} = \delta_{jk}1, \tag{1}$$

where $\{A, B\}$ *denotes the anticommutator* $AB + BA$.
The C^* algebra generated by these operators will be denoted by $\mathcal{A}$. $P_f(\mathbf{Z}^d)$ will denote the set of finite subsets of $\mathbf{Z}^d$. For each $\Lambda \in P_f(\mathbf{Z}^d)$, $\mathcal{A}_\Lambda$ denotes the subalgebra of $\mathcal{A}$ algebraically generated by $\{a_j, a_j^* \quad j \in \Lambda\}$. $\mathcal{A}_\Lambda$ is $*$isomophic to the tensur product algebra $\otimes_{i\in\Lambda} M_2(\mathbf{C})$, where $M_2(\mathbf{C})$ is the full matrix algebra of 2$\times$ 2 complex matrices. If w is a state of $\mathcal{A}$, then it will induce a restricted state w_Λ on $\mathcal{A}_\Lambda$ for any finite region Λ of $\mathbf{Z}^d$. w_Λ is determined by a unique density matrix D_Λ satisfying

$$w(a) = \mathrm{T}r_\Lambda(D_\Lambda a)$$

for all $a \in \Lambda$ and the matrix trace of $\mathcal{A}$.
The *entropy* $S_\Lambda(w)$ of w is defined by

$$S_\Lambda(w) \equiv -\mathrm{T}r_\Lambda D_\Lambda \log D_\Lambda.$$

The strong subadditibity property is related to the composition of three different systems. Let $\Lambda_i \in P_f(\mathbf{Z}^d)_{(i=1,2,3)}$ be mutally disjoint. For a given state w of $\mathcal{A}$, we denote the entropy $S_\Lambda(w)$ for $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3, \Lambda_1 \cup \Lambda_2, \Lambda_1 \cup \Lambda_3, \Lambda_1$ by $S_{1,2,3}, S_{1,2}, S_{1,3}, S_1$, respectively.

Our result is the following theorem.

Theorem(SSA property of entropy in the fermion systems)

$$S_{1,2,3} + S_1 \leq S_{1,2} + S_{1,3} \tag{2}$$

E.H.Lieb and M.B.Ruskai([3],[4]) already proved the same type of result. However, they considered the case where the combined system 1,2,3 is described by the tensor product of the algebras of composed systems 1,2,3.In the present case, this is not the case, for example, annihilatition operators in the systems 1,2,3 mutally anticommute ,i.e. they do not commute.

2 Proof of Theorem

2.1 Relative Entropy and Conditional expectation

Let w and φ are states of $\mathcal{A}$,the entropy of w relative to φ is defined by

$$S(w,\varphi) \equiv \mathrm{Tr} D_w(\log D_w - \log D_\varphi)$$

Suppose that $\Lambda \subseteq \Lambda\prime (\in P_f(\mathbf{Z}^d))$, and that τ is a normalized trace on $\mathcal{A}_{\Lambda'}$. As is well known, there existe a unique linear mapping$E : \mathcal{A}_{\Lambda'}. \mapsto \mathcal{A}_\Lambda$ such that

$$\begin{aligned}
&(1) E(x^*x) \geq 0, \\
&(2) E(y) = y \text{ for every } y \in \mathcal{A}_\Lambda \ , \\
&(3) E(xy) = E(x)y \text{ holds for every } x \in \mathcal{A}_{\Lambda'} \ y \in \mathcal{A}_\Lambda \ , \\
&(4) \tau(E(x)) = \tau(x) \text{ for every } \ x \in \mathcal{A}_{\Lambda'} \ .
\end{aligned} \tag{3}$$

This mapping is usually called a *conditional expectation.*

It is known that

$$S(w \circ E|\mathcal{A}_{\Lambda'}, \varphi \circ E|\mathcal{A}_{\Lambda'}) \geq S(w|\mathcal{A}_\Lambda, \varphi|\mathcal{A}_\Lambda) \tag{4}$$

This is the monotonicity property under the conditional expectations. (see [6]Ohya,Petz)

2.2 Proof

Let α be a one to one mapping from $\{1, 2, \cdots, \ |\Lambda_1 \cup \Lambda_2 \cup \Lambda_3|\}$ onto $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Thus, $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 = \{\alpha(k)\}_{k=1,\cdots,|\Lambda_1 \cup \Lambda_2 \cup \Lambda_3|}$. Here $|\Lambda|$ denotes the number of points in Λ . Now we fix α, and denote $\alpha(k)$ by k. Let τ be a normalized trace on $\mathcal{A}_{1,2,3}$. (For simplicity ,we drop Λ and write $\mathcal{A}_{1,2,3}$ for $\mathcal{A}_{\Lambda_1\cup\Lambda_2\cup\Lambda_3}$, etc.) Let F, G be a conditional expectation from $\mathcal{A}_{1,2,3}$ onto $\mathcal{A}_{1,3}$, and from $\mathcal{A}_{1,2}$ onto $\mathcal{A}_1$, respectively. We assert first that $G = F|_{\mathcal{A}_{1,2}}$. From (1)(the cannonical anticommutation relations), it is easy to see that $\mathcal{A}_{1,2,3}$ consists of linear combinations of Id and products

$$\{\prod_{k\in\Lambda_1\cup\Lambda_3} b(k)\}\{\prod_{k\in\Lambda_2} b(k)\},$$
$$where\ b(k) \in \{a_k^* a_k,\ a_k a_k^*,\ a_k,\ a_k^*,\ I\}$$

Set

$$f(k) = \begin{cases} 2, & \text{if } b(k) = a_k^* a_k, \text{ or } a_k a_k^* \\ 1, & \text{if } b(k) = a_k, \text{ or } a_k^* \\ 0, & \text{if } b(k) = I \end{cases}$$

From (3),

$$F\Big(\{\prod_{k\in\Lambda_1\cup\Lambda_3} b(k)\}\{\prod_{k\in\Lambda_2} b(k)\}\Big) = \{\prod_{k\in\Lambda_1\cup\Lambda_3} b(k)\}\cdot F\Big(\{\prod_{k\in\Lambda_2} b(k)\}\Big)$$

Thus our aim is to determine $F\Big(\{\prod_{k\in\Lambda_2} b(k)\}\Big)$.

