数理解析研究所講究録
1035 巻 1998 年 155-169

A Note on Transformations on White Noise Functions — Hida's Whiskers Revisited —

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Introduction

Given a real Gelfand triple $E \subset H \subset E^*$ let $O(E;H)$ denote the group of all linear homeomorphisms of E which preserve the norm of H. This $O(E;H)$, called the *infinite dimensional rotation group*, was first introduced by Yoshizawa around 1961 in a series of his lectures (see [10], [29]) and has offered an interesting aspect in analysis of Brownian functionals, or more generally, of white noise functions. That $O(E;H)$ is the group of automorphisms of the original Gelfand triple is to be in contrast to the full orthogonal group $O(H)$. The complex case is considered similarly and the *infinite dimensional unitary group* plays a role in analysis of complex white noise.

Our discussion here is mostly concerned with the particular Gelfand triples:

$$E=\mathcal{S}(\mathbf{R})\subset H=L^2(\mathbf{R})\subset E^*=\mathcal{S}'(\mathbf{R}), \tag{0.1}$$

where $\mathcal{S}(\mathbf{R})$ is the space of rapidly decreasing functions, $L^2(\mathbf{R})$ the Hilbert space of square-integrable functions, and $\mathcal{S}'(\mathbf{R})$ the space of tempered distributions; and its "second quantization" known also as a white noise triple:

$$\mathcal{W}\subset L^2(E^*,\mu)\cong \Gamma(H_{\mathbf{C}})\subset \mathcal{W}^*, \tag{0.2}$$

where μ is the Gaussian measure on E^* defined by

$$e^{-|\xi|_H^2/2}=\int_{E^*} e^{i\langle x,\xi\rangle}\mu(dx), \qquad \xi\in E.$$

The space $\mathcal{W}^*$ consists of generalized Gaussian random variables or white noise distributions. The underlying manifold $\mathbf{R}$ of the Gelfand triple (0.1) plays a role of time; the white noise process is realized in $\mathcal{W}^*$ as $W_t(x)=\langle x,\delta_t\rangle$ and the family of L^2-random variables

$$B_t=\int_0^t W_s\,ds, \qquad t\geq 0,$$

is a (realization) of the Brownian motion.

In 1969 Hida–Kubo–Nomoto–Yoshizawa [10] investigated a group-theoretical interpretation of the projective invariance of Brownian motion by constructing a finite dimensional subgroup of $O(E;H)$; in fact, one-parameter subgroups of $O(E;H)$ arising from the shift and the dilation of $\mathbf{R}$ played an essential role. More generally, one-parameter subgroups of $O(E;H)$ arising from one-parameter diffeomorphism groups of $\mathbf{R}$, which were named *whiskers* by Hida [8], have been expected to be a clue to study structure of the infinite dimensional rotation group, for some attempts see [12], [25], [26]. On the other hand, the idea of whiskers is also applied to a study of multi-parameter Brownian motion, see [9], [27], [28]. Thus it is interesting to characterize those whiskers among one-parameter subgroups of $O(E;H)$; however, this question is not yet solved and we report some preliminary consideration in this note.

Transformations on white noise functions have been also discussed from somewhat different aspects, e.g., in connection with Cauchy problems [2], [3], [4]; group-theoretical properties of the Kuo–Fourier–Mehler transforms [16] and infinite dimensional Laplacians [11], [18], [20].

General Notation For locally convex spaces $\mathfrak{X}, \mathfrak{Y}$ let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ be the space of continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$ equipped with the topology of bounded convergence. Let $GL(\mathfrak{X}) \subset \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ be the group of all linear homeomorphisms from $\mathfrak{X}$ onto itself. In this note no topology of $GL(\mathfrak{X})$ is considered. When $\mathfrak{X}$ is a real space, we denote by $\mathfrak{X}_{\mathbf{C}}$ the complexification.

1 One-parameter diffeomorphism groups of R

The group of diffeomorphisms of $\mathbf{R}$ is denoted by $\mathrm{Diff}(\mathbf{R})$. Each $\gamma \in \mathrm{Diff}(\mathbf{R})$ is a $\mathbf{R}$-valued function defined on $\mathbf{R}$ such that
(i) γ is a C^∞-function;
(ii) $\gamma(\mathbf{R}) = \mathbf{R}$;
(iii) γ' does not vanish on $\mathbf{R}$.
For any $\gamma \in \mathrm{Diff}(\mathbf{R})$ the derivative γ' is always positive or always negative. Put

$$\mathrm{Diff}^+(\mathbf{R}) = \{\gamma \in \mathrm{Diff}(\mathbf{R})\,;\, \gamma'(x) > 0 \text{ for all } x \in \mathbf{R}\}.$$

Then $\mathrm{Diff}^+(\mathbf{R})$ is a normal subgroup of $\mathrm{Diff}(\mathbf{R})$ and $\mathrm{Diff}(\mathbf{R}) = \mathrm{Diff}^+(\mathbf{R}) \cup \tau\,\mathrm{Diff}^+(\mathbf{R})$, where τ is the inversion, i.e., $\tau(x) = -x$.

By a *one-parameter diffeomorphism group* of $\mathbf{R}$ we mean a map $\theta \mapsto \gamma_\theta \in \mathrm{Diff}(\mathbf{R})$, $\theta \in \mathbf{R}$, or simply $\{\gamma_\theta\} \subset \mathrm{Diff}(\mathbf{R})$, such that
(i) $(\theta, x) \mapsto \gamma_\theta(x)$ is a C^∞-map from $\mathbf{R} \times \mathbf{R}$ onto $\mathbf{R}$;
(ii) $\gamma_{\theta_1+\theta_2} = \gamma_{\theta_1} \circ \gamma_{\theta_2}$ for any $\theta_1, \theta_2 \in \mathbf{R}$;
(iii) γ_0 is the identity diffeomorphism.
By continuity any one-parameter diffeomorphism group is a subgroup of $\mathrm{Diff}^+(\mathbf{R})$. With a one-parameter diffeomorphism group $\{\gamma_\theta\}$ we associate a vector field

$$F(x)\frac{d}{dx} \equiv F(x)D, \quad \text{where} \quad F(x) = \left.\frac{d}{d\theta}\right|_{\theta=0} \gamma_\theta(x). \tag{1.1}$$

Since $\mathbf{R}$ is not compact, not all vector fields are obtained in the above manner. We shall investigate a necessary and sufficient condition.

