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Perturbation Problem of Embedded Eigenvalues in Quantum Field Models and Representations of Canonical Commutation Relations

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Abstract

We review a general theory of a new type of representation of the canonical commutation relations over a Hilbert space in connection with perturbation problem of embedded eigenvalues in a class of quantum field models.

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1 Introduction—physical background and motivation

As is well known, a nonrelativistic quantum particle with mass $m > 0$ moving in the d-dimensional Euclidean space $\mathbf{R}^d$ under the influence of a scalar potential V (a real-valued Borel measurable function on $\mathbf{R}^d$) is described by the Schrödinger Hamiltonian

$$H_{\mathrm{p}} := -\frac{\Delta}{2m} + V \tag{1.1}$$

acting in the Hilbert space $L^2(\mathbf{R}^d)$, where Δ is the d-dimensional generalized Laplacian. We assume that H_{p} is essentially self-adjoint and denote its closure by $\bar{H}_{\mathrm{p}}$. Suppose that the particle can interact with a quantum field. Then one must replace the Hamiltonian H_{p} by another Hamiltonian H, taking into account the interaction between the particle

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and the quantum field. Indeed there are physical phenomena that can be explained only if such a consideration is made, e.g., the *Lamb shift* and the *spontaneous emission of light* in atoms (e.g., [17, Chapter 6]).

A standard description of a quantum field can be made in terms of a Fock space. To be concrete, let us consider a Bose quantum field whose one-particle states are described by a complex Hilbert space $\mathcal{K}$. The Hilbert space of state vectors of the quantum field may be taken to be the symmetric (boson) Fock space over $\mathcal{K}$

$$\mathcal{F}_{\mathrm{s}}(\mathcal{K}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathcal{K}, \tag{1.2}$$

where $\otimes_{\mathrm{s}}^{n}\mathcal{K}$ denotes the n-fold symmetric tensor product Hilbert space of $\mathcal{K}$ with $\otimes_{\mathrm{s}}^{0}\mathcal{K} := \mathbf{C}$. Then the free Hamiltonian of the quantum field (the Hamiltonian in the case where the quantum field has no interactions) is given by the second quantization operator

$$d\Gamma_{\mathcal{K}}(h) := \bigoplus_{n=0}^{\infty} h^{(n)}, \tag{1.3}$$

on $\mathcal{F}_{\mathrm{s}}(\mathcal{K})$, where h is a self-adjoint operator on $\mathcal{K}$ describing the one free boson and $h^{(n)}$ is the closure of the operator

$$\sum_{j=1}^{n} I \otimes \cdots I \otimes \overset{j}{\check{h}} \otimes I \cdots \otimes I$$

($h^{(0)} := 0$; the symbol I denotes identity operator) (for more details, see, e.g., [23, §VIII.10, Example 2], [16, §5.2]). A Hamiltonian H of the system of the above mentioned quantum particle interacting with the quantum field is given by the following form:

$$H := H_0 + H_I \tag{1.4}$$

acting in the tensor product Hilbert space $L^2(\mathbf{R}^d) \otimes \mathcal{F}_{\mathrm{s}}(\mathcal{K})$, where

$$H_0 := \bar{H}_{\mathrm{p}} \otimes I + I \otimes d\Gamma_{\mathcal{K}}(h) \tag{1.5}$$

and H_I is a symmetric operator describing an interaction between the quantum particle and the quantum field. Then an important task is to investigate the spectrum of H. But, here, we meet a difficult problem as explained below.

For a linear operator A on a Hilbert space, we denote its spectrum (resp. point spectrum) by $\sigma(A)$ (resp. $\sigma_{\mathrm{p}}(A)$). For simplicity, suppose that the spectrum of $\bar{H}_{\mathrm{p}}$ is given as follows:

$$\sigma_{\mathrm{p}}(\bar{H}_{\mathrm{p}}) = \{E_n\}_{n=0}^{\infty}, \quad E_0 < E_1 < \cdots < E_n < E_{n+1} < \cdots < \Sigma,$$
$$\sigma(\bar{H}_{\mathrm{p}}) = \sigma_{\mathrm{p}}(\bar{H}_{\mathrm{p}}) \cup [\Sigma, \infty),$$

where $\Sigma \in \mathbf{R}$ is a constant.

As for h, we suppose that

$$\sigma(h) = [M, \infty), \quad \sigma_{\mathrm{p}}(h) = \emptyset \tag{1.6}$$

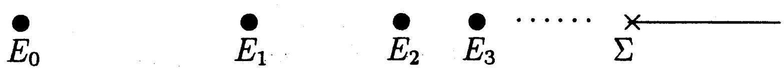

Figure 1: The spectrum of $\bar{H}_{\mathrm{p}}$

with $M \geq 0$ a constant. Then we have

$$\sigma_{\mathrm{p}}(d\Gamma_{\mathcal{K}}(h)) = \{0\}, \quad \sigma(d\Gamma_{\mathcal{K}}(h)) = \{0\} \cup [M, \infty). \tag{1.7}$$

It follows that

$$\sigma_{\mathrm{p}}(H_0) = \{E_n\}_{n=0}^{\infty}, \quad \sigma(H_0) = \{E_n\}_{n=0}^{\infty} \cup [E_0 + M, \infty). \tag{1.8}$$

This shows that all the eigenvalues E_n of H_0 with $E_n \geq E_0 + M$ are *embedded* in its continuous spectrum. In particular, if $M = 0$, then all the eignvalues of H_0 are embedded ones. Thus to analyze the spectrum of H includes a perturbation problem of embedded eigenvalues, which are difficult to solve in general.

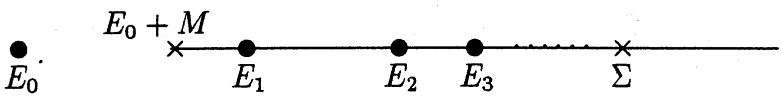

Figure 2: The spectrum of H_0

In the case where the quantum particle is a harmonic oscillator, i.e., V is of the form $V(x) = \mu x^2$ ($x \in \mathbf{R}^d$; $\mu > 0$ is a constant), mathematically rigorous studies on this problem have been made in a series of papers [2]–[9]. Recently more general cases and other types of models including the spin-boson model have been discussed [22], [18], [19], [20], [21], [13], [14] (see also [11], [12], [27]).

