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A PRIORI ESTIMATES AND EXISTENCE THEOREMS FOR THE LINDBLAD EQUATION WITH UNBOUNDED TIME-DEPENDENT COEFFICIENTS

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ABSTRACT. We prove new *a priori* estimates for the resolvent of a class of minimal Quantum Dynamical Semigroup (QDS). These estimates simplify the proof of the unital property of QDS and suggest a continuity condition for time-dependent infinitesimal generators to ensure existence and conservativity of the Markov master evolution equation.

§1. INTRODUCTION.

The theory of the Markov master equation has been intensively studied during the recent years [1-10]. Important applications of this theory were contributed to quantum chemistry [3] and quantum optics [4–6]. Numerical Monte–Carlo and Runge–Kutta algorithms for these equations are discussed in [7–8]. One of efficient analytical tools in the theory of master equation is the interaction representation technique [6–7], where the Markov master equations with time-dependent coefficients arise in a natural way.

To study Markov master equations with unbounded time-dependent coefficients describing evolution of observables from the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators or states from the algebra $\mathcal{T}(\mathcal{H})$ of operators with finite trace, we introduce the continuity conditions for the generator.

We recall that one parameter contraction semigroup $P_t(\cdot)$ acting in $\mathcal{B}(\mathcal{H})$ is called a *Quantum Dynamical Semigroup* (QDS) if it is completely positive, normal, conservative, and ultraweak continuous [2, 11]. Normal and ultraweak continuity properties mean respectively that l.u.b. $P_t(X_n) = P_t(\text{l.u.b. } X_n)$ and $\mathrm{Tr}\{\rho(P_t(B)-B)\} \to 0$ as $t \to 0$ for any $\rho \in \mathcal{T}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. In the Heisenberg representation, the conservative (or unital) property means the conservation of the unit operator I in the algebra $\mathcal{B}(\mathcal{H})$ of observables: $P_t(I) \equiv I\ \forall t \geq 0$; in the Schrödinger representation, it means the conservation of the trace of an initial state $\rho \in \mathcal{T}(\mathcal{H})$ during the evolution: $\mathrm{Tr}\, T_t(\rho) \equiv \mathrm{Tr}\, \rho\ t \geq 0$, where $T_t\colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ and $P_t\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are dual semigroups: $\mathrm{Tr}\{T_t(\rho)B\} = \mathrm{Tr}\{\rho P_t(B)\}$. The semigroup $T_t = P_t^\dagger$ is called the *predual* of the semigroup P_t. The dagger "†" is used to denote predual operators and maps. The map $P(\cdot)$ with the predual $P^\dagger(\cdot)$ is referred to as a *completely positive* (*CP*) if

$$\sum_{i,j} \mathrm{Tr}\{P(B_i^* B_j)\sigma_j\sigma_i^*\} \geq 0 \quad \text{or equivalently} \quad \sum_{i,j} \mathrm{Tr}\{B_i^* B_j P^\dagger(\sigma_j\sigma_i^*)\} \geq 0 \tag{1.1}$$

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for any finite sequences $\{B_j\} \in \mathcal{B}(\mathcal{H})$, $\{\sigma_j\} \in \mathcal{T}_2(\mathcal{H})$, where $\mathcal{T}_2(\mathcal{H})$ is the Banach algebra of Hilbert–Schmidt operators. For a bounded *CP*-map $P(\cdot)$, this definition is clearly equivalent to the standard definition of the cone $CP(\mathcal{H})$ [11]. Indeed, the substitution $\sigma_j = |\psi_j\rangle\langle\varphi|$, $\psi_j, \varphi \in \mathcal{H}$, $||\varphi|| = 1$ reduces (1.1) to the standard definition of *CP*-map: $\sum(\psi_i, P(B_i^* B_j)\psi_j) \geq 0$. On the other hand, the standard definition of *CP*-property and the definition of a trace implies the identity

$$\sum_{i,j} \mathrm{Tr}\{P_t(B_i^* B_j)\sigma_j\sigma_i^*\} = \sum_k \sum_{i,j} (\psi_{i,k}, P_t(B_i^* B_j)\psi_{j,k}) \geq 0, \quad \psi_{j,k} = \sigma_j h_k$$

for any complete orthonormal system $\{h_k\}$ in a separable Hilbert space $\mathcal{H}$.

Under natural assumptions on the operator-valued coefficients of the formal generator $\mathcal{L}(\,\cdot\,)$, it was shown in [2, 12–15, 18] that there exists a *minimal* QDS $P_t^{\min}(\,\cdot\,) = \exp\{t\mathcal{L}(\,\cdot\,)\}$ defined by the Dyson series; the conservativity of this QD-S is necessary and sufficient for the nonexistence of any other conservative QDS with the same formal generator [12, 13, 16, 18]. Therefore, the generator of the semigroup is called *regular* if the minimal QDS is conservative (unital).

In the present paper we show that it is possible, by an appropriate choice of continuity conditions, to guarantee the existence and conservativity of the minimal solution of the Cauchy problem for the Markov evolution equation (also known as the *Lindblad equation*) with time-dependent coefficients

$$\partial_t P_t(B) = \mathcal{L}_t(P_t(B)), \qquad P_t(B)\big|_{t=0} = B,$$

corresponding to the formal infinitesimal operator $\mathcal{L}_t(\,\cdot\,)$:

$$\mathcal{L}_t(B) = \Phi_t(B) - \Phi_t(I) \circ B + i[H_t, B], \qquad t \in \mathbb{R}_+$$

where H_t is a family of symmetric operators, and $\Phi_t(\,\cdot\,)$ is a time-dependent family of completely positive maps.

Consider the scale of Hilbert spaces $\dots \subseteq \mathcal{H}_n \subseteq \mathcal{H}_{n-1} \subseteq \dots$ with the inner product $(\psi, \varphi)_n = (\Lambda^{n/2}\psi, \Lambda^{n/2}\varphi)$ and the Banach scale of trace-class operator algebras $\dots \subseteq \mathcal{T}_{\Lambda^n}(\mathcal{H}) \subseteq \mathcal{T}_{\Lambda^{n-1}}(\mathcal{H}) \subseteq \dots$ generated by a positive invertible self-adjoint "reference" operator $\Lambda : \mathcal{T}_{\Lambda^n}(\mathcal{H}) \to \mathcal{T}_{\Lambda^{n-2}}(\mathcal{H})$:

$$\mathcal{T}_{\Lambda^n}(\mathcal{H}) = \{\rho : \ \rho = \Lambda^{-n/2}\sigma\Lambda^{-n/2}, \ \sigma \in \mathcal{T}(\mathcal{H}), \ ||\rho||_{\mathcal{T}_{\Lambda^n}(\mathcal{H})} = ||\sigma||_{\mathcal{T}(\mathcal{H})}\}.$$

By $\mathcal{B}_{\Lambda^n}(\mathcal{H})$ we denote the completion of $\mathcal{B}(\mathcal{H})$ with respect to weak* topology generated by duality between $\mathcal{B}(\mathcal{H})$ and $\mathcal{T}_{\Lambda^n}(\mathcal{H})$. Thus, if $\rho \in \mathcal{T}_{\Lambda^n}(\mathcal{H})$ then $\rho : \mathcal{H}_{-n} \to \mathcal{H}_n$ is a bounded operator and $\Lambda \in \mathcal{B}_\Lambda(\mathcal{H})$.

The article consists of six sections. In §2, we describe a class $CPn(\mathcal{B}_\Lambda)$ of unbounded completely positive normal maps $\Phi(\cdot)$ with the predual $\Phi^\dagger(\cdot) : \mathcal{T}_\Lambda(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ which enters the generator of the Lindblad equation. We discuss in §2 properties of the trace-form $\mathrm{Tr}_*\{\Phi(B)\rho\}$ which are similar to properties of a quadratic form, and prove the characteristic property for these maps:

$$\lim_{\varepsilon \to 0} \mathrm{Tr}\{\Phi^{(\varepsilon)}(B)\rho\} = \mathrm{Tr}\{\widetilde{\Phi}(B)\sigma\} = \mathrm{Tr}_*\{\Phi(B)\rho\},$$

where $\Phi^{(\varepsilon)}(B)$ is the sequence of bounded normal completely positive operators which converges with respect to a special locally convex topology to the operator $\Phi(B)$ on $\operatorname{dom}\Lambda$, $\tilde{\Phi}(B) = \Lambda^{-1/2}\Phi(B)\Lambda^{-1/2}$, $\Phi(B) \in \mathcal{B}_\Lambda(\mathcal{H})$, and $\rho \in \mathcal{T}_\Lambda(\mathcal{H})$, $\rho = \Lambda^{-1/2}\sigma\Lambda^{-1/2}$, $\sigma \in \mathcal{T}(\mathcal{H})$. This notion plays an important role in the present paper because the set of pure states, which enter as arguments of sesquilinear forms, is not invariant under irreversible quantum evolution of these forms.

The main analytical assumptions on coefficients of $\mathcal{L}_t(\cdot)$ are introduced in §3. We describe there an algebraic background of the theory based on properties of minimal solutions of the resolvent equation [17, 19, 20]. In §3 and §4, we discuss *a priori* estimates for the minimal solutions of the homogeneous and nonhomogeneous Lindblad equations.