Consider a vector field $F(x)D$, where $F \in C^{\infty}(\mathbf{R})$. Then $\{x \in \mathbf{R}\,;\, F(x) \neq 0\}$ is a countable union of mutually disjoint open intervals (α_n, β_n), where the end points are possibly $\pm\infty$. On each (α_n, β_n), $F(x)$ is always positive or always negative. Choosing an arbitrary point $\gamma_n \in (\alpha_n, \beta_n)$, we put

$$\eta_n(x) = \int_{\gamma_n}^{x} \frac{dy}{F(y)}, \qquad x \in (\alpha_n, \beta_n),$$

and

$$p_n = \lim_{x \downarrow \alpha_n} \eta_n(x) = -\int_{\alpha_n}^{\gamma_n} \frac{dy}{F(y)}, \qquad q_n = \lim_{x \uparrow \beta_n} \eta_n(x) = \int_{\gamma_n}^{\beta_n} \frac{dy}{F(y)}.$$

Then η_n is a diffeomorphism from (α_n, β_n) onto (p_n, q_n) or onto (q_n, p_n) according as $F(x) > 0$ or $F(x) < 0$ on (α_n, β_n). In particular, η_n is a diffeomorphism from (α_n, β_n) onto $\mathbf{R}$ if and only if

$$\lim_{x \downarrow \alpha_n} \eta_n(x) = -\int_{\alpha_n}^{\gamma_n} \frac{dy}{F(y)} = \mp\infty, \qquad \lim_{x \uparrow \beta_n} \eta_n(x) = \int_{\gamma_n}^{\beta_n} \frac{dy}{F(y)} = \pm\infty, \tag{1.2}$$

where $\mp\infty$ and $\pm\infty$ are taken according as $\pm F(x) > 0$ on (α_n, β_n).

Proposition 1. 1 *Notations being as above, a vector field $F(x)D$ is obtained from a one-parameter diffeomorphism group as in (1.1) if and only if (1.2) holds for all n.*

PROOF. Suppose that (1.2) holds for all n. Then for any $\theta, x \in \mathbf{R}$ we may define

$$\gamma_\theta(x) = \begin{cases} \eta_n^{-1}(\eta_n(x) + \theta), & x \in (\alpha_n, \beta_n), \\ x, & \text{otherwise}. \end{cases} \tag{1.3}$$

It is then easy to check that $(\theta, x) \mapsto \gamma_\theta(x)$ is continuous; for any fixed θ, the map $x \mapsto \gamma_\theta(x)$ is surjective; and $\gamma_{\theta+\theta'} = \gamma_\theta \circ \gamma_{\theta'}$. Namely, $\{\gamma_\theta\}$ is a one-parameter group of homeomorphisms of $\mathbf{R}$. Since

$$\frac{d}{d\theta}\gamma_\theta(x) = F(\gamma_\theta(x)), \qquad x \in \mathbf{R},$$

we see that $\theta \mapsto \gamma_\theta(x)$ is a C^∞-function. We need show that $x \mapsto \gamma_\theta(x)$ is also a C^∞-function. To this end it is sufficient to show the identity:

$$\gamma_\theta(x) - \gamma_\theta(0) = \int_0^x \left\{ \exp \int_0^\theta F'(\gamma_s(y))ds \right\} dy, \qquad x, \theta \in \mathbf{R}. \tag{1.4}$$

This is proved step-by-step following the argument of Sato [26, Proposition 2], where the discussion was carried out under the assumtion that $F'(x)$ is bounded and $F(x) = 0$ though these are redundant only to prove (1.4). It then follows that $\{\gamma_\theta\}$ is a one-parameter diffeomorphism group satisfying (1.1).

Conversely, suppose we are given a one-parameter diffeomorphism group $\{\gamma_\theta\}$. Since the argument is similar, assuming that $F(x) > 0$ on (α_n, β_n) and that

$$q_n \equiv \lim_{x \to \beta_n} \eta_n(x) = \int_{\gamma_n}^{\beta_n} \frac{dy}{F(y)} < \infty,$$

we shall show contradiction. Put

$$\tilde{\gamma}_\theta(x) = \eta_n^{-1}(\eta_n(x) + \theta), \quad \text{for} \quad p_n < \eta_n(x) + \theta < q_n.$$

Suppose x is fixed and put $\psi(\theta) = \tilde{\gamma}_\theta(x)$. By group property $\psi(\theta)$ satisfies the differential equation:

$$\psi'(\theta) = F(\psi(\theta)), \qquad \psi(0) = x.$$

Then by the uniqueness of a local solution we obtain $\tilde{\gamma}_\theta(x) = \gamma_\theta(x)$, from which contradiction follows by letting $\theta \to q_n - \eta_n(x)$. ∎

Remark Relation (1.3) appears also in a discussion of ceratin functional equations, see [1, Chapter 6]. If $a'(x)$ is bounded, condition (1.2) is satisfied, see [26, §1].