In this paper we present a brief review of the paper [10] which gives a unified approach, from a representation-theoretic point of view, to perturbation problem of embedded eigenvalues in a class of models considered in [2]–[9]. This approach is based on a new type of representation of the canonical commutaiton relations (CCR) over a Hilbert space and non-perturbative, making it possible to analyze exactly the spectrum of the Hamiltonian under consideration. Typical examples to which our method can be applied are as follows (th symbol $\otimes$ for operator tensor product is omitted):

(1) *The Schwabl-Thirring model* [2, 3] .

$$H = -\frac{1}{2m}\Delta + \frac{m\omega_0^2}{2}x^2 + \int_{\mathbf{R}^d} a(k)^* a(k)\omega(k)dk + \lambda \sum_{j=1}^{d} x_j \phi(g_j)$$

acting in $L^2(\mathbf{R}^d)\otimes\mathcal{F}_s(L^2(\mathbf{R}^d))$, where $a(f)=\int_{\mathbf{R}^d}a(k)f(k)^*dk$, $f\in L^2(\mathbf{R}^d)$, are the annihilation operators on $\mathcal{F}_s(L^2(\mathbf{R}^d))$ (e.g., [24, §X.7], [16, §5.2]), $\phi(g_j):=(a(g_j)+a(g_j)^*)/\sqrt{2}$, $g_j\in L^2(\mathbf{R}^d)$, $\omega(k)$ is a nonnegative function denoting a dispersion relation of one boson with momentum $k\in\mathbf{R}^d$, $\omega_0>0$ is a constant, $\lambda\in\mathbf{R}$ is a coupling constant and $x=(x_1,\cdots,x_d)\in\mathbf{R}^d$. The symbol $\int_{\mathbf{R}^d}a(k)^*a(k)\omega(k)dk$ is a formal expression of $d\Gamma_{L^2(\mathbf{R}^d)}(\omega)$.

A standard example of ω is: (i) (relativistic case) $\omega(k)=\sqrt{k^2+M^2}$, $k\in\mathbf{R}^d$ ($M\geq 0$ is a constant) ; (ii) (nonrelativistic case) $\omega(k)=k^2/2M$.

(2) *The RWA model* [5] .

$$H=\sum_{j=1}^{N}\omega_jA_j^*A_j+\int_{\mathbf{R}^d}a(k)^*a(k)\omega(k)dk+\lambda\sum_{j=1}^{N}[A_j^*a(g_j)+A_ja(g_j)^*]$$

acting in $\mathcal{F}_s(\mathbf{C}^N)\otimes\mathcal{F}_s(L^2(\mathbf{R}^d))$, where each $\omega_j>0$ is a constant and $A(z):=\sum_{j=1}^N A_jz_j^*$, $z=(z_1,\cdots,z_N)\in\mathbf{C}^N$, are the annihilation operators on $\mathcal{F}_s(\mathbf{C}^N)$: $[A_j,A_k^*]=\delta_{jk}$, $[A_j,A_k]=0$.

(3) *A generalized Schwabl-Thirring model* .

$$H=\frac{1}{2m}\sum_{j=1}^{d}(-iD_j-\alpha x_j)^2+\int_{\mathbf{R}^d}a(k)^*a(k)\omega(k)dk+\lambda\sum_{j=1}^{d}x_j\phi(g_j)$$

acting in $L^2(\mathbf{R}^d)\otimes\mathcal{F}_s(L^2(\mathbf{R}^d))$, where D_j is the generalized partial differential operator in x_j and $\alpha\in\mathbf{R}$ is a constant.

(4) *The Pauli-Fierz model in th dipole approximation* [1, 4, 9] (see also [27])

$$H=\frac{1}{2m}\sum_{j=1}^{d}(-iD_j-qA_j(\varrho))^2+\frac{m\omega_0^2}{2}x^2+\sum_{r=1}^{d-1}\int_{\mathbf{R}^d}a_r(k)^*a_r(k)\omega(k)dk,$$

acting in $L^2(\mathbf{R}^d)\otimes\mathcal{F}_s(\oplus_{r=1}^{d-1}L^2(\mathbf{R}^d))$, where $q\in\mathbf{R}$ is a constant denoting the electric charge of the particle and $A(\varrho)=(A_1(\varrho),\cdots,A_d(\varrho))$ is the quantized radiation field on $\mathcal{F}_s(\oplus_{r=1}^{d-1}L^2(\mathbf{R}^d))$ smeared out by a function ϱ with suitable regularity.

For other models, see [6] and references therein.

A basic observation for our method is in the fact that we have a natural identification

$$L^2(\mathbf{R}^d)=\mathcal{F}_s(\mathbf{C}^d),$$

so that

$$L^2(\mathbf{R}^d)\otimes\mathcal{F}_s(\mathcal{K})=\mathcal{F}_s(\mathbf{C}^d)\otimes\mathcal{F}_s(\mathcal{K})=\mathcal{F}_s(\mathbf{C}^d\oplus\mathcal{K})$$

Thus the quantum system consisting of a particle and a quantum field may be described in terms of *one* (extended) quantum field whose one-particle Hilbert space is $\mathbf{C}^d\oplus\mathcal{K}$. With this observation, we consider in an abstract form a quantum field theory on the Fock space $\mathcal{F}_s(\mathcal{M}\oplus\mathcal{K})$ as a representaion theory of CCR ($\mathcal{M}$ is a Hilbert space).

2 A new type of representation of the CCR over a Hilbert space

For a linear operator A on a Hilbert space, we denote its domain by $D(A)$.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$ (complex linear in the second variable) and norm $\|\cdot\|_{\mathcal{H}}$. We denote by CCR($\mathcal{H}$) the abstract $*$-algebra (with unit element I) generated by elements $a(f), a(f)^*$ ($f \in \mathcal{H}$) satisfying the CCR over $\mathcal{H}$

$$[a(f), a(g)^*] = (f,g)_{\mathcal{H}} I, \quad [a(f), a(g)] = 0 = [a(f)^*, a(g)^*], \quad f, g \in \mathcal{H}, \tag{2.1}$$

with the property that the mapping $a: f \to a(f)$ from $\mathcal{H}$ to CCR($\mathcal{H}$) is anti-linear, where $[A,B] := AB - BA$.