The important observation is that the spaces $\mathcal{T}_\Lambda(\mathcal{H})$ and $\mathcal{B}_\Lambda(\mathcal{H})$ are invariant under QDS provided there exist a constant $c \in \mathbb{R}$ and a self-adjoint "reference" operator $\Lambda \geq \Phi(I)$ such that the following relative bound for quadratic forms holds true: $\mathcal{L}(\Lambda)_* \leq c\Lambda_*$. More precise, we assume that

$$\operatorname{Tr}_*\{\Lambda\mathcal{L}^\dagger(\rho)\} \leq c\operatorname{Tr}_*\{\Lambda\rho\} \quad \forall\rho \in \mathcal{T}^+_{\Lambda^2}(\mathcal{H}). \tag{1.2}$$

This bound implies the uniqueness of a solution of the Cauchy problem in the class of dynamical semigroups under additional assumptions on the domain of Λ and on the coefficients of the infinitesimal operator $\mathcal{L}(\,\cdot\,)$. This solution can be constructed as the minimal QSD, and for the corresponding representations we have

$$||T_t^{min}(\rho)||_{\mathcal{T}_\Lambda} \leq e^{ct}||\rho||_{\mathcal{T}_\Lambda}, \quad ||P_t^{min}(X)||_{\mathcal{B}_\Lambda} \leq e^{ct}||X||_{\mathcal{B}_\Lambda}.$$

The *continuity conditions* for the family of CCP-maps $\mathcal{L}(\cdot)$ with the predual $\mathcal{L}_t^\dagger : \mathcal{T}_{\Lambda^2}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ are the following:

$$\mathcal{L}_t^\dagger(\cdot) - \mathcal{L}_s^\dagger(\cdot) \text{ can be extended as a bounded map from } \mathcal{T}_\Lambda(\mathcal{H}) \text{ to } \mathcal{T}(\mathcal{H}), \tag{1.3}$$

$$\lim_{h\to=0}\ \sup_{||Y||\leq I} ||\mathcal{L}_{t+h}(Y) - \mathcal{L}_t(Y)||_{\mathcal{B}_\Lambda} = 0 \quad \forall t \geq 0,\ Y \in \mathcal{B} \tag{1.4}$$

are introduced in §5 to describe a class of infinitesimal operators with time-dependent coefficients as a completion of the set of infinitesimal maps $\mathcal{L}_t(\,\cdot\,)$ with piecewise constant coefficients $\Phi_t(\,\cdot\,)$ and H_t satisfying the sufficient conservativity condition at every point $t \in \mathbb{R}_+$. The typical temporal dependence of coefficients of the generator in the interaction representation. The main result of this article is the derivation of global sufficient conservativity conditions from local sufficient conservativity conditions for equation with time-dependent coefficients. In §6, we consider examples illustrating the main result.

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§2. BASIC DEFINITIONS AND CONSTRUCTION OF UNBOUNDED *CP*-MAPS.

In what follows, the subscript "$*$" is used to denote quadratic, bilinear or trace forms; the arguments of these forms are specified in square brackets or braces. For example, $\Lambda_*[\varphi,\psi] = (\Lambda^{1/2}\varphi, \Lambda^{1/2}\psi)$ is the bilinear form generated by a positive self-adjoint operator Λ, $\operatorname{dom}\Lambda_* = \operatorname{dom}\Lambda^{1/2}$ (see [21, 22]), and $\Lambda_*[\varphi]$ is the corresponding quadratic form. We write $\Phi(I) \leq \Lambda$ if $\operatorname{dom}\Lambda \subseteq \operatorname{dom}\Phi(I)$ and $(\psi, \Phi(I)\psi) \leq (\psi, \Lambda\psi)\ \forall\psi \in \operatorname{dom}\Lambda$. This inequality can be extended by continuity to all $\psi \in \operatorname{dom}\Lambda^{1/2} \subseteq \operatorname{dom}\Phi(I)^{1/2}$: $||\Phi^{1/2}(I)\psi|| \leq ||\Lambda^{1/2}\psi||$. It is also convenient to associate the bilinear form $\Phi(B)_*$ with the *CP*-map $\Phi(B) = \sum A_k^* B A_k$:

$$\Phi(B)_*[\varphi,\psi] = \sum(A_k\varphi, B\,A_k\psi) = \mathrm{Tr}\{B\rho\}, \quad \rho = \sum |A_k\varphi\rangle\langle A_k\psi|.$$

Let Λ be a positive self-adjoint operator in $\mathcal{H}$ such that $I \leq \Phi(I) \leq \Lambda$ and let $\mathcal{H}_\Lambda = \mathcal{H}_1$ be the Hilbert space with the norm $||h||^2_{\mathcal{H}_\Lambda} = \Lambda_*[h]$. Consider the cone $CPn(\mathcal{H}_\Lambda)$ of completely positive normal maps defined in [12] as a completion of the set $CPn(\mathcal{H})$ of completely positive normal bounded maps with respect to the locally convex topology generated by the system of seminorms

$$\sigma_{A,B}(\Phi) = \sup_{X\in B,\psi\in A} \left|\Phi_*(B)[\psi]\right|, \tag{2.1}$$

where A and B are absolutely convex compact subsets of the Hilbert space $\mathcal{H}_\Lambda = \mathcal{H}_1$ and the Banach algebra $\mathcal{B}(\mathcal{H})$ respectively endowed with the strong operator topology. In what follows we assume that the coefficient $\Phi(\cdot)$ of the infinitesimal map $\mathcal{L}(\cdot)$ is an element of $CPn(\mathcal{H}_\Lambda)$. This definition implies the normal property of $\Phi(\cdot)_*[\psi]$ for any $\psi \in \operatorname{dom}\Phi(I)_*$, that is

$$\sup_n \Phi(B_n)_*[\psi] = \Phi(\underset{n}{\text{l.u.b.}}\ B_n)_*[\psi] \quad \forall\psi \in \mathcal{H}_\Lambda \tag{2.2}$$

for any uniformly bounded increasing sequence of operators $B_n \in \mathcal{B}(\mathcal{H})$.

Let us start from "internal" the definition of the algebra $\mathcal{T}_\Lambda(\mathcal{H})$ which does not involve the scale of Hilbert spaces.

Definition 2.1. By $\mathcal{T}_\Lambda(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})$ we denote a completion with respect to the norm $||\cdot||_{\mathcal{T}_\Lambda}$ of the linear span of the cone $\mathcal{T}^+_\Lambda(\mathcal{H})$ of positive operators ρ such that $\sigma = \Lambda^{1/2}\rho^{1/2}$ is a Hilbert–Schmidt operator, i.e. $\operatorname{range}\rho^{1/2} \subseteq \operatorname{dom}\Lambda^{1/2}$ and $\sigma \in \mathcal{T}_2(\mathcal{H})$. We denote by $||\rho||_{\mathcal{T}_\Lambda} = \mathrm{Tr}_*\{\rho\Lambda\} = \mathrm{Tr}\{\sigma^*\sigma\}$ for $\rho \in \mathcal{T}^+_\Lambda$,

$$||\rho||_{\mathcal{T}_\Lambda} = \inf_{\xi\in\mathcal{T}^+_\Lambda,\ \xi-\rho\in\mathcal{T}^+_\Lambda} \mathrm{Tr}_*\{(2\xi-\rho)\Lambda\} \text{ for a Hermitian operator } \rho$$

$||\rho||_{\mathcal{T}_\Lambda} = \frac{1}{2}\left(||\rho+\rho^*||_{\mathcal{T}_\Lambda} + ||\rho-\rho^*||_{\mathcal{T}_\Lambda}\right)$ for arbitrary operator $\rho \in \operatorname{Span}\mathcal{T}^+_\Lambda(\mathcal{H})$. For $\rho \in \mathcal{T}^+_\Lambda(\mathcal{H})$ we set

$$\mathrm{Tr}_*\{\rho\Lambda\} = \inf_{\xi\in\mathcal{T}^+_\Lambda,\ \xi-\rho\in\mathcal{T}^+_\Lambda} \mathrm{Tr}_*\{(\xi+\xi-\rho)\Lambda\} \geq \inf_{\xi-\rho\in\mathcal{T}^+_\Lambda} \mathrm{Tr}_*\{\xi\Lambda\} = \mathrm{Tr}_*\{\rho\Lambda\}$$

and $\rho+\rho^*=2\rho$, $\rho-\rho^*=0$. Hence, the norm is well defined because all three equations take the same value on the cone $\mathcal{T}_\Lambda^+(\mathcal{H})$ of positive operators, and the last two clearly coincide on subspace of the Hermitian operators.

Note that $\mathrm{Tr}\{|\rho|\Lambda\}$ is a well defined candidate to be the norm in subalgebra $\mathcal{T}_\Lambda(\mathcal{H})\subseteq\mathcal{T}(\mathcal{H})$, but the triangle inequality is violated for entangled states ρ.