There are two basic examples of one-parameter diffeomorphism groups. For $\theta \in \mathbf{R}$ we define the shift and dilation respectively by

$$\sigma_\theta(x) = x + \theta, \qquad \tau_\theta(x) = e^\theta x,$$

and their corresponding vector fields are given by

$$D = \frac{d}{dx}, \qquad xD = x\,\frac{d}{dx},$$

respectively. Proposition 1. 1 has many applications and we here mention the following

Proposition 1. 2 *Let $\{\gamma_\theta\}$ be a one-parameter diffeomorphism group of* $\mathbf{R}$ *associated with a vector field $F(x)D$. Then $\{\gamma_\theta\}$ is conjugate to the shift $\{\sigma_\theta\}$, i.e., there exists $\lambda \in \mathrm{Diff}(\mathbf{R})$ such that $\gamma_\theta = \lambda^{-1}\sigma_\theta\lambda$ for all $\theta \in \mathbf{R}$, if and only if $F(x)$ does not vanish on* $\mathbf{R}$, *that is, $F(x) > 0$ for all $x \in \mathbf{R}$ or $F(x) < 0$ for all $x \in \mathbf{R}$.*

Proposition 1. 3 *Let $\{\gamma_\theta\}$ be a one-parameter diffeomorphism group of* $\mathbf{R}$ *associated with a vector field $F(x)D$. Then $\{\gamma_\theta\}$ is conjugate to the dilation $\{\tau_\theta\}$ if and only if (i) there exists a unique x_0 such that $F(x_0) = 0$; (ii) $F(x) > 0$ for $x > x_0$ and $F(x) < 0$ for $x < x_0$, or conversely; (iii) the integral*

$$\int_I \frac{dy}{F(y)}$$

is divergent for the intervals $I = (-\infty, x_0 - 1), (x_0 - 1, x_0), (x_0, x_0 + 1), (x_0 + 1, +\infty)$.

2 Transformations on $\mathcal{S}(\mathbf{R})$

The topology of $\mathcal{S}(\mathbf{R})$ is given by the family of norms:

$$\|\,\xi\,\|_{\alpha,\beta} = \sup_{x\in\mathbf{R}} |x^\alpha \xi^{(\beta)}(x)|, \qquad \alpha, \beta = 0, 1, 2, \cdots. \tag{2.1}$$

A function $f : \mathbf{R} \to \mathbf{R}$ is called *of polynomial growth* if there exist $p \geq 0$ and $C \geq 0$ such that

$$|f(x)| \leq C(1 + |x|^p) \quad \text{for all } x \in \mathbf{R}.$$

A C^∞-function f is called *slowly increasing* if it is of polynomial growth together with all its derivatives $f^{(m)}$, $m = 0, 1, 2, \cdots$.

Proposition 2.1 *For $\gamma \in \mathrm{Diff}(\mathbf{R})$ put*

$$G\xi(x) = \xi(\gamma^{-1}(x)), \qquad \xi \in \mathcal{S}(\mathbf{R}).$$

Then $G \in GL(\mathcal{S}(\mathbf{R}))$ if and only if both γ and γ^{-1} are slowly increasing.

PROOF. Assume that $G \in GL(\mathcal{S}(\mathbf{R}))$. In view of $G \in \mathcal{L}(\mathcal{S}(\mathbf{R}), \mathcal{S}(\mathbf{R}))$, we choose $\alpha \geq 0$ and $C_{jk} \geq 0$ for $0 \leq j, k \leq \alpha$ such that

$$\| G\xi \|_{1,0} \leq \sum_{0 \leq j,k \leq \alpha} C_{jk} \| \xi \|_{j,k}. \tag{2.2}$$

By definition $\| G\xi \|_{1,0} = \sup_{x\in\mathbf{R}} |x\xi(\gamma^{-1}(x))| = \sup_{x\in\mathbf{R}} |\gamma(x)\xi(x)|$, hence (2.2) becomes

$$|\gamma(x)\xi(x)| \leq \sum_{0 \leq j,k \leq \alpha} C_{jk} \| \xi \|_{j,k}, \qquad x \in \mathbf{R}. \tag{2.3}$$

We shall obtain an estimate of γ by taking a particular function ξ. Choose $\rho \in C^\infty(\mathbf{R})$ satisfying

$$0 \leq \rho(x) \leq 1, \qquad \rho(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x \geq 1, \end{cases}$$

and set

$$M_k = \sup_{x\in\mathbf{R}} |\rho^{(k)}(x)| < \infty, \qquad k = 0, 1, 2, \cdots.$$

For $T \geq 0$ consider $\xi = \xi_T \in \mathcal{S}(\mathbf{R})$ defined by

$$\xi_T(x) = \begin{cases} 1, & 0 \leq |x| \leq T, \\ \rho(|x| - T), & T \leq |x| \leq T+1, \\ 0, & T+1 \leq |x|. \end{cases}$$

For this ξ_T we have

$$\| \xi_T \|_{j,k} = \sup_{|x| \leq T+1} |x^j \xi_T^{(k)}(x)| \leq (T+1)^j M_k.$$

Hence (2.3) becomes

$$|\gamma(x)\xi_T(x)| \leq \sum_{0 \leq j,k \leq \alpha} C_{jk}(T+1)^j M_k \leq C(T+1)^\alpha, \qquad x \in \mathbf{R}, \quad T \geq 0, \tag{2.4}$$

where $C = \sum_{0 \leq j,k \leq \alpha} C_{jk} M_k$. Since (2.4) is valid for any $x \in \mathbf{R}$ and $T \geq 0$, we come to

$$|\gamma(x)| \leq C(|x| + 1)^\alpha, \qquad x \in \mathbf{R},$$

which shows that γ is of polynomial growth.