Definition 2.1 A triple $\{\mathcal{F}, \mathcal{D}, \{a(f)|f \in \mathcal{H}\}\}$ consiting of a complex Hilbert space $\mathcal{F}$, a dense subspace $\mathcal{D}$ of $\mathcal{F}$ and an anti-linear mapping $a: f \to a(f)$ from $\mathcal{H}$ to the set of closed linear operators on $\mathcal{F}$ is called a representation of CCR($\mathcal{H}$) if the following (i) and (ii) hold: (i) $\mathcal{D} \subset \cap_{f\in\mathcal{H}} D(a(f)) \cap D(a(f)^*)$, $a(f)\mathcal{D} \subset \mathcal{D}$, $a(f)^*\mathcal{D} \subset \mathcal{D}$ for all $f \in \mathcal{H}$; (ii) $\{a(f)|f \in \mathcal{H}\}$ fulfil the CCR (2.1) on $\mathcal{D}$.

A standard example of representation of CCR($\mathcal{H}$) is given as follows. Let $\mathcal{F}_{\rm s}(\mathcal{H})$ be the symmetric Fock space over $\mathcal{H}$. We denote by $\Omega_{\mathcal{H}} := \{1, 0, 0, \cdots\}$ the Fock vacuum in $\mathcal{F}_{\rm s}(\mathcal{H})$ and by $a_{\mathcal{H}}(f), f \in \mathcal{H}$, the annihilation operators on $\mathcal{F}_{\rm s}(\mathcal{H})$ (anti-linear in f) (e.g., [24, §X.7], [16, §5.2]). Let

$$\mathcal{F}_{\rm fin}(\mathcal{H}) := \mathcal{L}\{\Omega_{\mathcal{H}},\ a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}} |\ n \geq 1,\ f_j \in \mathcal{H},\ j = 1, \cdots, n\}, \tag{2.2}$$

where $\mathcal{L}\{\cdots\}$ denotes the subspace algebraically spanned by the vectors in the set $\{\cdots\}$. Then $\mathcal{F}_{\rm fin}(\mathcal{H})$ is dense and $\{\mathcal{F}_{\rm s}(\mathcal{H}), \mathcal{F}_{\rm fin}(\mathcal{H}), \{a_{\mathcal{H}}(f)|f \in \mathcal{H}\}\}$ is a representation of CCR($\mathcal{H}$). This representation is called the *Fock representation* of CCR($\mathcal{H}$).

As is explained in the Introduction, we are concerned with the case where $\mathcal{H}$ is given by the direct sum of two Hilbert spaces $\mathcal{M}$ and $\mathcal{K}$ with $\mathcal{M} \neq \{0\}$ and $\mathcal{K} \neq \{0\}$:

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{K} = \{(v,u)|v \in \mathcal{M}, u \in \mathcal{K}\}. \tag{2.3}$$

Then we have the natural identification

$$\mathcal{F}_{\rm s}(\mathcal{H}) = \mathcal{F}_{\rm s}(\mathcal{M}) \otimes \mathcal{F}_{\rm s}(\mathcal{K}). \tag{2.4}$$

Remark 2.2 In applications to models of a quantum particle coupled to a quantum field, the Hilbert spaces $\mathcal{M}$ and $\mathcal{K}$ are taken as $\mathcal{M} = \mathbf{C}^N$, $\mathcal{K} = \oplus^m L^2(\mathbf{R}^d)$ with d, m, $N \in \mathbf{N}$. Then we have

$$\mathcal{F}_{\rm s}(\mathcal{H}) = \mathcal{F}_{\rm s}(\mathbf{C}^N) \otimes \mathcal{F}_{\rm s}(\oplus^m L^2(\mathbf{R}^d)) = L^2(\mathbf{R}^N) \otimes \mathcal{F}_{\rm s}(\oplus^m L^2(\mathbf{R}^d))$$

Let $J_{\mathcal{M}}$ and $J_{\mathcal{K}}$ be conjugations on $\mathcal{M}$ and $\mathcal{K}$ respectively and define

$$J_{\mathcal{H}} := J_{\mathcal{M}} \oplus J_{\mathcal{K}}, \tag{2.5}$$

which is a conjugation on $\mathcal{H}$. For a linear operator A on $\mathcal{H}$ and $f \in \mathcal{H}$, we set

$$A_c := J_{\mathcal{H}} A J_{\mathcal{H}}, \quad \bar{f} := J_{\mathcal{H}} f. \tag{2.6}$$

For two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we denote by $\mathsf{B}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and set $\mathsf{B}(\mathcal{H}_1) := \mathsf{B}(\mathcal{H}_1, \mathcal{H}_1)$.

Let S and T be elements in $\mathsf{B}(\mathcal{K}, \mathcal{H})$ which satisfy

$$S^*S - T^*T = I_{\mathcal{K}}, \quad S^*T_c - T^*S_c = 0, \tag{2.7}$$

where $I_{\mathcal{K}}$ denotes the identity operator on $\mathcal{K}$.

We denote by N_{b} the number operator on $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$ ([24, §X.7], [16, §5.2]). It is well known [16, §5.2] that, for all $f \in \mathcal{H}$, $D(N_{\mathrm{b}}^{1/2}) \subset D(a(f)^{\#})$ and

$$\|a(f)^{\#}\Psi\| \leq \|f\|_{\mathcal{H}} \|(N_{\mathrm{b}} + 1)^{1/2}\Psi\|, \quad \Psi \in D(N_{\mathrm{b}}^{1/2}), \tag{2.8}$$

where $a(f)^{\#}$ denotes either $a(f)$ or $a(f)^*$. For each $u \in \mathcal{K}$, we define an operator $b(u)$ acting in $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$ by

$$b(u) := a_{\mathcal{H}}(Su) + a_{\mathcal{H}}(T_c\bar{u})^*. \tag{2.9}$$

with $D(b(u)) = D(N_{\mathrm{b}}^{1/2})$. It follows that $D(N_{\mathrm{b}}^{1/2}) \subset D(b(u)^*)$ for all $u \in \mathcal{K}$. Hence $b(u)$ is closable. We denote its closure by the same symbol $b(u)$, so that $D(N_{\mathrm{b}}^{1/2}) \subset D(b(u))$. We have

$$b(u)^* = a_{\mathcal{H}}(Su)^* + a_{\mathcal{H}}(T_c\bar{u}) \tag{2.10}$$

on $D(N_{\mathrm{b}}^{1/2})$. The following fact can be easily proved.