As an example in the case $\mathcal{H}=C_2$, $\mathcal{B}(\mathcal{H})=M_2$ one can take the entangled state $\rho=\rho_a-\rho_b$, $\rho_a=|a\rangle\langle a|$, $\rho_b=|b\rangle\langle b|$ and any positive 2×2-matrix Λ such that $\langle a,b\rangle\neq 0$, $\langle\Lambda^{1/2}a,\Lambda^{1/2}b\rangle=0$. Straightforward computations give

$$|\rho_a-\rho_b|=\sqrt{1-|\langle a,b\rangle|^2}(\rho_a+\rho_c),\quad c=(b-\langle a,b\rangle a)/||b-\langle a,b\rangle a||;$$

$$\mathrm{Tr}\{|\rho_a-\rho_b|\Lambda\}=\left(||\Lambda^{1/2}a||^2+||\Lambda^{1/2}b||^2\right)/\sqrt{1-|\langle a,b\rangle|^2}$$

provided $\langle\Lambda^{1/2}a,\Lambda^{1/2}b\rangle=0$. Therefore, $\mathrm{Tr}\{|\rho_a-\rho_b|\Lambda\}>||\Lambda^{1/2}a||^2+||\Lambda^{1/2}b||^2=||\rho_a||_{\mathcal{T}_\Lambda}+||\rho_b||_{\mathcal{T}_\Lambda}$.

On the other hand, here is the proof of the triangle inequality for $||\cdot||_{\mathcal{T}_\Lambda}$ for Hermitian operators ρ:

$$\begin{aligned}&\inf_{\xi,\,\xi-\rho_1+\rho_2\in\mathcal{T}_\Lambda^+}\mathrm{Tr}_*\{(2\xi-\rho_1+\rho_2)\Lambda\}\\ \le&\inf_{\xi_1,\xi_2,\,\xi_1+\xi_2-\rho_1+\rho_2\in\mathcal{T}_\Lambda^+}\mathrm{Tr}_*\{(2\xi_1+\xi_2-\rho_1+\rho_2)\Lambda\}\\ =&\inf_{\xi_1,\,\xi_1-\rho_1\in\mathcal{T}_\Lambda^+}\mathrm{Tr}_*\{(2\xi-\rho_1)\Lambda\}+\inf_{\xi_2,\,\xi_2+\rho_2\in\mathcal{T}_\Lambda^+}\mathrm{Tr}_*\{(2\xi_2+\rho_2)\Lambda\}\end{aligned}$$

due to the identity $\inf_{x\in X,\,y\in Y}(f_1(x)+f_2(y))=\inf_{x\in X}f_1(x)+\inf_{y\in Y}f_2(y)$. The triangle inequality for arbitrary states from $\mathcal{T}_\Lambda$ clearly follows from here.

Remark 2.1. For $\rho\in\mathcal{T}_\Lambda$, $\rho=\Lambda^{-1/2}\sigma\Lambda^{-1/2}$, $\sigma\in\mathcal{T}$ the norm $||\rho||_{\mathcal{T}_\Lambda}$ is equivalent to the norm $||\sigma||_{\mathcal{T}}$. More precise, $2^{-1/2}||\rho||_{\mathcal{T}_\Lambda}\le||\sigma||_{\mathcal{T}}\le||\rho||_{\mathcal{T}_\Lambda}$, and $||\rho||_{\mathcal{T}_\Lambda}=||\sigma||_{\mathcal{T}}$ for Hermitian operators ρ. Thus, both norms describe the same topology. If ρ is a pure state $\rho=|\varphi\rangle\langle\varphi|$, $\varphi\in\mathrm{dom}\,\Lambda^{1/2}$, then clearly, $\mathrm{Tr}_*\{\rho\Lambda\}=||\Lambda^{1/2}\varphi||^2=\Lambda_*[\varphi]$.

Definition 2.1 will be used to extend the construction of a quadratic form, defined originally on the set of pure states, to the algebra of trace class operators. This generalization is important when the argument of the quadratic form evolves in time and the evolution does not preserve the set of pure states.

If both operators $\rho\Lambda$ and $\sigma^*\sigma$ are well defined, then the trace $\mathrm{Tr}\{\rho\Lambda\}$ and the trace-form coincide: $\mathrm{Tr}_*\{\rho\Lambda\}=\mathrm{Tr}\{\sigma^*\sigma\}$. But in the general case $\mathrm{dom}\,\rho\Lambda\subseteq\mathrm{dom}\,\sigma^*\sigma$, and the operator $\rho\Lambda$ may be ill-defined. For example, set $\mathcal{H}=\ell_2$, $(\Lambda\psi)_n=n\psi_n$, $\rho=|r\rangle\langle r|$, $r=(r_1,r_2,\ldots)$, and $r_n=n^{-1-\delta}/||r||$, $||r||=\left(\sum_k k^{-1-\delta}\right)^{1/2}$, $\delta>0$. Then ρ is a projection and $\rho^{1/2}=\rho$. Hence, $\{\rho\Lambda\psi\}_n=n^{-1-\delta}\sum_k k^{-\delta}\psi_k/||r||$ and $\{\Lambda^{1/2}\rho^{1/2}\psi\}_n=n^{-1/2-\delta}(r,\psi)/||r||$. That is $\mathrm{dom}\,\Lambda^{1/2}\rho^{1/2}=\mathcal{H}$. On the other hand

$$\mathrm{dom}\,\rho\Lambda=\{\psi:\sum|\psi_n|^2<\infty,\ \sum|n^{-\delta}\psi_n|<\infty\}\subset\mathcal{H}.$$

In particular, $\psi=\{\psi_n\}\in\mathcal{H}$, $\psi_n=n^{-1+\delta}$ does not belong to $\mathrm{dom}\,\rho\Lambda$ that is $||\rho\Lambda\psi||=\infty$. Thus, the aim of Definition 2.1 is to extend $\mathrm{Tr}\{\rho\Lambda\}$ to a larger set of ρ such that $\overline{\rho^{1/2}\Lambda^{1/2}}\in\mathcal{B}(\mathcal{H})$.

The next Lemma gives several alternative characterizations of $\mathcal{T}_\Lambda(\mathcal{H})$.

Lemma 2.1. *The following are equivalent:*

(a) $\rho \in \mathcal{T}_\Lambda^+(\mathcal{H}) = \{\rho:\ \rho^{1/2}: \mathcal{H} \to \operatorname{dom} \Lambda^{1/2},\ \sigma = \Lambda^{1/2}\rho^{1/2} \in \mathcal{T}_2(\mathcal{H})\}$;

(b) $\sup_{\varepsilon>0} \operatorname{Tr}\{\rho\Lambda_\varepsilon\} < \infty,\ \Lambda_\varepsilon = \Lambda(I+\varepsilon\Lambda)^{-1},\ \rho > 0$;

(c) *The densely defined operator* $s = \rho^{1/2}\Lambda^{1/2}$ *admits an extension to entire* $\mathcal{H}$ *as a Hilbert–Schmidt operator;*

(d) $\mathcal{T}_\Lambda^+(\mathcal{H})$ *is a completion with respect to* $\|\cdot\|_{\mathcal{T}_\Lambda}$*-norm of the envelope of the cone* $\mathcal{E}^+ = \{\rho_\psi = |\psi\rangle\langle\psi|,\ \psi \in \operatorname{dom}\Lambda_*\}$ *of extreme points of the unit ball* $\mathcal{T}(\mathcal{H})$*, that is*

$$\mathcal{T}_\Lambda^+(\mathcal{H}) = \left\{\rho:\ \rho = \sum r_k|\psi_k\rangle\langle\psi_k|,\ \sum r_k\Lambda_*[\psi_k] < \infty,\ r_k \geq 0\right\};$$

(e) $\rho \in \mathcal{T}_\Lambda^+(\mathcal{H}) = \{\rho:\ \rho = \Lambda^{-1/2}\sigma\Lambda^{1/2},\ \sigma \in \mathcal{T}^+(\mathcal{H})\}$.

Proof.

Let $\rho \in \mathcal{T}_\Lambda^+(\mathcal{H})$. Then $\operatorname{Tr}\{\rho\Lambda_\varepsilon\} = \operatorname{Tr}\{\sigma^*(I+\varepsilon\Lambda)^{-1}\sigma\} \leq \operatorname{Tr}\{\sigma^*\sigma\} < \infty$. Hence,

$$\operatorname{Tr}\{\rho\Lambda_\varepsilon\} = \sum \|(I+\varepsilon\Lambda)^{-1/2}\sigma\psi_k\|^2 \leq \sum \|\sigma\psi_k\|^2 = \operatorname{Tr}\{\sigma^*\sigma\} \tag{2.3}$$

converges uniformly in ε for any orthonormal system $\{\psi_n\}$. Therefore, it is possible to pass to the limit as $\varepsilon \to 0$ in each summand:

$$\sup_{\varepsilon\to 0} \operatorname{Tr}\{\sigma^*(I+\varepsilon\Lambda)^{-1}\sigma\} = \sum \sup_{\varepsilon\to 0} \|(I+\varepsilon\Lambda)^{-1}\sigma\psi_k\|^2 = \operatorname{Tr}\{\sigma^*\sigma\} < \infty.$$

Thus we prove that (a) implies (b). More precise, we proved the equation describing the regularization of trace-form by traces:

$$\operatorname{Tr}_*\{\rho\Lambda\} = \sup_{\varepsilon\to 0} \operatorname{Tr}\{\rho\Lambda_\varepsilon\} \qquad \rho \in \mathcal{T}_\Lambda^+. \tag{2.4}$$