Next we start with $G^{-1} \in \mathcal{L}(\mathcal{S}(\mathbf{R}), \mathcal{S}(\mathbf{R}))$. Choose $\beta \geq 0$ and $C'_{jk} \geq 0$ for $0 \leq j, k \leq \beta$ such that

$$\| G^{-1}\xi \|_{0,1} \leq \sum_{0 \leq j,k \leq \beta} C'_{jk} \| \xi \|_{j,k}. \tag{2.5}$$

In view of

$$\| G^{-1}\xi \|_{0,1} = \sup_{x\in\mathbf{R}} \left| \frac{d}{dx} \xi(\gamma(x)) \right| = \sup_{x\in\mathbf{R}} |\xi'(\gamma(x))\gamma'(x)| = \sup_{x\in\mathbf{R}} |\xi'(x)\gamma'(\gamma^{-1}(x))|,$$

In view of (2.5) we obtain

$$|\xi'(x)\gamma'(\gamma^{-1}(x))| \le \sum_{0\le j,k\le\beta} C'_{jk} \| \xi \|_{j,k}, \qquad x \in \mathbf{R}. \tag{2.6}$$

For $T \ge 1$ define $\eta_T \in \mathcal{S}(\mathbf{R})$ by

$$\eta'_T(x) = \begin{cases} -\rho(1-x), & 0 \le x \le 1, \\ -1, & 1 \le x \le T, \\ -\rho(x), & T \le x \le T+1, \\ 0, & T+1 \le x, \\ -\eta'_T(-x), & x \le 0. \end{cases}$$

Then,

$$\| \eta_T \|_{j,k} = \sup_{|x|\le T+1} |x^j \eta_T^{(k)}(x)| \le (T+1)^j \sup_{|x|\le T+1} |\eta_T^{(k)}(t)|. \tag{2.7}$$

It is obvious that for $k \ge 1$,

$$\sup_{|x|\le T+1} |\eta_T^{(k)}(x)| = \sup_{0\le x\le 1} |\rho^{(k-1)}(x)| = M_{k-1},$$

and for $k = 0$ we have

$$\sup_{|x|\le T+1} |\eta_T(x)| = \eta_T(0) = \int_{-(T+1)}^{0} \eta'_T(x)dx \le T+1.$$

Hence (2.7) becomes

$$\| \eta_T \|_{j,k} \le \begin{cases} (T+1)^j M_{k-1}, & k \ge 1, \\ (T+1)^{j+1}, & k = 0. \end{cases}$$

Thus, setting $\xi = \eta_T$ in (2.6) we come to

$$|\eta'_T(x)\gamma'(\gamma^{-1}(x))| \le C'(T+1)^{\beta+1}, \qquad x \in \mathbf{R}, \quad T \ge 1, \tag{2.8}$$

where $C' = \sum_{0\le j,k\le\beta} C'_{jk} M_{k-1}$ and $M_{-1} = 1$. Since (2.8) is valid for any $x \in \mathbf{R}$ and $T \ge 1$, we easily obtain

$$|\gamma'(x)| \le C'(|\gamma(x)|+1)^{\beta+1}, \qquad |\gamma(x)| \ge 1, \tag{2.9}$$

from which we see that γ' is of polynomial growth for $\{x \in \mathbf{R}; |\gamma(x)| \le 1\}$ is compact.

Now we show that $\gamma^{(n)}(x)$ is of polynomial growth by induction. Suppose that $\gamma^{(k)}(x)$ is of polynomial growth up to $k = n-1$. Note that $\gamma^{-1}(x)$ is also of polynomial growth as is easily seen from the first half of this proof. Hence $\gamma^{(k)}(\gamma^{-1}(x))$ is of polynomial growth for $0 \le k \le n-1$. On the other hand,

$$\frac{d^n}{dx^n}\xi(\gamma(x)) = \xi'(\gamma(x))\gamma^{(n)}(x) + \sum_{k=2}^{n} \xi^{(k)}(\gamma(x)) P_{n,k}(\gamma'(x), \cdots, \gamma^{(n-1)}(x)),$$

where $P_{n,k}$ is a polynomial. Since $G^{-1}\xi(x) = \xi(\gamma(x))$,

$$|\xi'(\gamma(x))\gamma^{(n)}(x)| \le \| G^{-1}\xi \|_{0,n} + \sum_{k=2}^{n} |\xi^{(k)}(\gamma(x)) P_{n,k}(\gamma'(x), \cdots, \gamma^{(n-1)}(x))|.$$

Then, by the continuity of G^{-1} and the assumption we obtain an estimate of the form:

$$|\xi'(\gamma(x))\gamma^{(n)}(x)| \le \sum_{0\le i,j\le\alpha} C_{ij} \| \xi \|_{i,j}, \qquad x \in \mathbf{R}.$$

Setting $\xi = \eta_T$ and repeating the above argument, we see that $\gamma^{(n)}$ is of polynomial growth. ∎

Let f be a $\mathbf{R}$-valued measurable function on $\mathbf{R}$ and let M_f be the corresponding multiplication operator acting on functions. For such a multiplication operator we have the following result, the proof of which is similar, see also [24, Chapter V].

Proposition 2. 2 *$M_f \in \mathcal{L}(\mathcal{S}(\mathbf{R}), \mathcal{S}(\mathbf{R}))$ if and only if f is slowly increasing. In particular, $M_f \in GL(\mathcal{S}(\mathbf{R}))$ if and only if both f and $1/f$ are slowly increasing.*

3 One-parameter transformation groups on a locally convex space

Throughout this section let $\mathfrak{X}$ denote a locally convex space with defining seminorms $\{\|\cdot\|_\alpha\}_{\alpha\in A}$ and the canonical bilinear form on $\mathfrak{X}^* \times \mathfrak{X}$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. A one-parameter subgroup $\{G_\theta\}_{\theta\in\mathbf{R}} \subset GL(\mathfrak{X})$ is called *differentiable* if there exists an operator $X \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ such that

$$X\xi = \lim_{\theta\to 0} \frac{G_\theta\xi - \xi}{\theta}, \qquad \xi \in \mathfrak{X}, \tag{3.1}$$

where the convergence of the right hand side is understood in the sense of $\mathfrak{X}$. As usual, this operator X is called the *infinitesimal generator* of $\{G_\theta\}$. A differentiable one-parameter subgroup is uniquely determined by its infinitesimal generator.