Proposition 2.3 *The triple*

$$\pi_b := \{\mathcal{F}_{\mathrm{s}}(\mathcal{H}), \mathcal{F}_{\mathrm{fin}}(\mathcal{H}), \{b(u) | u \in \mathcal{K}\}\} \tag{2.11}$$

is a representation of $\mathrm{CCR}(\mathcal{K})$.

The representation π_b is a basic object playing an important role in our theory.

Remark 2.4 Under the identification (2.4), we can identify $a_{\mathcal{H}}(f)^{\#}$, $f = (v, u) \in \mathcal{H}$, as

$$a_{\mathcal{H}}(f)^{\#} = a_{\mathcal{M}}(v)^{\#} \otimes I_{\mathcal{F}_{\mathrm{s}}(\mathcal{K})} + I_{\mathcal{F}_{\mathrm{s}}(\mathcal{M})} \otimes a_{\mathcal{K}}(u)^{\#} \tag{2.12}$$

on $\mathcal{F}_{\mathrm{fin}}(\mathcal{M}) \otimes_{\mathrm{alg}} \mathcal{F}_{\mathrm{fin}}(\mathcal{K})$, where $\otimes_{\mathrm{alg}}$ denotes algebraic tensor product. Then there exist operators $W, V \in \mathsf{B}(\mathcal{K})$ and $P, Q \in \mathsf{B}(\mathcal{K}, \mathcal{M})$ such that

$$Su = (Qu, Wu), \quad Tu = (Pu, Vu), \quad u \in \mathcal{K}. \tag{2.13}$$

The operators W and Q (resp. V and P) are uniquely determined by S (resp. T). Hence we have

$$\begin{aligned} b(u) &= a_{\mathcal{M}}(Qu) \otimes I_{\mathcal{F}_{\mathrm{s}}(\mathcal{K})} + I_{\mathcal{F}_{\mathrm{s}}(\mathcal{M})} \otimes a_{\mathcal{K}}(Wu) \\ &\quad + a_{\mathcal{M}}(P_c\bar{u})^* \otimes I_{\mathcal{F}_{\mathrm{s}}(\mathcal{K})} + I_{\mathcal{F}_{\mathrm{s}}(\mathcal{M})} \otimes a_{\mathcal{K}}(V_c\bar{u})^* \end{aligned} \tag{2.14}$$

on $\mathcal{F}_{\mathrm{fin}}(\mathcal{M}) \otimes_{\mathrm{alg}} \mathcal{F}_{\mathrm{fin}}(\mathcal{K})$. This is the original form of operators of the type $b(u)$ [8].

Remark 2.5 The triple $\{\mathcal{F}_s(\mathcal{H}), \mathcal{F}_{\rm fin}(\mathcal{H}), \{a_{\mathcal{H}}(0,u)|u \in \mathcal{K}\}\}$ is a representation of CCR($\mathcal{K}$). But this representation is not equivalent in general to the representation π_b (see Theorem 4.4 in §4 below).

Remark 2.6 The mapping $a_{\mathcal{H}}(0, \cdot) \to b(\cdot)$ may be regarded as a Bogoliubov transformation in the Fock space $\mathcal{F}_s(\mathcal{H})$. But this is a *different* type of Bogoliubov transformations from the usual ones as discussed in, e.g., [15], [25, 26].

Under additional conditions, one can express $a_{\mathcal{H}}(\cdot)$ in terms of $b(\cdot)$ and $b(\cdot)^*$:

Proposition 2.7 *Suppose that S and T satisfy, in addition to (2.7),*

$$SS^* - T_cT_c^* = I_{\mathcal{H}}, \quad T_cS_c^* - ST^* = 0. \tag{2.15}$$

Then, for all $f \in \mathcal{H}$,

$$a_{\mathcal{H}}(f) = b(S^*f) - b(T^*\bar{f})^*, \quad a_{\mathcal{H}}(f)^* = b(S^*f)^* - b(T^*\bar{f}). \tag{2.16}$$

on $D(N_b^{1/2})$.

Let

$$\phi_{\mathcal{H}}(f) := \frac{1}{\sqrt{2}}(a_{\mathcal{H}}(f) + a_{\mathcal{H}}(f)^*), \quad f \in \mathcal{H}, \tag{2.17}$$

which are called the Segal field operators and essentially self-adjoint on $\mathcal{F}_{\rm fin}(\mathcal{H})$[24, Theorem X.41]. We denote the closure of $\phi_{\mathcal{H}}(f)$ by $\overline{\phi_{\mathcal{H}}(f)}$.

An analogue of the Segal field operator is defined in the representation π_b:

$$\Phi(u) := \frac{1}{\sqrt{2}}(b(u) + b(u)^*), \quad u \in \mathcal{K}. \tag{2.18}$$

It can be proved [10] that $\Phi(u)$ is essentially self-adjoint on $\mathcal{F}_{\rm fin}(\mathcal{H})$ and

$$\overline{\Phi(u)} = \overline{\phi_{\mathcal{H}}(Su + T_c\bar{u})}, \quad u \in \mathcal{K}. \tag{2.19}$$

We set

$$C^\infty(N_b) := \cap_{k=1}^\infty D(N_b^k). \tag{2.20}$$

Then, for all $f \in \mathcal{H}$, $a_{\mathcal{H}}(f)^\#$ leaves $C^\infty(N_b)$ invariant and so does $b(u)^\#$ for all $u \in \mathcal{K}$.