Let (b) holds, that is $\sup_{\varepsilon\to 0} \operatorname{Tr}\{\rho\Lambda_\varepsilon\} = c < \infty$. Let us prove that $\operatorname{range}\rho^{1/2} \subseteq \operatorname{dom}\Lambda^{1/2}$ and $\operatorname{Tr}\{\sigma^*\sigma\} = c$ for $\sigma = \Lambda^{1/2}\rho^{1/2}$. For any $\psi \in \mathcal{H}$, $\|\psi\| = 1$ we have:

$$\|\Lambda_\varepsilon^{1/2}\rho^{1/2}\psi\|^2 \leq \operatorname{Tr}_*\{\rho\Lambda\} = c < \infty \tag{2.5}$$

for any $\varepsilon > 0$. Since Λ is a positive self-adjoint operator, $\Lambda_\varepsilon^{1/2}\varphi \to \Lambda^{1/2}\varphi$ as a resolvent for all $\varphi \in \operatorname{dom}\Lambda^{1/2}$, and $\|\Lambda_\varepsilon^{1/2}\varphi\| \to \|\Lambda^{1/2}\varphi\|$, we conclude that $\|\Lambda_\varepsilon^{1/2}\varphi\|$ is bounded if and only if $\varphi \in \operatorname{dom}\Lambda^{1/2}$. Hence (2.5) implies $\rho^{1/2}\psi \in \operatorname{dom}\Lambda^{1/2}\ \forall\psi \in \mathcal{H}$. The inequality $\operatorname{Tr}\{\rho^{1/2}(I+A)^{-1}\rho^{1/2}\} \geq \|(I+A)^{-1}\| \operatorname{Tr}\rho$ with $A \geq 0$ implies $\operatorname{Tr}\{\rho\Lambda_\varepsilon\} = \operatorname{Tr}\{\sigma^*(I+\varepsilon\Lambda)^{-1}\sigma\} \geq \|(I+\varepsilon\Lambda)^{-1}\| \operatorname{Tr}\{\sigma^*\sigma\}$. Hence,

$$c = \sup_{\varepsilon>0} \operatorname{Tr}\{\rho\Lambda_\varepsilon\} \geq \sup_{\varepsilon>0} \|I+\varepsilon\Lambda\|^{-1} \operatorname{Tr}\{\sigma^*\sigma\} = \operatorname{Tr}\{\sigma^*\sigma\},$$

that is $\sigma \in \mathcal{T}_2(\mathcal{H})$. Hence, we proved that (b) $\to$ (a).

Let us prove that (a) implies (c) and (c) implies (b). Since σ is a bounded operator, σ^* is also bounded and $\sigma^*\varphi = \rho^{1/2}\Lambda^{1/2}\varphi$, $\forall\varphi \in \operatorname{dom}\Lambda^{1/2}$. Hence $s = \rho^{1/2}\Lambda^{1/2}$ is densely defined and uniformly bounded. Therefore, the operator s has a unique bounded extension to $\mathcal{H}$: $\overline{\rho^{1/2}\Lambda^{1/2}} = \sigma^*$; $\sigma^* \in \mathcal{T}_2(\mathcal{H})$ because $\mathcal{T}_2(\mathcal{H})$ is a *-algebra and $\sigma \in \mathcal{T}_2(\mathcal{H})$. Thus, (a) $\to$ (c).

Assume (c) holds. Note that $s(I + \varepsilon\Lambda)^{-1/2} = \rho^{1/2}\Lambda_\varepsilon^{1/2}$, $(I + \varepsilon\Lambda)^{-1/2}s^* = \Lambda_\varepsilon^{1/2}\rho^{1/2}$. Since s is a Hilbert–Schmidt operator,

$$\infty > \mathrm{Tr}\{ss^*\} \geq \mathrm{Tr}\{s(I+\varepsilon\Lambda)^{-1}s^*\} = \mathrm{Tr}\{\rho\Lambda_\varepsilon\}$$

uniformly in $\varepsilon > 0$. Hence, $\rho \in \mathcal{T}_\Lambda^+(\mathcal{H})$ by (b).

Let $\{\psi_k\}$ be an orthonormal system in $\mathcal{H}$. Then $I = \sum |\psi_k\rangle\langle\psi_k|$, $\rho = \rho^{1/2}I\rho^{1/2} = \sum |h_k\rangle\langle h_k|$ with $h_k = \rho^{1/2}\psi_k \in \mathrm{dom}\,\Lambda_*$, and $\sum ||\Lambda^{1/2}h_k||^2 = \mathrm{Tr}_*\{\Lambda\rho\} < \infty$. Thus (a) $\to$ (d) with $r_k = 1$. On the other hand, if $\rho = \sum_k r_k|\psi_k\rangle\langle\psi_k|$ by (d) then $\mathrm{Tr}\{\rho\Lambda_\varepsilon\} \leq \sum r_k \,||\Lambda^{1/2}\psi_k||^2 < \infty$. Hence, (d) $\to$ (b).

Set $\rho_\Lambda = \Lambda^{-1/2}\rho\Lambda^{-1/2}$, $\rho \in \mathcal{T}^+(\mathcal{H})$. Then

$$\sup \mathrm{Tr}\{\rho\Lambda_\varepsilon\} = \sup \mathrm{Tr}\{\rho(1+\varepsilon\Lambda)^{-1}\} = \mathrm{Tr}\ \rho < \infty.$$

Thus, (e) $\to$ (b). On the other hand from (a) we have $\Lambda^{-1/2}\sigma = \rho^{1/2} \in \mathcal{T}_2^+(\mathcal{H})$ and from (c), $s\Lambda^{-1/2} = \rho^{1/2} \in \mathcal{T}_2^+(\mathcal{H})$. Therefore, $\rho = \Lambda^{-1/2}\sigma s\Lambda^{-1/2}$ with $\sigma s \in \mathcal{T}^+(\mathcal{H})$. That is{(a),(c)} $\to$ (e), where (a) $\to$ (c) was proved above. This finishes the proof of Lemma 2.1.

The assertion (b) of Lemma 2.1 holds true for any sequence $A(\varepsilon) \leq c_A\Lambda$ of bounded positive operators such that the sequence of quadratic forms $A_*(\varepsilon)$ generated by $A(\varepsilon)$ converges to A_* on $\mathrm{dom}\,\Lambda^{-1/2}$ and $A \leq c_A\Lambda$ on $\mathrm{dom}\,\Lambda$. Indeed, for $\rho \in \mathcal{T}_\Lambda^+$, $\rho = \Lambda^{-1/2}\sigma\Lambda^{-1/2}$, $\sigma \in \mathcal{T}$ we have $\mathrm{Tr}\{A(\varepsilon)\rho\} = \mathrm{Tr}\{\sigma A_\Lambda(\varepsilon)\} \to \mathrm{Tr}_*\{A\rho\}$, where $A_\Lambda(\varepsilon) = \Lambda^{-1/2}A(\varepsilon)\Lambda^{-1/2} \leq c_A\, I$ converges strongly and ultra weakly to the bounded positive operator $\overline{\Lambda^{-1/2}A^{1/2}}A^{1/2}\Lambda^{-1/2}$.

Definition 2.2. For any T such that $T = \sum_{k=0}^3 i^k T_k$, $0 \leq T_k \leq c_k\Lambda$, $\mathit{dom}\,\Lambda \subseteq \mathit{dom}\,T^k$ and for any $\rho \in \mathcal{T}_\Lambda$ we set

$$\mathrm{Tr}_*\{\rho T\} = \sum_{k,\ell=0}^{3} i^{k+\ell}\,\mathrm{Tr}_*\{T_k\rho_\ell\} = \sum_{k,\ell=0}^{3} i^{k+\ell}\,\mathrm{Tr}_*\{\sigma_\ell\widetilde{T}_k\sigma_\ell\},$$

where $\widetilde{T}_k = \overline{\Lambda^{-1/2}T_k^{1/2}}T_k^{1/2}\Lambda^{-1/2} \in \mathcal{B}(\mathcal{H})$, and $\sigma_\ell = \Lambda^{1/2}\rho_\ell^{1/2} \in \mathcal{T}_2^+(\mathcal{H})$, $\rho_\ell = \Lambda^{-1/2}\sigma_\ell\Lambda^{-1/2}$. $\mathrm{Tr}_*\{\rho T\}$ is referred to as a *trace-form* and the set of operators T as above is denoted by $\mathcal{B}_\Lambda(\mathcal{H})$:

$$||T||_{\mathcal{B}_\Lambda} = \inf_{0\leq T_k\leq c_k\Lambda,\ T=\sum_{k=0}^3 i^kT_k} \sum_{k=0}^{3}|c_k|.$$

Clearly, all operators $\Phi(B)$, $B \in \mathcal{B}(\mathcal{H})$ are elements of $\mathcal{B}_\Lambda(\mathcal{H})$ provided $\Phi(I) \leq \Lambda$.

Remark 2.2. Operators from $\mathcal{B}_\Lambda(\mathcal{H})$ can be uniquely extended as bounded operators from $\mathcal{H}_\Lambda$ into $\mathcal{H}_{\Lambda^{-1}}$. Indeed, for $\forall\psi \in \mathrm{dom}\ \Lambda$ we have $(\psi, T\psi) = \sum_k i^k||T_k^{1/2}\psi||^2$, and the right-hand side of this equation is a bounded quadratic form on dom $\Lambda^{1/2}$. Hence, for the extension $\overline{T}$, we have $||\overline{T}||_{\mathcal{H}_1\to\mathcal{H}_{-1}} \le \sum |c_k|$.