Remark If the Banach–Steinhaus theorem holds for $\mathfrak{X}$, (for example, if $\mathfrak{X}$ is a Barreled space, in particular, a Fréchet space), the existence of $\lim_{\theta\to 0}(G_\theta\xi - \xi)/\theta$ for any $\xi \in \mathfrak{X}$ with respect to the topology of $\mathfrak{X}$ ensures that the infinitesimal generator X is continuous, i.e., $X \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$. Moreover, the convergence (3.1) is uniform on every compact subset of $\mathfrak{X}$, namely,

$$\lim_{\theta\to 0} \sup_{\xi\in K} \left\| \frac{G_\theta\xi - \xi}{\theta} - X\xi \right\|_\alpha = 0 \tag{3.2}$$

for any $\alpha \in A$ and any compact subset $K \subset \mathfrak{X}$. When $\mathfrak{X}$ is a nuclear Fréchet space, every bounded closed subset of $\mathfrak{X}$ is compact. Therefore, in that case (3.2) is valid for any bounded subset $K \subset \mathfrak{X}$.

In general, not every $X \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ can be an infinitesimal generator of a differentiable one-parameter subgroup of $GL(\mathfrak{X})$, e.g., consider $X = 1 + x^2 - (d/dx)^2$ on $\mathfrak{X} = \mathcal{S}(\mathbf{R})$.

Proposition 3. 1 (Hida–Obata–Saitô [12]) *Let $X \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ and assume that there exists $R > 0$ such that $\{(RX)^n/n!\}_{n=0}^\infty$ is equicontinuous, namely, for every $\alpha \in A$ there exist $C = C(\alpha) \ge 0$ and $\beta = \beta(\alpha) \in A$ such that*

$$\sup_{n\ge 0} \frac{1}{n!} \| (RX)^n\xi \|_\alpha \le C \| \xi \|_\beta, \qquad \xi \in \mathfrak{X}.$$

Then there exists a differentiable one-parameter subgroup $\{G_\theta\}_{\theta\in\mathbf{R}}$ of $GL(\mathfrak{X})$ with infinitesimal generator X.

An outline of the proof is as follows: By assumption, the series

$$G_\theta \xi = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} X^n \xi, \qquad \xi \in \mathfrak{X}, \quad |\theta| < R, \tag{3.3}$$

is convergent in $\mathfrak{X}$ and $\| G_\theta \xi \|_\alpha \le C(1 - |\theta|/R)^{-1} \| \xi \|_\beta$, namely, $G_\theta \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ for $|\theta| < R$. Furthermore, $G_0 = I$ and $G_{\theta_1 + \theta_2} = G_{\theta_1} G_{\theta_2}$ whenever $|\theta_1|, |\theta_2|, |\theta_1 + \theta_2| < R$. We now define G_θ for all $\theta \in \mathbf{R}$. For a given $\theta \in \mathbf{R}$ choose a positive integer n such that $|\theta/n| < R$ and put $G_\theta = (G_{\theta/n})^n$. As is easily seen, this definition is independent of the choice of n, and therefore $G_{\theta_1+\theta_2} = G_{\theta_1} G_{\theta_2}$ for all $\theta_1, \theta_2 \in \mathbf{R}$. Finally, from the estimate

$$\left\| \frac{G_\theta \xi - \xi}{\theta} - X\xi \right\|_\alpha \le \sum_{n=2}^{\infty} \frac{|\theta|^{n-1}}{n!} \| X^n \xi \|_\alpha \le |\theta| C R^{-2} \left(1 - \frac{|\theta|}{R} \right)^{-1} \| \xi \|_\beta, \qquad |\theta| < R,$$

it follows that $\{G_\theta\}_{\theta \in \mathbf{R}}$ is a differentiable one-parameter subgroup of $GL(\mathfrak{X})$ with infinitesimal generator X.

Being based on the power series (3.3), the above argument is more natural in the complex context. Suppose that $\mathfrak{X}$ is a locally convex space over $\mathbf{C}$ and consider a "complex" one-parameter subgroup $\{\Omega_z\}_{z \in \mathbf{C}}$ of $GL(\mathfrak{X})$, i.e., $\Omega_z \in GL(\mathfrak{X})$ for any $z \in \mathbf{C}$ and

$$\Omega_{z_1} \Omega_{z_2} = \Omega_{z_1 + z_2}, \quad z_1, z_2 \in \mathbf{C}; \qquad \Omega_0 = I \quad \text{(identity operator)}.$$

It is called *holomorphic* if there exists an operator $\Xi \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ such that

$$\Xi \xi = \lim_{z \to 0} \frac{\Omega_z \xi - \xi}{z}, \qquad \xi \in \mathfrak{X}. \tag{3.4}$$

Again Ξ is called the *infinitesimal generator* of $\{\Omega_z\}$.

Lemma 3.2 (Obata [22]) *For $\Xi \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ the following four conditions are equivalent:*

(i) *there exists some $R > 0$ such that $\{(R\Xi)^n/n!\,;\, n = 0, 1, 2, \cdots\}$ is equicontinuous;*

(ii) *$\{(R\Xi)^n/n!\,;\, n = 0, 1, 2, \cdots\}$ is equicontinuous for any $R > 0$;*

(iii) *Ξ is the infinitesimal generator of some holomorphic one-parameter subgroup $\{\Omega_z\}$ of $GL(\mathfrak{X})$ such that $\{\Omega_z\,;\, |z| < R\}$ is equicontinuous for some $R > 0$.*

(iv) *Ξ is the infinitesimal generator of some holomorphic one-parameter subgroup $\{\Omega_z\}$ of $GL(\mathfrak{X})$ such that $\{\Omega_z\,;\, |z| < R\}$ is equicontinuous for any $R > 0$.*

Moreover, in that case, for any $\alpha \in A$ there exists $\beta \in A$ such that

$$\lim_{N \to \infty} \sup_{\| \phi \|_\beta \le 1} \left\| \Omega_z \phi - \sum_{n=0}^{N} \frac{z^n}{n!} \Xi^n \phi \right\|_\alpha = 0, \qquad z \in \mathbf{C},$$

$$\lim_{z \to 0} \sup_{\| \phi \|_\beta \le 1} \| \Omega_z \phi - \phi \|_\alpha = 0,$$

$$\lim_{z \to 0} \sup_{\| \phi \|_\beta \le 1} \left\| \frac{\Omega_z \phi - \phi}{z} - \Xi \phi \right\|_\alpha = 0. \tag{3.5}$$

In particular,

$$\Omega_z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Xi^n, \quad z \in \mathbf{C}; \qquad \lim_{z \to 0} \Omega_z = I, \qquad \lim_{z \to 0} \frac{\Omega_z - I}{z} = \Xi,$$

with respect to the topology of $\mathcal{L}(\mathfrak{X}, \mathfrak{X})$.