We denote by $\mathcal{I}_2(\mathcal{K}, \mathcal{H})$ the space of Hilbert-Schmidt operators from $\mathcal{K}$ to $\mathcal{H}$.

Definition 2.8 Let $S, T \in \mathsf{B}(\mathcal{K}, \mathcal{H})$. We say that the pair $\langle S, T\rangle$ is in the set $\mathsf{S}(\mathcal{K}, \mathcal{H})$ if S and T satisfy (2.7), (2.15) and $T \in \mathcal{I}_2(\mathcal{K}, \mathcal{H})$.

The fundamental properties of the representation π_b are summarized in the following theorem.

Theorem 2.9 [10, Theorem 2.5]. *Let $\langle S, T\rangle \in \mathsf{S}(\mathcal{K}, \mathcal{H})$. Then there exist a unit vector $\Psi_0 \in \mathcal{F}_s(\mathcal{H})$ and a unitary transformation $U : \mathcal{F}_s(\mathcal{H}) \to \mathcal{F}_s(\mathcal{K})$ such that the following* (a)–(d) *hold:*

(a) $\Psi_0 \in C^\infty(N_b)$ *and, for all* $u \in \mathcal{K}$, $b(u)\Psi_0 = 0$.

(b) *The subspace* $\mathcal{L}\{\Psi_0, b(u_1)^* \cdots b(u_n)^*\Psi_0 | n \geq 1, u_j \in \mathcal{K}, j = 1, \cdots, n\}$ *is dense in* $\mathcal{F}_s(\mathcal{H})$.

(c) $U\Psi_0 = \Omega_\mathcal{K}$ *and* $Ub(u_1)^* \cdots b(u_n)^*\Psi_0 = a_\mathcal{K}(u_1)^* \cdots a_\mathcal{K}(u_n)^*\Omega_\mathcal{K}$ *for all* $n \geq 1$, $u_j \in \mathcal{K}$, $j = 1, \cdots, n$.

(d) *For all* $u \in \mathcal{K}$,

$$U\overline{\Phi(u)}U^{-1} = \overline{\phi_\mathcal{K}(u)}, \quad Ub(u)^*U^{-1} = a_\mathcal{K}(u)^*.$$

Moreover, Ψ_0 *is the only one (up to scalar multiples) of vectors* Ψ *such that* $\Psi \in D(N_b^{1/2})$ *and* $b(u)\Psi = 0$ *for all* $u \in \mathcal{K}$.

3 Construction of a Hamiltonian

By using the representation π_b given by (2.11), we can construct a self-adjoint Hamiltonian acting in $\mathcal{F}_s(\mathcal{H})$ whose spectrum can be exactly identified. In application to perturbation problem of embedded eigenvalues in quantum systems of quantum particles interacting with quantum fields, this class of Hamiltonians gives a class of exactly solvable models [7, 8].

For every $K \in \mathcal{I}_2(\mathcal{H}) := \mathcal{I}_2(\mathcal{H}, \mathcal{H})$, there exist (not necessarily complete) orthonormal sets $\{\psi_n\}_{n=1}^M$ and $\{\phi_n\}_{n=1}^M$ in $\mathcal{H}$ (M may be finite or infinite) and positive real numbers $\{\lambda_n\}_{n=1}^M$ such that $\sum_{n=1}^M \lambda_n^2 < \infty$,

$$K = \sum_{n=1}^{M} \lambda_n(\psi_n, \ \cdot \ )\phi_n, \tag{3.1}$$

where, in the case $M = \infty$, the sum in (3.1) converges in operator norm (e.g., [23, Theorem VI.17, Theorem VI.22]). We define for a finite positive integer N

$$\langle a_\mathcal{H}^* | K_N | a_\mathcal{H}^* \rangle = \sum_{n=1}^{\min\{M,N\}} \lambda_n a_\mathcal{H}(\bar{\psi}_n)^* a_\mathcal{H}(\phi_n)^* \tag{3.2}$$

and

$$\langle a_\mathcal{H} | K_N | a_\mathcal{H} \rangle = \sum_{n=1}^{\min\{M,N\}} \lambda_n a_\mathcal{H}(\psi_n) a_\mathcal{H}(\bar{\phi}_n). \tag{3.3}$$

Then we can show that, for all $\Psi \in \mathcal{F}_{\text{fin}}(\mathcal{H})$, the strong limits

$$\langle a_\mathcal{H}^* | K | a_\mathcal{H}^* \rangle \Psi := \text{s-}\lim_{N\to\infty} \langle a_\mathcal{H}^* | K_N | a_\mathcal{H}^* \rangle \Psi \tag{3.4}$$

and

$$\langle a_\mathcal{H} | K | a_\mathcal{H} \rangle \Psi := \text{s-}\lim_{N\to\infty} \langle a_\mathcal{H} | K_N | a_\mathcal{H} \rangle \Psi \tag{3.5}$$

exist. Moreover, the operator $\langle a_\mathcal{H}^\# | K | a_\mathcal{H}^\# \rangle$ defined on $\mathcal{F}_{\text{fin}}(\mathcal{H})$ is closable and

$$\langle a_\mathcal{H}^* | K | a_\mathcal{H}^* \rangle^* = \langle a_\mathcal{H} | K^* | a_\mathcal{H} \rangle \tag{3.6}$$

on $\mathcal{F}_{\text{fin}}(\mathcal{H})$. We denote the closure of $\langle a^{\#}_{\mathcal{H}}|K|a^{\#}_{\mathcal{H}}\rangle$ by the same symbol.

For a densely defined closed linear operator A on $\mathcal{H}$, we denote by $d\Gamma_{\mathcal{H}}(A)$ the second quantization operator on $\mathcal{F}_{\text{s}}(\mathcal{H})$ [23, p.302, Example 2], which is the closed linear operator on $\mathcal{F}_{\text{s}}(\mathcal{H})$ such that $d\Gamma_{\mathcal{H}}(A)\Omega_{\mathcal{H}} = 0$ and

$$d\Gamma_{\mathcal{H}}(A)a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}} = \sum_{j=1}^{n} a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(Af_j)^* \cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}},$$

for all $f_1, \cdots, f_n \in D(A)$ and $n \geq 1$.