Remark 2.3. The values of $\mathrm{Tr}_*\{\rho T_k\}$ depend only on restrictions of components T_k to dom Λ. That is $\mathrm{Tr}_*\{\rho T_k\}$ does not depend on any choice of a self-adjoint extension of the positive densely defined operator $T_k|_{\mathrm{dom}\,\Lambda}$.

The independence of $\mathrm{Tr}_*\{\rho T\}$ on a choice of positive components ρ_ℓ and T_k follows from the representation of each summand:

$$\mathrm{Tr}_*\,\rho T = \sup_{\varepsilon>0} \mathrm{Tr} \sum_{k,\ell=0}^{3} i^{k+\ell}\,\mathrm{Tr}_*\{T_k^{(\varepsilon)}\rho_\ell\} = \mathrm{Tr}_*\{\sum_{k,\ell=0}^{3} i^{k+\ell} T_k^{(\varepsilon)}\rho_\ell\}$$

with $T_k^{(\varepsilon)} = (I+\varepsilon\Lambda)^{-1}T_k(I+\varepsilon\Lambda)^{-1}$ (see Lemma 2.1).

Corollary 2.2. *If $\Phi_*(\cdot) \in CPn_*(\mathcal{H}_\Lambda)$ and $\Phi(I) \le \Lambda$, then*

$$\sum_k |\Phi(B)_*[\rho^{1/2}\psi_k]| \le ||B||\,||\rho||_{\mathcal{T}_\Lambda}, \quad \forall \rho \in \mathcal{T}_\Lambda^+(\mathcal{H}), \quad B \in \mathcal{B}(\mathcal{H}) \tag{2.6}$$

for any orthonormal system $\{\psi_k\}$.

To prove the normality of the map $B \to \mathrm{Tr}_*\{\Phi(B)\rho\}$, consider a sequence B_n converging weakly* to 0. We have $\sup_n ||B_n|| = b < \infty$ and

$$|\Phi(B_n)_*[\rho^{1/2}\psi_k]| \le ||B_n||\Phi(I)_*[\rho^{1/2}\psi_k] \le bc\Lambda_*[\rho^{1/2}\psi_k].$$

Since $\sum_k \Lambda_*[\rho^{1/2}\psi_k] < \infty$ and $\Phi(B_n)_*[\rho^{1/2}\psi_k] \to 0$ as $n \to \infty$, by the Lebesgue theorem we have

$$\lim_n \sum_k \Phi(B_n)_*[\rho^{1/2}\psi_k] = \sum_k \lim_n \Phi(B_n)_*[\rho^{1/2}\psi_k] = 0.$$

Since the bilinear form $(B,\rho) = \mathrm{Tr}\{B\rho\}$ separates points of algebras $\mathcal{B}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$, the estimate (2.6) justifies the definition of the predual CP-map $\Phi^\dagger(\cdot)$ as a *contraction* ranging $\mathcal{T}_\Lambda(\mathcal{H})$ to $\mathcal{T}(\mathcal{H})$:

$$\mathrm{Tr}_*\{\Phi(B)\rho\} = \mathrm{Tr}\{B\Phi^\dagger(\rho)\} \quad \forall B \in \mathcal{B}(\mathcal{H}). \tag{2.7}$$

Now we are in a position to give a simple equivalent definition of $CPn(\mathcal{H}_\Lambda)$.

Definition 2.3. For each contractive CP-map $\Phi^\dagger(\cdot) : \mathcal{T}_\Lambda(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ we denote by $\Phi(\cdot) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}_\Lambda(\mathcal{H})$ the dual CP-map defined by (2.7).

The key advantage of this definition is that the normal property (2.2) of $\Phi(\cdot) \in CPn(\mathcal{H}_\Lambda)$ becomes obvious, because $\Phi^\dagger(\rho)$ in (2.7) is a bounded trace class operator.

Theorem 2.3. *There exists one-to-one correspondence defined by (2.7) between the subset $\{\Phi(\cdot) \in CPn(\mathcal{H}_\Lambda),\ \Phi(I) \leq \Lambda\}$ and the subset of contractive CP-maps $\Phi^\dagger(\cdot)$: $\mathcal{T}_\Lambda(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$.*

Proof. Corollary 2.1 shows that if $\Phi(\cdot) \in CPn(\mathcal{H}_\Lambda)$ and $\Phi(I) \leq \Lambda$, then $\Phi^\dagger(\cdot)$ is a contractive CP-map ranging $\mathcal{T}_\Lambda(\mathcal{H})$ into $\mathcal{T}(\mathcal{H})$. To prove the converse, consider the sequence of bounded CPn-maps $\Phi^{(\varepsilon)}(\cdot)$:

$$\mathrm{Tr}\{\Phi^{(\varepsilon)}(B)\rho\} = \mathrm{Tr}\{B\Phi^\dagger(\rho_\varepsilon)\}, \quad \rho_\varepsilon = (I+\varepsilon\Lambda)^{-1}\rho(I+\varepsilon\Lambda)^{-1} \in \mathcal{T}_\Lambda^+(\mathcal{H}), \tag{2.8}$$

where $\mathrm{Tr}_*\{\Phi^\dagger(\rho_\varepsilon)\} = \mathrm{Tr}\{\Lambda_\varepsilon\rho(I+\varepsilon\Lambda)^{-1}\} \leq ||\Lambda_\varepsilon||\,\mathrm{Tr}\,\rho \leq \varepsilon^{-1}\,\mathrm{Tr}\,\rho$. Therefore, $||\Phi^{(\varepsilon)}(\cdot)|| \leq \varepsilon^{-1}$.

The map $\Phi^{(\varepsilon)}(\cdot)$ is normal because $\mathrm{Tr}_*\{\Phi^\dagger(\rho_\varepsilon)\}$ is finite [16, Theorem 2.4.21]. Since $\mathcal{T}_\Lambda(\mathcal{H})$ is dense in $\mathcal{T}(\mathcal{H})$ with respect to the trace norm, equation (2.8) defines $\Phi^{(\varepsilon)}(\cdot)$ uniquely as an element of $CPn(\mathcal{H})$. To complete the proof we need to verify convergence $\Phi^{(\varepsilon)}(\cdot) \to \Phi(\cdot)$ with respect to topology (2.1). Consider the seminorm

$$\begin{aligned}\sigma_{A,B}(\Phi^{(\varepsilon)} - \Phi) &= \sup_{X\in B,\ \psi\in A}\left|\Phi^{(\varepsilon)}(X)_*[\psi] - \Phi(X)_*[\psi]\right| \\ &\leq 2\sup_{X\in B}||\Lambda^{-1/2}\Phi(X)\Lambda^{-1/2}||\ \sup_{\psi\in A}||\rho-\rho_\varepsilon||_{\mathcal{T}_\Lambda},\end{aligned}$$

where $\rho = |\psi\rangle\langle\psi|$, $\rho_\varepsilon = |(I+\varepsilon\Lambda)^{-1}\psi\rangle\langle(I+\varepsilon\Lambda)^{-1}\psi|$ where $\psi \in \mathrm{dom}\,\Lambda^{1/2}$. Let us evaluate $\delta(\varepsilon) = ||\rho-\rho_\varepsilon||_{\mathcal{T}_\Lambda}$. For Hermitian operators, by Definition 2.1 we have:

$$\delta(\varepsilon) = \inf_{\xi,\ \xi-\rho+\rho_\varepsilon\in\mathcal{T}_\Lambda^+}\mathrm{Tr}\{(2\xi-\rho+\rho_\varepsilon)\Lambda\} = \inf_{\widetilde{\xi},\ \widetilde{\xi}-\widetilde{\rho}+\widetilde{\rho}_\varepsilon\in\mathcal{T}+}\mathrm{Tr}\{(2\widetilde{\xi}-\widetilde{\rho}+\widetilde{\rho}_\varepsilon)\}$$

with $\widetilde{\xi} = (\xi^{1/2}\Lambda^{1/2})^*(\xi^{1/2}\Lambda^{1/2}) \in \mathcal{T}(\mathcal{H})$, $|\psi_\Lambda\rangle\langle\psi_\Lambda| \in \mathcal{T}(\mathcal{H})$, $\psi_\Lambda = \Lambda^{1/2}\psi \in \mathcal{H}$, $\rho_\varepsilon = |\psi_\Lambda^\varepsilon\rangle\langle\psi_\Lambda^\varepsilon| \in \mathcal{T}_\mathcal{H}$, $\psi_\Lambda^\varepsilon = (I+\varepsilon\Lambda)^{-1}\psi_\Lambda \in \mathcal{H}$, $\widetilde{\rho} = |\psi\rangle\langle\psi|$, $\widetilde{\rho}_\varepsilon = |\psi^\varepsilon\rangle\langle\psi^\varepsilon|$.