An operator $\Xi \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ satisfying one of the conditions in Lemma 3.2 is called an *equicontinuous generator*. A one-parameter subgroup $\{\Omega_z\}$ is called *locally equicontinuous* if $\{\Omega_z\,;\,|z| < R\}$ is equicontinuous for any $R > 0$. Obviously the idea of an equicontinuous generator is a variant of the standard terminology of an equicontinuous semigroup (see e.g., Yosida [30]), and our main consequence is the establishment of a one-to-one correspondence via the exponential map between the equicontinuous generators and the locally equicontinuous holomorphic one-parameter subgroups.

Note that the convergence in the sense of (3.5) is somewhat stronger than (3.2). If for any $\alpha \in A$ there exists $\beta \in A$ such that (3.5) holds, the one-parameter subgroup $\{\Omega_z\}_{z\in\mathbf{C}}$ is called *regular*. This notion is used also for a differentiable one-parameter subgroup $\{G_\theta\}_{\theta\in\mathbf{R}}$, see [12]. However, algebraic operation for equicontinuous or regular generators has not been investigated satisfactorily.

4 Cochran–Kuo–Sengupta space — White noise triple

Following Cochran–Kuo–Sengupta [5] we review the construction of white noise triples, see also [23]. For a positive sequence $\{\alpha(n)\}_{n=0}^{\infty}$ we consider the following three conditions

(A1) $\alpha(0) = 1$ and $\gamma \equiv \sup \alpha^{-1}(n) < \infty$;

(A2) the associated exponential generating function:

$$G_\alpha(t) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!}\, t^n \tag{4.1}$$

is entire holomorphic, i.e., has an infinite radius of convergence;

(A3) $\displaystyle\limsup_{n\to\infty} \frac{n^2}{(n!\alpha(n))^{1/n}} \left\{ \inf_{t>0} \frac{G_\alpha(t)^{1/n}}{t} \right\} < \infty$, or equivalently

$$\widetilde{G}_\alpha(t) = \sum_{n=0}^{\infty} t^n\, \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\} \tag{4.2}$$

has a positive radius of convergence.

Given such a sequence $\{\alpha(n)\}$, with a Hilbert space H one may associate a variant of (Boson) Fock space:

$$\Gamma_\alpha(H) = \left\{ (f_n)\,;\, f_n \in H^{\widehat{\otimes} n},\ \sum_{n=0}^{\infty} n!\alpha(n)\, |\, f_n\,|^2 < \infty \right\}.$$

Obviously, $\Gamma_\alpha(H)$ becomes a Hilbert space with the norm

$$\|\,(f_n)\,\|^2 = \sum_{n=0}^{\infty} n!\alpha(n)\, |\, f_n\,|^2.$$

By definition (and our convention, e.g., [17], [19]) the usual Fock space, denoted by $\Gamma(H)$, is the case of $\alpha(n) = 1$ for all n. By condition (A1), $\mathbf{C}$ is isometrically isomorphic to the zero-particle space of $\Gamma_\alpha(H)$, and $\Gamma_\alpha(H)$ is continuously imbedded in $\Gamma(H)$.

We now go back to the Gelfand triple:

$$E = \mathcal{S}(\mathbf{R}) \subset H = L^2(\mathbf{R}) \subset E^* = \mathcal{S}'(\mathbf{R}). \tag{4.3}$$

Recall that E is a countably Hilbert nuclear space with the defining Hilbertian norms: $|\xi|_p = |A^p\xi|_0$, where $A = 1 + t^2 - d^2/dt^2$. For $p \in \mathbf{R}$ let E_p be the completion of $\mathcal{S}(\mathbf{R})$ with respect to the norm $|\cdot|_p$. By definition we put

$$\Gamma_\alpha(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\alpha(E_p).$$

The topology is given by the family of norms:

$$\|(f_n)\|_{p,+}^2 = \sum_{n=0}^{\infty} n!\alpha(n)\,|f_n|_p^2, \qquad p \geq 0.$$

We say that $\Gamma_\alpha(E)$ is the *Fock space over E associated with* $\{\alpha(n)\}$. The dual space of $\Gamma_\alpha(E)$ is described easily. The space $\Gamma_{\alpha^{-1}}(E_{-p})$ is defined in a similar manner as above, the norm of which is given by

$$\|(f_n)\|_{-p,-}^2 = \sum_{n=0}^{\infty} n!\,\alpha^{-1}(n)\,|f_n|_{-p}^2, \qquad p \geq 0.$$

It is proved by a standard argument that

$$\Gamma_\alpha(E)^* \cong \operatorname*{ind\,lim}_{p\to\infty} \Gamma_{\alpha^{-1}}(E_{-p}),$$

where $\Gamma_\alpha(E)^*$ carries the strong dual topology. Finally, taking the complexification, we obtain a chain of Fock spaces: $\Gamma_\alpha(E_\mathbf{C}) \subset \Gamma_\alpha(H_\mathbf{C}) \subset \Gamma(H_\mathbf{C}) \subset \Gamma_\alpha(H_\mathbf{C})^* \subset \Gamma_\alpha(E_\mathbf{C})^*$. Since $\|A^{-q}\|_{HS}^2 = \sum_{j=0}^{\infty}(2j+2)^{-2q}$ can be less than one for a sufficiently large $q \geq 0$, the space $\Gamma_\alpha(E_\mathbf{C})$ is nuclear and

$$\Gamma_\alpha(E_\mathbf{C}) \subset \Gamma(H_\mathbf{C}) \subset \Gamma_\alpha(E_\mathbf{C})^* \tag{4.4}$$

is a Gelfand triple, see [5].