Let $\langle S,\, T\rangle \in \mathsf{S}(\mathcal{K}, \mathcal{H})$ and h be a nonnegative self-adjoint operator on $\mathcal{K}$ such that $h = h_c$ and the following properties (h.1)–(h.3) hold:

(h.1) The subspace $\mathcal{H}_0 := \{f \in \mathcal{H} | S^*f, T_c^*f \in D(h)\}$ is dense in $\mathcal{H}$.

(h.2) ThS^* and $Th^{1/2}$ respectively define a Hilbert-Schmidt operator on $\mathcal{H}$ and from $\mathcal{K}$ to $\mathcal{H}$.

(h.3) The subspace $D_S(h) := \{u \in D(h) | S^*Su \in D(h)\}$ is a core of h.

It follows that $ShS^* + T_chT_c^*$ is densely defined, hence a symmetric operator on $\mathcal{H}$ and $D(ShT^*)$ is dense and defines a Hilbert-Schmidt operator on $\mathcal{H}$.

We define

$$H := d\Gamma_{\mathcal{H}}(\overline{ShS^* + T_chT_c^*}) + \langle a_{\mathcal{H}}|\overline{ThS^*}|a_{\mathcal{H}}\rangle + \langle a_{\mathcal{H}}|\overline{ThS^*}|a_{\mathcal{H}}\rangle^*, \tag{3.7}$$

and set

$$E := -\|Th^{1/2}\|_{\text{HS}}^2, \tag{3.8}$$

where $\|\cdot\|_{\text{HS}}$ denotes Hilbert-Schmidt norm. The operator H gives an abstract form unifying Hamiltonians of models of a quantum harmonic oscillator coupled to a quantized field [2]–[9] (see the Introduction).

Let

$$\mathcal{F}_{\text{fin}}(\mathcal{H}_0) = \mathcal{L}\{\Omega_{\mathcal{H}},\, a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}} |\, n \geq 1, f_j \in \mathcal{H}_0,\, j = 1, \cdots, n\} \tag{3.9}$$

Obviously $\mathcal{F}_{\text{fin}}(\mathcal{H}_0) \subset D(H)$. Hence H is a symmetric operator. We can prove the following fact.

Theorem 3.1 [10, Theorem 3.1]. *The operator H is essentially self-adjoint on $\mathcal{F}_{\text{fin}}(\mathcal{H}_0)$ and its closure $\bar{H}$ is unitarily equivalent to $d\Gamma_{\mathcal{K}}(h) + E$ under the unitary transformation U given in Theorem 2.9: $U\bar{H}U^{-1} = d\Gamma_{\mathcal{K}}(h) + E$.*

As a corollary to Theorem 3.1, we can identify the spectrum of $\bar{H}$:

Corollary 3.2

$$\begin{aligned} \sigma(\bar{H}) &= \sigma(d\Gamma_{\mathcal{K}}(h) + E), \quad \sigma_{\text{ac}}(\bar{H}) = \sigma_{\text{ac}}(d\Gamma_{\mathcal{K}}(h) + E), \\ \sigma_{\text{s}}(\bar{H}) &= \sigma_{\text{s}}(d\Gamma_{\mathcal{K}}(h) + E), \quad \sigma_{\text{p}}(\bar{H}) = \sigma_{\text{p}}(d\Gamma_{\mathcal{K}}(h) + E), \end{aligned}$$

where σ_{s} and σ_{ac} denote singular continuous spectrum and absolutely continuous spectrum respectively. The multiplicity of each eigenvalue of $\bar{H}$ is the same as that of the corresponding one of $d\Gamma_{\mathcal{K}}(h) + E$. In particular, $\bar{H}$ has a unique ground state given by the vector Ψ_0 (up to constant multiples) with the ground state energy E.

In concrete models, the unperturbed Hamiltonian H_0 is of the form

$$H_0 = d\Gamma_{\mathcal{H}}(\ell \oplus h) = d\Gamma_{\mathcal{M}}(\ell) \otimes I_{\mathcal{F}_s(\mathcal{K})} + I_{\mathcal{F}_s(\mathcal{M})} \otimes d\Gamma_{\mathcal{K}}(h), \tag{3.10}$$

where ℓ is a self-adjoint operator on $\mathcal{M}$ bounded from below (see the examples given in the Introduction). We write

$$H = H_0 + H_I \tag{3.11}$$

with

$$H_I = d\Gamma_{\mathcal{H}}(\overline{ShS^* + T_c h T_c^*}) - d\Gamma_{\mathcal{H}}(\ell \oplus h) + \langle a_{\mathcal{H}} | \overline{ThS^*} | a_{\mathcal{H}} \rangle + \langle a_{\mathcal{H}} | \overline{ThS^*} | a_{\mathcal{H}} \rangle^*. \tag{3.12}$$

For this form of H, Corollary 3.2 implies the following. For simplicity, consider the case where $\sigma(h)$ is purely continuous as is given by (1.6) and $\sigma(\ell)$ is purely discrete so that

$$\sigma(d\Gamma_{\mathcal{M}}(\ell)) = \sigma_{\mathrm{p}}(d\Gamma_{\mathcal{M}}(\ell)) = \{E_n\}_{n=0}^{\infty}$$

with $E_0 < E_1 < E_2 < \cdots$ (E_n is determined by $\sigma(\ell)$). Then we have (1.7) and hence (1.8). Thus each E_n is an eigenvalue of H_0 and the eigenvalues $E_n \geq E_0 + M$ are embedded in the continuous spectrum of H_0. On the other hand, Corollary 3.2 implies that

$$\sigma(\bar{H}) = \{E\} \cup [E + M, \infty), \quad \sigma_{\mathrm{p}}(\bar{H}) = \{E\}.$$

Hence all the embedded eigenvalues $E_n \geq E_0 + M$ turn out to disappear under the perturbation H_I, i.e., they are unstable under the perturbation H_I (we may regard $E_n < E_0 + M$ as eigenvalues changing to E or $E + M$ under the perturbation H_I). Thus $\bar{H}$ gives, in an abstract from, a class of self-adjoint operators acting in the Fock space $\mathcal{F}_s(\mathcal{H})$, which describe the instability phenomenon of embedded eigenvalues.