Set $c_\varepsilon = ||\psi_\Lambda||^2||\psi_\Lambda^\varepsilon||^{-2} \geq 1$ and $\widetilde{\rho}_\varepsilon^{nor} = c_\varepsilon|\psi_\Lambda^\varepsilon\rangle\langle\psi_\Lambda^\varepsilon| \geq \widetilde{\rho}_\varepsilon$. Since the positive operators $\widetilde{\rho}_\varepsilon^{nor}$, $\widetilde{\rho}$ have the same traces, the straightforward computations implies

$$\begin{aligned}\delta(\varepsilon) &= \inf_{\xi,\ \xi-\widetilde{\rho}+\widetilde{\rho}_\varepsilon^{nor}\in\mathcal{T}+}\mathrm{Tr}\{(2\xi-\widetilde{\rho}+\widetilde{\rho}_\varepsilon^{nor})\} \\ &\leq \mathrm{Tr}\{(2|\widetilde{\rho}-\widetilde{\rho}_\varepsilon| - \widetilde{\rho}+\widetilde{\rho}_\varepsilon^{nor})\} = 2\,\mathrm{Tr}\{|\widetilde{\rho}-\widetilde{\rho}_\varepsilon^{nor}|\} = 4||\psi_\Lambda||\sqrt{||\psi_\Lambda||^2 - |(\psi_\Lambda,\psi_\Lambda^\varepsilon)|^2}.\end{aligned}$$

Therefore, $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ because $\psi_\Lambda^\varepsilon \to \psi_\Lambda$ in $\mathcal{H}$ for any $\psi \in \mathcal{H}_\Lambda$. This finishes the proof of the theorem.

Hence, we may set $\mathrm{Tr}_*\{\Phi(B)\rho\} = \lim_{\varepsilon\to 0}\mathrm{Tr}\{\Phi^{(\varepsilon)}(B)\rho\}$, where $\Phi^{(\varepsilon)}(\cdot)$ is any sequence from $CPn(\mathcal{H})$ converging to $\Phi(\cdot)$ with respect to topology (2.1), or equivalently, $\Phi^\dagger(\cdot)$ is a contraction ranging $\mathcal{T}_\Lambda$ to $\mathcal{T}$.

§3. A PRIORI ESTIMATES OF QDS AND CONDITIONS SUFFICIENT FOR CONSERVATIVITY.

Let us describe assumptions on coefficients of the formal infinitesimal map (1.2) for fixed $t \in \mathbb{R}_+$. From now on we assume that there exist the generator $-G$ of the strongly continuous one parameter semigroup of contractions $\exp\{-Gt\} = s - \lim_{N\to\infty}\left(I - \frac{t}{N}G\right)^{-N} = W_t : \mathcal{H} \to \mathcal{H}$, a vector subspace $\mathcal{D} = \operatorname{dom} G^N \subseteq \mathcal{H}$ such that $\mathcal{D} \subseteq \operatorname{dom} H \cap \operatorname{dom} \Phi(I)$ for some $N \geq 2$, and

$$G\psi = iH\psi + \frac{1}{2}\Phi(I)\psi, \quad H\psi = H^*\psi \qquad \forall\psi \in \operatorname{dom}\Lambda. \tag{3.1}$$

By $\mathcal{T}_\mathcal{D}(\mathcal{H})$ we denote the linear span of the set of pure states $|\psi\rangle\langle\psi|$, $\psi \in \mathcal{D}$. Clearly, $\mathcal{T}_\mathcal{D}(\mathcal{H}) \subseteq \mathcal{T}_{\Lambda^2}(\mathcal{H})$. Let $\Lambda \geq \Phi(I) \geq I$ be a reference self-adjoint "reference" operator such that $\Lambda \geq \Phi(I) \geq I$. In sequel we assume that $\mathcal{D}$ is a core for $\Lambda^{1/2}$ and

$$G \in \mathcal{B}_\Lambda(\mathcal{H}), \quad \operatorname{dom} G^N \subseteq \operatorname{dom}\Lambda \subseteq \operatorname{dom} G \subseteq \operatorname{dom}\Lambda^{1/2} \subseteq \mathcal{H}, \tag{3.2}$$

$$\Phi^\dagger(\cdot) : \mathcal{T}_{\Lambda^k}(\mathcal{H}) \to \mathcal{T}_{\Lambda^{k-2}}(\mathcal{H}), \quad k = 1, 2 \tag{3.3}$$

is a CP-*contraction* for $k = 1$ and CP-*continuous* mapping for $k = 2$;

$$\mathcal{L}^\dagger(\cdot) : \mathcal{T}_{\Lambda^k}(\mathcal{H}) \to \mathcal{T}_{\Lambda^{k-1}}(\mathcal{H}), \quad k = 1, 2 \tag{3.4}$$

is a CCP-*continuous* mapping such that

$$\operatorname{Tr}_*\{\Lambda\mathcal{L}^\dagger(\rho)\} \leq c\operatorname{Tr}_*\{\rho\Lambda\}. \tag{3.5}$$

In §2 we noted that the predual *CP*-map $\Phi^\dagger(\cdot)$ is well defined on $\mathcal{T}_\Lambda(\mathcal{H})$. The more so it is well defined on $\mathcal{T}_\mathcal{D}(\mathcal{H}) \subseteq \mathcal{T}_\Lambda(\mathcal{H})$ or on $\mathcal{T}_{\Lambda^2}(\mathcal{H})$. Similarly, $G\rho \in \mathcal{T}(\mathcal{H})$ for $\rho \in \mathcal{T}_{\Lambda^2}(\mathcal{H})$, and $\rho G^* \in \mathcal{T}(\mathcal{H})$ because the vector space $\mathcal{B}_\Lambda(\mathcal{H})$ is *-invariant. Hence, the predual infinitesimal map $\mathcal{L}^\dagger(\cdot)$ is well defined on $\mathcal{T}_\mathcal{D}(\mathcal{H})$:

$$\mathcal{L}^\dagger(\rho) = \Phi^\dagger(\rho) - \overline{\rho G^*} - G\rho \in \mathcal{T}_\Lambda(\mathcal{H}) \qquad \forall\rho \in \mathcal{T}_{\Lambda^2}(\mathcal{H}) \tag{3.6}$$

and therefore, $|\operatorname{Tr}_*\{\Lambda\mathcal{L}^\dagger(\rho)\}| < \infty$. Note that the main property of $\mathcal{T}_\mathcal{D}(\mathcal{H})$ is that the image of this set under the Schrödinger evolution belongs to $\operatorname{dom}\Lambda$.

The algebraic assumption (3.5) arises as a result of series of improvements (see [12-19]) of one-sided relative bounds for commutators introduced originally in [11].

For equations with time-dependent generator $\mathcal{L}_t(\cdot)$, we assume that the coefficients of $\mathcal{L}_t^\dagger(\cdot)$ are continuous in the trace-form sense:

$$\mathcal{L}_t^\dagger(\cdot) - \mathcal{L}_s^\dagger(\cdot) \text{ can be extended as a bounded map from } \mathcal{T}_\Lambda(\mathcal{H}) \text{ to } \mathcal{T}(\mathcal{H}), \tag{3.7}$$

$$\lim_{h\to=0} \sup_{\|Y\|\leq I} \|\mathcal{L}_{t+h}(Y) - \mathcal{L}_t(Y)\|_{\mathcal{B}_\Lambda} = 0 \quad \forall t \geq 0,\ Y \in \mathcal{B} \tag{3.8}$$

Let us start with preliminary review of algebraic ideas of the proof that in the Heisenberg picture, the quantum dynamical semigroup $P_t^{min}(\cdot)$, is conservative under assumption (3.6) implies, and in the Schrödinger picture, the subalgebra of trace class operators $\mathcal{T}_\Lambda(\mathcal{H})$ is invariant under action of QDS.

In [13–14] we introduced a convenient form of the resolvent equation for the quantum dynamical semigroup $P_t(\cdot) = \exp\{t\mathcal{L}\}(\cdot)$ on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ with time-independent generator $\mathcal{L}(B) = \Phi(B) - \Phi(I) \circ B + i[H, B]$. This resolvent equation reads

$$X = A_\lambda(B) + Q_\lambda(X), \quad B \in \mathcal{B}(\mathcal{H}), \quad X = R_\lambda(B) = \int_0^\infty dt\, e^{-\lambda t} P_t(B) \qquad (3.9)$$

where $A_\lambda(\cdot)$ and $Q_\lambda(\cdot)$ are completely positive contraction maps,

$$A_\lambda(B) = \int_0^\infty dt\, e^{-\lambda t} W_t^* B W_t, \qquad Q_\lambda(B) = \int_0^\infty dt\, e^{-\lambda t} W_t^* \Phi(B) W_t, \qquad (3.10)$$

and $W_t = \exp\{-Gt\}$ is a strongly continuous one-parameter contraction semigroup in $\mathcal{H}$ with the formal generator $-G$, $G\psi = iH\psi + \frac{1}{2}\Phi(I)\psi \; \forall\psi \in \operatorname{dom} H \cap \operatorname{dom} \Phi(I)$. It was proved in [14] that the series $\sum Q_\lambda^k(A_\lambda(\cdot))$ converges strongly to the resolvent

$$R_\lambda^{\min}(B) = \sum_{k=0}^{\infty} Q_\lambda^k(A_\lambda(B)) \qquad \forall B \in \mathcal{B}(\mathcal{H}) \qquad (3.11)$$

of the minimal dynamical semigroup [14, 15]. The minimal property means that, for the resolvent $R_\lambda(\cdot)$ of any other dynamical semigroup with the same formal generator $\mathcal{L}(\cdot)$, the difference $R_\lambda(\cdot) - R_\lambda^{\min}(\cdot)$ is a completely positive map.