Let μ be the standard Gaussian measure on E^* and $L^2(E^*, \mu)$ the Hilbert space of $\mathbf{C}$-valued L^2-functions on E^*. Then through the Wiener–Itô–Segal isomorphism (4.4) gives rise to a Gelfand triple:

$$\mathcal{W} \subset L^2(E^*, \mu) \cong \Gamma(H_\mathbf{C}) \subset \mathcal{W}^*, \tag{4.5}$$

which is referred to as the *Cochran–Kuo–Sengupta space.* In particular, (4.5) is called the *Hida-Kubo-Takenaka space* [15] or the *Kondratiev–Streit space* [13] according as $\alpha(n) = 1$ or $\alpha(n) = (n!)^\beta$, $0 \leq \beta < 1$, see also [17]. The canonical bilinear form on $\mathcal{W} \times \mathcal{W}^*$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. Then we have

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n!\,\langle F_n, f_n\rangle, \qquad \Phi \sim (F_n) \in \mathcal{W}^*, \quad \phi \sim (f_n) \in \mathcal{W}.$$

A non-trivial example of a sequence $\{\alpha(n)\}$ satisfying (A1)–(A3) is the *Bell numbers of degree k* defined by the generating function:

$$G_{\mathrm{Bell}(k)}(t) = \frac{\exp(\overbrace{\exp(\exp(\cdots(\exp}^{k \text{ times}} t)\cdots)))}{\exp(\exp(\exp(\cdots(\exp 0)\cdots)))} = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n, \tag{4.6}$$

for more details see [5], [14].

5 Infinite dimensional rotation group and Fock representation

Let $X \in \mathcal{L}(E_{\mathbf{C}}, E_{\mathbf{C}})$ be given. For $\phi \sim (f_n) \in \mathcal{W}$ we put

$$(\Gamma(X)\phi) \sim (X^{\otimes n} f_n), \qquad (d\Gamma(X)\phi) \sim (n(X \otimes I^{\otimes(n-1)}) f_n). \tag{5.1}$$

It is easily verified that both $\Gamma(X)$ and $d\Gamma(X)$ belong to $\mathcal{L}(\mathcal{W}, \mathcal{W})$. Their symbols are easily obtained:

$$\widehat{\Gamma(X)}(\xi, \eta) = \langle\!\langle \Gamma(X)\phi_\xi, \phi_\eta \rangle\!\rangle = e^{\langle X\xi, \eta\rangle}, \quad \widehat{d\Gamma(X)}(\xi, \eta) = \langle X\xi, \eta\rangle\, e^{\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E_{\mathbf{C}},$$

where $\phi_\xi \sim (\xi^{\otimes n}/n!)$ is an exponential vector.

Theorem 5. 1 *Let $\{G_\theta\}_{\theta\in\mathbf{R}}$ be a regular one-parameter subgroup of $GL(E)$ with infinitesimal generator X. Then, $\{\Gamma(G_\theta)\}_{\theta\in\mathbf{R}}$ is a regular one-parameter subgroup of $GL(\mathcal{W})$ with infinitesimal generator $d\Gamma(X)$.*

Theorem 5. 2 *Let $\{G_z\}_{z\in\mathbf{C}}$ be a holomorphic one-parameter subgroup of $GL(E_{\mathbf{C}})$ with equicontinuous generator X. Then, $\{\Gamma(G_z)\}_{z\in\mathbf{C}}$ is a holomorphic one-parameter subgroup of $GL(\mathcal{W})$ with equicontinuous generator $d\Gamma(X)$.*

The proof is a simple modification of the arguments in [12], [19]; however, it is rather long and is omitted here.

Let $g \in O(E; H)$. Then g^* becomes a topological isomorphism of E^* and the Gaussian measure μ is kept invariant under the action of g^*. Therefore, $(\Gamma, L^2(E^*, \mu))$ is a unitary representation of $O(E; H)$ and it holds that

$$(\Gamma(g)\phi)(x) = \phi(g^* x), \qquad \phi \in L^2(E^*, \mu), \quad x \in E^*.$$

Note also that $\Gamma(g) \in GL(\mathcal{W})$.

As is easily seen, if X is the infinitesimal generator of a differentiable one-parameter subgroup of $O(E; H)$, it is skew-symmetric in the sense that

$$\langle X\xi, \eta\rangle = -\langle \xi, X\eta\rangle, \qquad \xi, \eta \in E. \tag{5.2}$$

In general, if $X \in \mathcal{L}(E, E)$ is skew-symmetric in the sense of (5.2), there exists a skew-symmetric distribution $\kappa \in E \otimes E^*$ such that

$$d\Gamma(X) = \int_{\mathbf{R}\times\mathbf{R}} \kappa(s,t)(a_s^* a_t - a_t^* a_s)\, ds dt. \tag{5.3}$$

In fact, $\kappa \in (E \otimes E)^*$ defined by

$$\langle \kappa, \eta \otimes \zeta\rangle = \frac{1}{2}\langle \eta, X\zeta\rangle, \qquad \eta, \zeta \in E,$$

has the desired property. Moreover, using the notion of an integral kernel operator, we have

$$d\Gamma(X) = 2\Xi_{1,1}(\kappa).$$

For a comprehensive account of integral kernel operators see [19]. Combining the above discussion we come to

Theorem 5. 3 *Let X be an infinitesimal generator of a regular one-parameter subgroup $\{G_\theta\}$ of $O(E;H)$. Then, $\{\Gamma(G_\theta)\}$ is a regular one-parameter subgroup of $GL(\mathcal{W})$ with the infinitesimal generator $d\Gamma(X)$. Moreover, $d\Gamma(X)$ is given by*

$$d\Gamma(X) = \int_{\mathbf{R}\times\mathbf{R}} \kappa(s,t)(a_s^* a_t - a_t^* a_s)\, dsdt = \int_{\mathbf{R}\times\mathbf{R}} \kappa(s,t)(W_s a_t - W_t a_s)\, dsdt,$$

where $\kappa \in E \otimes E^*$ is a skew-symmetric distribution and $\{W_t\}$ is the white noise.