4 Structure of the representation π_b

We write each vector $f \in \mathcal{H}$ as

$$f = (f_{\mathcal{M}}, f_{\mathcal{K}}), \quad f_{\mathcal{M}} \in \mathcal{M}, f_{\mathcal{K}} \in \mathcal{K}.$$

For $A \in \mathsf{B}(\mathcal{K}, \mathcal{H})$, we define $\tilde{A} \in \mathsf{B}(\mathcal{H})$ by

$$\tilde{A} f := A f_{\mathcal{K}}, \quad f \in \mathcal{H}. \tag{4.1}$$

Then we have

$$\tilde{A}^* f = (0, A^* f), \quad f \in \mathcal{H}. \tag{4.2}$$

It is easy to show that, for all $A, B \in \mathsf{B}(\mathcal{K}, \mathcal{H})$,

$$\tilde{A}\tilde{B}^* = AB^*, \quad \tilde{B}^*\tilde{A} f = (0, B^* A f_{\mathcal{K}}), f \in \mathcal{H}. \tag{4.3}$$

Let $\langle S, T \rangle \in \mathsf{S}(\mathcal{K}, \mathcal{H})$ and $P_{\mathcal{K}}$ be the orthognal projection from $\mathcal{H}$ onto $\mathcal{K}$. Then we have

$$\tilde{S}^*\tilde{S} - \tilde{T}^*\tilde{T} = P_{\mathcal{K}}, \quad \tilde{S}^*\tilde{T}_c - \tilde{T}^*\tilde{S}_c = 0, \tag{4.4}$$

$$\tilde{S}\tilde{S}^* - \tilde{T}_c\tilde{T}_c^* = I_{\mathcal{H}}, \quad \tilde{T}_c\tilde{S}_c^* - \tilde{S}\tilde{T}^* = 0. \tag{4.5}$$

Let $L \in \mathsf{B}(\mathcal{H})$ be such that

$$L^*L = P_{\mathcal{K}}, \quad LL^* = I_{\mathcal{H}}. \tag{4.6}$$

Then L is a partial isometry on $\mathcal{H}$ with initial space $\mathcal{K}$ and final space $\mathcal{H}$.
We define $X, Y \in \mathsf{B}(\mathcal{H})$ by

$$X := \tilde{S}L^*, \quad Y := \tilde{T}L^*.$$

Then one can prove the following fact.

Lemma 4.1 [10, Lemma 4.1]. *The following relations hold:*

$$X^*X - Y^*Y = I_{\mathcal{H}}, \qquad X^*Y_c - Y^*X_c = 0, \tag{4.7}$$
$$XX^* - Y_cY_c^* = I_{\mathcal{H}}, \qquad Y_cX_c^* - XY^* = 0. \tag{4.8}$$

Moreover, $Y \in \mathcal{I}_2(\mathcal{H})$.

For each $f \in \mathcal{H}$, we define an opeartor $c(f)$ by

$$c(f) := a_{\mathcal{H}}(Xf) + a_{\mathcal{H}}(Y_c\bar{f})^* \tag{4.9}$$

with $D(c(f)) = D(N_{\mathrm{b}}^{1/2})$ Then $c(f)$ is closable. We denote its closure by the same symbol.

Theorem 4.2 [10, Theorem 4.2]. *The mapping* $\{a_{\mathcal{H}}, a_{\mathcal{H}}^*\} \to \{c, c^*\}$ *is a proper Bogoliubov (canonical) transformation on* $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$*, i.e., there exists a unitary operator* $U_{\mathcal{H}}$ *on* $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$ *such that, for all* $f \in \mathcal{H}$*,*

$$c(f) = U_{\mathcal{H}}a_{\mathcal{H}}(f)U_{\mathcal{H}}^{-1}, \quad c(f)^* = U_{\mathcal{H}}a_{\mathcal{H}}(f)^*U_{\mathcal{H}}^{-1}$$

As a corollary to Theorem 4.2, we have the following.

Corollary 4.3 *For all* $u \in \mathcal{K}$*,*

$$b(u) = U_{\mathcal{H}}a_{\mathcal{H}}(L(0,u))U_{\mathcal{H}}^{-1}, \quad b(u)^* = U_{\mathcal{H}}a_{\mathcal{H}}(L(0,u))^*U_{\mathcal{H}}^{-1}. \tag{4.10}$$

We next consider expressing $a_{\mathcal{H}}(L(0,\cdot))$ as a transformation of $a_{\mathcal{H}}(0,\cdot)$.
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $C \in \mathsf{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction operator, i.e., $\|C\| \leq 1$. Then we can define a contraction operator $\Gamma_{\mathcal{H}_1,\mathcal{H}_2}(C) : \mathcal{F}_{\mathrm{s}}(\mathcal{H}_1) \to \mathcal{F}_{\mathrm{s}}(\mathcal{H}_2)$ by

$$\Gamma_{\mathcal{H}_1,\mathcal{H}_2}(C) := \oplus_{n=0}^{\infty}(\otimes^n C) \tag{4.11}$$

with $\otimes^0 C := 1$, where $\otimes^n C$ denotes the n-fold tensor product of C.
In the case where C is a contraction operator on a single Hilbert space $\mathcal{H}_1$, we set

$$\Gamma_{\mathcal{H}_1}(C) := \Gamma_{\mathcal{H}_1,\mathcal{H}_1}(C). \tag{4.12}$$

We have

$$\Gamma_{\mathcal{H}}(L)\Gamma_{\mathcal{H}}(L)^* = I_{\mathcal{F}_{\mathrm{s}}(\mathcal{H})}, \quad \Gamma_{\mathcal{H}}(L)^*\Gamma_{\mathcal{H}}(L) = \Gamma_{\mathcal{H}}(P_{\mathcal{K}}). \tag{4.13}$$