Integration by parts of in (3.10) yields the identities

$$A_\lambda(I) + \lambda^{-1} Q_\lambda(I) \equiv \lambda^{-1} I, \qquad \sum_{k=0}^{n} Q_\lambda^k(A_\lambda(I)) + \lambda^{-1} Q_\lambda^{n+1}(I) \equiv \lambda^{-1} I \qquad (3.12)$$

for any $n > 1$. If $\Phi(I) \geq I$, then $A_\lambda(I) \leq (\lambda+1)^{-1}$ and $Q_\lambda(I) \leq I$. Since *CP*-maps $A_\lambda(\cdot)$ and $Q_\lambda(\cdot)$ are contractions, the definition (3.10) and commutation property (2.2) prove that $A_\lambda(\cdot)$ and $Q_\lambda(\cdot)$ are normal. Thus, $A_\lambda(\cdot)$, $Q_\lambda(\cdot) \in CPn(\mathcal{H})$.

¿From (3.12) follows three important assertions:

(a) The sequence $Q_\lambda^n(I)$ is monotone decreasing;
(b) There exists uniform a priori estimate $\sum Q_\lambda^k(A_\lambda(I)) \leq \lambda^{-1} I$ for any partial sum of the series;
(c) The resolvent $R_\lambda(I)$ can be represented by the series:

$$R_\lambda(I) = \sup_n \sum_{k=0}^{n} Q_\lambda^k(A_\lambda(I)) + \lambda^{-1} \inf_n Q_\lambda^n(I) = R_\lambda^{\min}(I) + \lambda^{-1} \inf_n Q_\lambda^n(I). \quad (3.13)$$

The last identity shows that $R_\lambda^{\min}(I) = \lambda^{-1} I$ and $P_t^{\min}(I) \equiv I$ (that is QDS $P_t(\cdot)$ is *unital or conservative*) if and only if $\inf_n Q_\lambda^n(I) = 0$ for any $\lambda > 0$ (see [10–13]). Otherwise, there exist a stationary point $X_\lambda = \inf_n Q_\lambda^n(I) \geq 0$ which is the *maximal* eigenoperator of $Q_\lambda(\cdot)$ in the class of positive operators with unit norm.

If $R_\lambda^{\min}(I) \le \lambda^{-1} I$, then any QDS $P_t(\cdot)$ with the same formal generator $\mathcal{L}(\cdot)$ is conservative if and only if $R_\lambda(I) - R_\lambda^{\min}(I) = X_\lambda$.

Suppose that the generator $\mathcal{L}(B)$ satisfies the condition $\mathcal{L}(\Lambda)_* \le c\Lambda_*$, $c \in \mathbb{R}_+$, $\Lambda \ge \Phi(I)$ for some positive self-adjoint "reference" operator $\Lambda \in \mathcal{C}(\mathcal{H})$. Then the following *a priori* estimates hold for the semigroup and for its resolvent:

$$P_t(\Lambda) = \Lambda + t\mathcal{L}(\Lambda) + \frac{t^2}{2!}\mathcal{L}^2(\Lambda) + \cdots \le \Lambda + ct\Lambda + \frac{(ct)^2}{2!}\Lambda + \cdots \le \Lambda e^{ct},$$
$$R_\lambda(\Lambda) = \int_0^\infty dt\, e^{-\lambda t} P_t(\Lambda) \le (\lambda - c)^{-1}\Lambda \qquad \forall \lambda > c. \tag{3.14}$$

Note that $A_\lambda(\Phi(I)) \equiv Q_\lambda(I)$. Since $\Phi(I) \le \Lambda$, the estimate for sum of the series

$$\sum_1^\infty Q_\lambda^k(I) = R_\lambda(\Phi(I)) \le R_\lambda(\Lambda) \le (\lambda - c)^{-1}\Lambda \qquad \forall \lambda > c$$

follows from (3.14) and from the decomposition (3.11) of the resolvent. As we shall see later, these estimates remain true for the resolvent of *minimal* QDS with unbounded coefficients:

$$\sum_1^\infty Q_\lambda^k(I) = R_\lambda^{\min}(\Phi(I)) \le (\lambda - c)^{-1}\Lambda \qquad \forall \lambda > c. \tag{3.15}$$

The weak convergence of the series (3.15) implies that $Q_\lambda^n(I) \xrightarrow{w} 0$ as $n \to \infty$ on the dense subset $\operatorname{dom}\Lambda_* \in \mathcal{H}$. Since the sequence of positive operators $Q_\lambda^n(I)$ is uniformly bounded, it converges strongly on $\mathcal{H}$. Hence, the condition (3.5) is sufficient for the conservativity of the minimal QDS (see [14–15]).

A priori bound (3.14) for the minimal resolvent and condition (3.6) were considered in [17] for the simplest and the most natural choice $\Lambda = \Phi(I)$. An important observation that it is possible to use operators majoring $\Phi(I)$ was made in [18] and independently in [20], where a condition similar to (3.6) was used as a sufficient conservativity condition together with assumption $\|H\varphi\| \le \|\Lambda\varphi\|$ on a dense set *es-s dom* Λ. In [14] and the present paper we show how to avoid excessive assumptions like explicit relative bounds for the Hamiltonian H.

The estimate (3.14) and trace-form regularization (see Lemma 2.1, (b)) mean that the subalgebra $\mathcal{T}_\Lambda(\mathcal{H})$ is T_t-invariant. Indeed,

$$\mathrm{Tr}_*\{T_t(\rho)\Lambda\} = \sup_{\varepsilon>0} \mathrm{Tr}\{T_t(\rho)\Lambda_\varepsilon\} = \sup_{\varepsilon>0} \mathrm{Tr}\{\rho P_t(\Lambda_\varepsilon)\}, \quad \forall \rho \in \mathcal{T}_\Lambda^+(\mathcal{H})$$
$$\mathrm{Tr}\{T_t(\rho)\Lambda_\varepsilon\} = \mathrm{Tr}\{\rho P_t(\Lambda_\varepsilon)\} \le e^{ct}\,\mathrm{Tr}\{\rho\Lambda_\varepsilon\} \le e^{ct}\,\mathrm{Tr}_*\{\rho\Lambda\} < \infty. \tag{3.16}$$

Simple bounds (3.15)–(3.16) give an algebraic hint to analytical estimates considered in the next Section. The estimate (3.16) can be extended to semigroups with piecewise constant generators satisfying condition (3.6) at pointwise and remains true even for time-dependent generators. We divide the rigorous proof of these statements into a series of auxiliary lemmas.

§4. A priori estimates and sufficient conservativity conditions. Analytical part of the theory

We start from a simple observation on monotone property of the regularization map $\Lambda \to \Lambda_\varepsilon = \Lambda(I+\varepsilon\Lambda)^{-1}$.

Lemma 4.1. *Let A, Λ be positive self-adjoint operators. Then $A_\varepsilon \le \Lambda_\varepsilon$.*

Proof. For bounded positive operators A and Λ, the proof follows from the inequality $(I+\varepsilon\Lambda)^{-1} \le (I+\varepsilon A)^{-1}$ (see [21]). For bounded A and self-adjoint Λ it was proved in [14, Lemma 2]. Let now $A \ge 0$ be a self-adjoint operator. Since $(A_\varepsilon)_\mu = A_{\varepsilon+\mu} \le A \quad \forall \varepsilon, \mu > 0$ (see [14]) and $A_{\varepsilon+\mu}$ is a bounded operator,

$$0 \le A_{\varepsilon+\mu} \le \Lambda_\varepsilon \tag{4.1}$$

for any positive self-adjoint operator $A \le \Lambda$. The family of bounded operators $A_{\varepsilon+\mu}$ is strongly continuous in μ, $\mu \to +0$, as a resolvent. Hence, (3.7) holds for $\mu = 0$. Therefore,

$$0 \le A_\varepsilon \le \Lambda_\varepsilon. \tag{4.2}$$

Thus, we extend the assertion [14, Lemma 2] for positive self-adjoint operators.

Lemma 4.2. *Let the conditions* (3.1–3.4) *are satisfied; then for $\rho_t = W_t\rho W_t^*$, $\rho \in \mathcal{T}_\Lambda$ we have*

$$||\rho_t||_{\mathcal{T}_\Lambda} \le e^{ct}||\rho||_{\mathcal{T}_\Lambda}, \quad ||A_\lambda^\dagger(\rho)||_{\mathcal{T}_\Lambda} \le (\lambda - c)^{-1}||\rho||_{\mathcal{T}_\Lambda}. \tag{4.3}$$

Proof. By the assertion (d) of Lemma 2.1, it suffices to prove estimates (4.3) for a total set of pure states $\rho = |\psi\rangle\langle\psi| \in \mathcal{T}_{\Lambda^2}$, $\psi \in \operatorname{dom} G^N$. In this case $\psi_\lambda = (I+\lambda^{-1}G)\psi \in \operatorname{dom}\Lambda^{1/2}$ and $\rho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda| \in \mathcal{T}_\Lambda$. By assumption (3.4) we have

$$c||\rho||_{\mathcal{T}_\Lambda} = c\,\mathrm{Tr}_*\{\Lambda\rho\} \ge \mathrm{Tr}_*\{\Lambda\mathcal{L}(\rho)\} \ge -2\,\mathrm{Re}\,\mathrm{Tr}_*\{\Lambda|G\psi\rangle\langle\psi|\} = -2\,\mathrm{Re}\,\Lambda_*[G\psi,\psi].$$