Now consider a one-parameter difeomorphism group $\{\gamma_\theta\}_{\theta\in\mathbf{R}}$ of $\mathbf{R}$ and put

$$(G_\theta \xi)(x) = \xi(\gamma_\theta(x))\sqrt{\gamma_\theta'(x)}.$$

Assume that $\{G_\theta\}$ is a one-parameter subgroup of $GL(E)$. For example, this holds if $\gamma_\theta(x)$ is slowly increasing for all $\theta \in \mathbf{R}$, see Propositions 2. 1 and 2. 2. (This condition seems also necessary but we have no proof.) Then $\{G_\theta\}$ is a one-parameter subgroup of $O(E;H)$ and is called a *whisker* after Hida [8]. The infinitesimal generator of $\{G_\theta\}$ is given by

$$X = F(x)D + \frac{1}{2}F'(x), \tag{5.4}$$

where $F(x)D$ is the vector field corresponding to $\{\gamma_\theta\}$. Using the symbol M_F for the multiplication operator by $F(x)$, we have

$$X = \frac{1}{2}(DM_F + M_F D).$$

In the early 1970's Goldin [6], Grodnik– Sharp [7] and others introduced the particle flux density (or the momentum density)

$$\frac{1}{2i}\{a_x^*(\nabla a_x) - (\nabla a_x^*)a_x\}, \qquad \nabla = \left(\frac{d}{dx_1}, \cdots, \frac{d}{dx_n}\right), \qquad x \in \mathbf{R}^n, \tag{5.5}$$

in connection with unitary representation of diffeomorphism groups, or more precisely, of Lie algebras of vector fields. Now we consider the case of $\mathbf{R}^n = \mathbf{R}$ for notational simplicity. For $\zeta \in E_{\mathbf{C}}$ define an integral kernel operator

$$J(\zeta) = \Xi_{1,1}(-(1\otimes\zeta)\partial_1\tau + (\zeta\otimes 1)\partial_2\tau),$$

where ∂_k is the partial derivative with respect to the k-th coordinate variable and $\tau \in (E\otimes E)^*$ is defined by $\langle \tau, \xi\otimes\eta\rangle = \langle \xi, \eta\rangle$. By partial integration in an integral kernel operator [21] we may write

$$J(\zeta) = \int_{\mathbf{R}} \zeta(x)\{a_x^*(\nabla a_x) - (\nabla a_x^*)a_x\}\, dx.$$

We first note the following

Proposition 5. 4 *$J(\zeta) \in \mathcal{L}(\mathcal{W},\mathcal{W})$ for any $\zeta \in E_{\mathbf{C}}$, i.e., the particle flux density (5.5) is a $\mathcal{L}(\mathcal{W},\mathcal{W})$-valued distribution.*

In fact, the above assertion is proved in [21] for the case of Hida–Kubo–Takenaka space, but the proof for the general case is similar.

Let $K \in \mathcal{L}(E, E)$ be the corresponding operator to $\kappa = -(1 \otimes \zeta)\partial_1\tau + (\zeta \otimes 1)\partial_2\tau$ under the canonical isomorphism $(E \otimes E)^* \cong \mathcal{L}(E, E)$. Then, as is easily seen, we have

$$K = -(DM_\zeta + M_\zeta D) \quad \text{and} \quad J(\zeta) = \Xi_{1,1}(\kappa) = d\Gamma(K).$$

Hence from Theorem 5. 1 it follows that $J(\zeta)$ is an infinitesimal generator of a regular one-parameter subgroup of $GL(\mathcal{W})$ if and only if K is an infinitesimal generator of a regular one-parameter subgroup of $GL(E)$. Summarizing the above discussion we state

Proposition 5. 5 *Let $\{\gamma_\theta\}$ be a one-parameter diffeomorphism group of $\mathbf{R}$ such that $\gamma_\theta(x)$ is slowly increasing for all $\theta \in \mathbf{R}$ and let $\{G_\theta\}$ be the corresponding whisker. Then $\{\Gamma(G_\theta)\}$ is a regular one-parameter subgroup of the infinite dimensional unitary group $U(\mathcal{W}, L^2(E^*, \mu))$ if and only if $\{G_\theta\}$ is a regular one-parameter subgroup of $O(E; H)$. In that case, the infinitesimal generator of $\Gamma(G_\theta)$ is given by*

$$-\frac{1}{2}\int_{\mathbf{R}} F(x)\{a_x^*(\nabla a_x) - (\nabla a_x^*)a_x\}\, dx. \tag{5.6}$$

Remark It is an interesting open question to find a necessary and sufficient condition for $F \in C^\infty(\mathbf{R})$ in order that (5.6) is regular, or equivalently, in order that X in (5.4) is regular.

As is expected from below, $J(\zeta)$ seldom happens to be an equicontinuous generator. The shift $\{\sigma_\theta\}$ gives rise to a whisker $\{S_\theta\}$ of which infinitesimal generator is the differential operator D. If D were an equicontinuous generator, every $\xi \in \mathcal{S}(\mathbf{R})$ should have a holomorphic extension. On the other hand, the regularity is obvious from the direct estimate:

$$\left\| \frac{S_\theta\xi - \xi}{\theta} - D\xi \right\|_{\alpha,\beta} \le \frac{\theta}{2} \| \xi \|_{\alpha,\beta+2}, \qquad \xi \in E,$$

which is verified by the mean value theorem in elementary calculus.

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