It is easy to see that $\Gamma_{\mathcal{H}}(P_{\mathcal{K}})$ is the orthogonal projection onto the closed subspace $\mathcal{F}_s(\{0\}\oplus \mathcal{K}) = \mathbf{C}\otimes\mathcal{F}_s(\mathcal{K})$. Hence $\Gamma_{\mathcal{H}}(L)$ is a partial isometry on $\mathcal{F}_s(\mathcal{H})$. Let

$$V_{\mathcal{H}} := U_{\mathcal{H}}\Gamma_{\mathcal{H}}(L). \tag{4.14}$$

Then

$$V_{\mathcal{H}}V_{\mathcal{H}}^* = I_{\mathcal{F}_s(\mathcal{H})}, \quad V_{\mathcal{H}}^*V_{\mathcal{H}} = \Gamma_{\mathcal{H}}(P_{\mathcal{K}}), \tag{4.15}$$

which imply that $V_{\mathcal{H}}$ is a partial isometry on $\mathcal{F}_s(\mathcal{H})$ with initial space $\mathbf{C}\otimes\mathcal{F}_s(\mathcal{K})$ and final space $\mathcal{F}_s(\mathcal{H})$. We can prove the following fact..

Theorem 4.4 [10, Corollary 4.5]. *For all $u\in\mathcal{K}$,*

$$b(u) = V_{\mathcal{H}}a_{\mathcal{H}}(0,u)V_{\mathcal{H}}^*, \quad b(u)^* = V_{\mathcal{H}}a_{\mathcal{H}}(0,u)^*V_{\mathcal{H}}^*. \tag{4.16}$$

This theorem shows that the representation π_b is a transformation of the representation $\{a_{\mathcal{H}}(0,u)|u\in\mathcal{K}\}$ by the *partial isometry* $V_{\mathcal{H}}$, which is a composition of the partial isometry $\Gamma_{\mathcal{H}}(L)$ and the proper Bogoliubov transformation $U_{\mathcal{H}}$.

References

[1] Arai A, Self-adjointness and spectrum of Hamiltonians in nonrelativistic quantum electrodynamics, *J. Math. Phys.* **22**(1981), 534-537.

[2] Arai A, On a model of a harmonic oscillator coupled to a quantized, massless, scalar field. I, *J. Math. Phys.* **22**(1981), 2539-2548.

[3] Arai A., On a model of a harmonic oscillator coupled to a quantized, massless, scalar field. II, *J. Math. Phys.* **22**(1981), 2549-2552.

[4] Arai A., Rigorous theory of spectra and radiation for a model in quantum electrodynamics, *J. Math. Phys.* **24**(1983), 1896-1910.

[5] Arai A., Spectral analysis of a quantum harmonic oscillator coupled to infinitely many scalar bosons, *J. Math. Anal. Appl.* **140**(1989), 270-288.

[6] Arai A., Long-time behavior of two-point functions of a quantum harmonic oscillator interacting with bosons, *J. Math. Phys.* **30**(1989), 1277–1288.

[7] Arai A., Perturbation of embedded eigenvalues : a general class of exactly soluble models in Fock spaces, *Hokkaido Math. Jour.* **19**(1990), 1–34.

[8] Arai A., Noninvertible Bogoliubov transformations and instability of embedded eigenvalues, *J. Math. Phys.* **32**(1991), 1838–1846.

[9] Arai A., Long-time behavior of an electron interacting with a quantized radiation field, *J. Math. Phys.* **32**(1991), 2224–2242.

[10] Arai A., A class of representations of the $*$-algebra of the canonical commutation relations over a Hilbert space and instability of embedded eigenvalues in quantum field models, *J. Nonlinear Math. Phys.* **4**(1997), 338–349.

[11] Arai A. and Hirokawa M., On the spin-boson model, *RIMS Kokyuroku* No. 957(1996), 16–35.

[12] Arai A. and Hirokawa M., On the existence and uniqueness of ground states of a generalized spin-boson model, to be published in *J. Funct. Anal.*

[13] Bach V., Fröhlich J. and Sigal I., Mathematical theory of nonrelativistic matter and radiation, *Lett. Math. Phys.* **34**(1995), 183-201.

[14] Bach V., Fröhlich J. and Sigal I., Quantum Electrodynamics of confined non-relativistic particles, 1996, Preprint.

[15] Berezin F. A., *The Method of Second Quantization*, Academic Press, 1966.

[16] Bratteli, O. and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics 2*, Second Edition, Springer, Berlin, Heidelberg, 1997

[17] Healy, W. P., *Non-relativistic Quantum Electrodynamics*, Academic Press, New York, 1982.

[18] Hübner M. and Spohn H., Spectral properties of the spin-boson Hamiltonian, *Ann. Inst. H. Poincaré* **62**(1995), 289–323.

[19] Hübner M. and Spohn H., Radiative decay: nonperturbative approach, *Rev. Math. Phys.* **7**(1995), 363–387.

[20] V. Jakšić and C. A. Pillet, On a model for quantum friction. I. Fermi's golden rule and dynamics at zero temperature, *Ann. Inst. H. Poincaré* **62**(1995), 47–68.

[21] V. Jakšić and C. A. Pillet, On a model for quantum friction. II. Fermi's golden rule and dynamics at positive temperature, *Commun. Math. Phys.* **176**(1996), 619–644.

[22] Okamoto T. and Yajima K, Complex scaling technique in non-relativistic massive QED, *Ann. Inst. H. Poincaré* **42**(1985), 311–327.

[23] Reed M. and Simon B., *Methods of Modern Mathematical Physics Vol.I*, Academic Press, New York, 1972.

[24] Reed M. and Simon B., *Methods of Modern Mathematical Physics Vol.II*, Academic Press, New York, 1974.

[25] Ruijsenaars S. N. M., On Bogoliubov transformations for systems of relativistic charged particles, *J. Math. Phys.* **18**(1977), 517–526.

[26] Ruijsenaars S. N. M., On Bogoliubov transformations II. The general case, *Ann. Phys.* (NY) **116**(1978), 105–134.

[27] Spohn H., Asymptotic completeness for Rayleigh scattering, *J. Math. Phys.* **38**(1997), 2281–2296.