If $\sum_{k\in\Lambda_2} f(k) = 2\mathbb{Z}$ we have, from (1), that

$$a\{\prod_{k\in\Lambda_2} b(k)\} = \{\prod_{k\in\Lambda_2} b(k)\}a, \quad \text{for each } a \in \mathcal{A}_{1,3}.$$

From (3),

$$aF\Big(\{\prod_{k\in\Lambda_2} b(k)\}\Big) = F\Big(\{\prod_{k\in\Lambda_2} b(k)\}\Big)a.$$

Thus $F\Big(\{\prod_{k\in\Lambda_2} b(k)\}\Big)$ is in the center . And by (3), we have

$$F\Big(\{\prod_{k\in\Lambda_2} b(k)\}\Big) = \tau\Big(\{\prod_{k\in\Lambda_2} b(k)\}\Big)\cdot I. \tag{5}$$

If $\sum_{k\in\Lambda_3} f(k) = 2\mathbb{Z}+1$, there is a $m \in \Lambda_2$ such that $f(m) = 1$. Now, let m be the smallest number with the above property.
By (1)

$$\prod_{k\in\Lambda_2} b(k) = b(m)\cdot\{\prod_{k\neq m\ k\in\Lambda_2} b(k)\}$$

$$\sum_{k\neq m\ k\in\Lambda_2} f(k) = 2\mathbb{Z}.$$

Next, note that

$$\begin{aligned} a_j(2a_j^* a_j - 1) &= 2a_j a_j^* a_j - a_j \\ &= 2a_j(1 - a_j a_j^*) - a_j \end{aligned}$$

$$
\begin{aligned}
&= 2a_j - a_j \\
&= a_j \\
\text{and} \qquad (2a_j^* a_j - 1)a_j &= -a_j
\end{aligned}
$$

$$
\begin{aligned}
\text{Similary} \qquad a_j^*(2a_j a_j^* - 1) &= a_j^* \\
(2a_j a_j^* - 1)a_j^* &= -a_j^*
\end{aligned}
$$

Thus

$$
b(m) \cdot \{2b(m)^* b(m) - 1\} = b(m)
$$

$$
\{2b(m)^* b(m) - 1\} \cdot b(m) = -b(m)
$$

For each $a \in \mathcal{A}_{1,3}$, we have

$$
\begin{aligned}
&\tau\Big(a \cdot b(m) \{ \prod_{k \neq m \ k \in \Lambda_2} b(k)\}\Big) \\
&= \tau\Big(a \cdot b(m) \cdot \{2b(m)^* b(m) - 1\} \{ \prod_{k \neq m \ k \in \Lambda_2} b(k)\}\Big) \\
&= \tau\Big(a \cdot b(m) \{ \prod_{k \neq m \ k \in \Lambda_2} b(k)\} \cdot \{2b(m)^* b(m) - 1\}\Big) \\
&= \tau\Big(\{2b(m)^* b(m) - 1\} \cdot a \cdot b(m) \{ \prod_{k \neq m \ k \in \Lambda_2} b(k)\}\Big) \\
&= \tau\Big(a \cdot \{2b(m)^* b(m) - 1\} \cdot b(m) \{ \prod_{k \neq m \ k \in \Lambda_2} b(k)\}\Big) \\
&= -\tau\Big(a \cdot b(m) \{ \prod_{k \neq m \ k \in \Lambda_2} b(k)\}\Big)
\end{aligned}
$$

So, $\tau\Big(a \cdot b(m) \{\prod_{k \neq m \ k \in \Lambda_2} b(k)\}\Big) = 0$. From this, we have

$$
b(m) \cdot \{ \prod_{k \neq m \ k \in \Lambda_2} b(k)\} = 0 \tag{6}
$$

From (5) and (6), we conclude that

$$
G = F|_{\mathcal{A}_{1,3}}. \tag{7}
$$

Let w be a state of $\mathcal{A}$, and $D_{1,2,3}, D_{1,2}, D_{1,3}, D_1$ denote the density of the state $w_{1,2,3}, w_{1,2}, w_{1,3}, w_{1,2}$, and w_1, respectively. It is easy to see that

$$
\begin{aligned}
S(D_{1,2,3}, D_{1,2} \otimes D_3) &= S_{1,2} + S_3 - S_{1,2,3} \\
S(D_{1,3}, D_1 \otimes D_3) &= S_1 + S_3 - S_{1,3}
\end{aligned} \tag{8}
$$

By definition,

$$
F(D_{1,2,3}) = D_{1,3}, \quad G(D_{1,2}) = D_1. \tag{9}
$$

Apparently,

$$D_3 = F(D_3) \tag{10}$$

From (7),(8),(9),(10)

$$\begin{aligned} & (S_{1,2} + S_{1,3}) - (S_1 + S_{1,2,3}) \\ = \; & S(D_{1,2,3}, D_{1,2} \otimes D_3) - S(F(D_{1,2,3}), F(D_{1,2}) \otimes F(D_3)) \end{aligned}$$

Thus, it follows from (4) (monotonicity property of relative entropy) that $S_1 + S_{1,2,3} \leq S_{1,2} + S_{1,3}$. □

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