Therefore, $2\lambda^{-1}\,\mathrm{Re}\,\Lambda_*[G\psi,\psi] \ge c\lambda^{-1}c||\rho||_{\mathcal{T}_\Lambda}$ and

$$\begin{aligned} ||\rho_\lambda||_{\mathcal{T}_\Lambda} &= \Lambda_*[\psi_\lambda] = \Lambda_*[\psi] + \lambda^{-2}\Lambda_*[G\psi] + 2\lambda^{-1}\,\mathrm{Re}\,\Lambda_*[G\psi,\psi] \\ &\ge (1-c\lambda^{-1})\Lambda_*[\psi] = (1-c\lambda^{-1})||\rho||_{\mathcal{T}_\Lambda}. \end{aligned}$$

Hence, $\mathrm{Tr}_*\{\Lambda\rho_\lambda\} \le (1-c\lambda^{-1})||\rho||_{\mathcal{T}_\Lambda}$ and from the strong convergence $W_t = s-\lim_{N\to\infty}(I-\frac{t}{N}G)^{-N}$ we have

$$||\rho_t||_{\mathcal{T}_\Lambda} = \lim_{N\to\infty}\mathrm{Tr}_*\{\Lambda(I-\frac{t}{N}G)^{-N}\rho\,(I-\frac{t}{N}G)^{-N}\} \le e^{ct}||\rho||_{\mathcal{T}_\Lambda}.$$

The second estimate obviously follows from here. This finishes the proof.

Corollary 4.1. *For dual mappings, the following estimates hold true:*

$$W_t^*\Lambda W_t \le \Lambda e^{ct}, \quad A_\lambda(\Lambda) \le (\lambda-c)^{-1}\Lambda \quad \text{on dom}\,\Lambda \tag{4.4}$$

Lemma 4.3. *Assume the conditions* (3.2–3.6) *are satisfied; then*

$$||(R_\lambda^{(n)})^\dagger(\rho)||_{\mathcal{T}_\Lambda} \le (\lambda - c)^{-1}||\rho||_{\mathcal{T}_\Lambda}, \quad R_\lambda^{(n)}(\Lambda) \le (\lambda - c)^{-1}\Lambda. \tag{4.5}$$

Proof. The assertion of this lemma is fulfilled for $n = 0$ (Lemma 4.2 and Corollary 4.1). Assume that (4.5) is fulfilled for $n = 1, \dots, k$ and prove it for $n = k + 1$. Set $\rho = |\psi\rangle\langle\psi| \in \mathcal{T}_{\Lambda^2}$, $\psi \in \operatorname{dom} G^N$. Then

$$\begin{aligned}
\mathrm{Tr}_*\{\Lambda_\varepsilon R_\lambda^{(k+1)}(\rho)\} &= \mathrm{Tr}_*\{\Lambda_\varepsilon A_\lambda^\dagger(\rho)\} + \int_0^\infty e^{-\lambda t}dt\ \mathrm{Tr}_*\{R_\lambda^{(k)}(\Lambda_\varepsilon)\Phi^\dagger(\rho_t)\} \\
&\le \mathrm{Tr}_*\{\Lambda A_\lambda^\dagger(\rho)\} + (\lambda - c)^{-1}\int_0^\infty e^{-\lambda t}dt\ \mathrm{Tr}_*\{\Lambda\Phi^\dagger(\rho_t)\} \\
&= \mathrm{Tr}_*\{\Lambda A_\lambda^\dagger(\rho)\} + (\lambda - c)^{-1}\int_0^\infty e^{-\lambda t}dt\ \mathrm{Tr}_*\{\Lambda\mathcal{L}^\dagger(\rho_t) + G\rho_t + \rho_t G^*\},
\end{aligned} \tag{4.6}$$

where $\rho_t = W_t\rho W_t^* \in \mathcal{T}_{\Lambda^2}(\mathcal{H})$. Using assumption (3.5) for the first summand in integral (4.6) and integrating by parts in the second summand, we obtain

$$\mathrm{Tr}_*\{\Lambda_\varepsilon R_\lambda^{(k+1)}(\rho)\} \le \left(1 + \frac{c}{\lambda - c} - \frac{\lambda}{\lambda - c}\right)\mathrm{Tr}_*\{\Lambda A_\lambda^\dagger(\rho)\} + \frac{1}{\lambda - c}||\rho||_{\mathcal{T}_\Lambda}.$$

The assertions of the lemma follow from here.

We recall that

$$(R_\lambda^{min})^\dagger(\rho) = \operatorname*{l.u.b.}_n (R_\lambda^{(n)})^\dagger(\rho) = A_\lambda^\dagger(\rho) + \sum_1^\infty (Q_\lambda^\dagger)^n(A_\lambda^\dagger(\rho)) = \int_0^\infty e^{-\lambda t}dt\, T_t^{min}(\rho)$$

is the resolvent of the minimal quantum dynamical semigroup in the Schrödinger picture (see [2], [12–15]), and on the other hand

$$T_t^{min}(\rho) = w^* - \lim_{N\to\infty} \underbrace{\lambda(R_\lambda^{min})^\dagger \cdots \lambda(R_\lambda^{min})^\dagger}_{N}(\rho)\Big|_{\lambda = tN^{-1}}.$$

Definition 4.1. For $X \in \mathcal{B}_\Lambda(\mathcal{H})$, we define $P_t(X) : \mathcal{H}_1 \to \mathcal{H}_{-1}$ as an image $\Lambda^{\frac{1}{2}}B_t\Lambda^{\frac{1}{2}}$ of the bounded operator B_t such that $\mathrm{Tr}\{B_t\sigma\} = \mathrm{Tr}\{B\widetilde{\rho}_t\}$, for all $\sigma \in \mathcal{T}(\mathcal{H})$, where $\widetilde{\rho}_t = \Lambda^{\frac{1}{2}}T_t(\rho)\Lambda^{\frac{1}{2}} \in \mathcal{T}(\mathcal{H})$, $\rho = \Lambda^{-\frac{1}{2}}\sigma\Lambda^{-\frac{1}{2}}$.

Now as a simple corollary of Lemma 4.2, we obtain

Theorem 4.4. *Under assumptions* (3.2–3.5) *we have a priori estimates for the quantum dynamical semigroup*

$$||T_t^{min}(\rho)||_{\mathcal{T}_\Lambda} \le e^{ct}||\rho||_{\mathcal{T}_\Lambda}, \quad ||P_t^{min}(X)||_{\mathcal{B}_\Lambda} \le e^{ct}||X||_{\mathcal{B}_\Lambda}. \tag{4.8}$$

Proof. The first inequality (4.8) follows from the uniform estimate (4.7):

$$||R_\lambda^{min}||_{\mathcal{T}_\Lambda} = \lim_{n\to\infty} ||R_\lambda^{(n)}||_{\mathcal{T}_\Lambda} = (\lambda - c)^{-1},$$

$$||T_t^{min}(\rho)||_{\mathcal{T}_\Lambda} \le \lim_{N\to\infty} ||\lambda(R_\lambda^{min})^\dagger||_{\lambda = tN^{-1}}^N = \lim_{N\to\infty}\left(tN^{-1}(tN^{-1} - c)\right)^N = e^{ct}.$$

Since for $B \in \mathcal{B}(\mathcal{H})$, $\sigma \in \mathcal{T}(\mathcal{H})$, $X = \Lambda^{\frac{1}{2}}B\Lambda^{\frac{1}{2}}$, $\rho = \Lambda^{-\frac{1}{2}}\sigma\Lambda^{-\frac{1}{2}}$ we have $\mathrm{Tr}\{B\sigma\} = \mathrm{Tr}_*\{X\rho\}$, the second estimate (4.8) follows from here and from Definition 4.1.

Corollary 4.2. *Conditions* (3.2–3.6) *are sufficient for the minimal dynamical semigroup to be conservative.*

This statement follows from the uniform upper bound for the decreasing sequence of bounded operators $Q_\lambda^{(n)}(I)$ and from the weak convergence of the series

$$\sum Q_\lambda^k(I) \leq R_\lambda^{\min}(\Lambda) \leq (\lambda - c)^{-1}\Lambda$$

on the dense set $\operatorname{dom}\Lambda \in \mathcal{H}$.

Theorem 4.5. *Let* $\rho \in \mathcal{T}_\Lambda^+(\mathcal{H})$ *and conditions* (3.2–3.6) *are fulfilled, then*
1) $R_t = \mathrm{Tr}_*\{\rho P_t(\Lambda)\} = \mathrm{Tr}\{\sigma_t^*\sigma_t\}$ *where* $\sigma_t = \Lambda^{1/2}\{T_t(\rho)\}^{1/2}$ *is a scalar function continuous in* $t \geq 0$.
2) If A *is a positive self-adjoint operator such that* $A \leq c_A\Lambda$ *then the scalar function*

$$A_t = \mathrm{Tr}_*\{\rho P_t(A)\} \leq c_A e^{ct}\,\mathrm{Tr}_*\{\rho\Lambda\}$$

is bounded and continuous.

Proof. 1) Let $\{\psi_k\}$ be an orthonormal system in $\mathcal{H}$. The series

$$\delta_{s,t} = R_s - R_t = \sum\left(\|\Lambda^{1/2}\rho_s^{1/2}\psi_k\|^2 - \|\Lambda^{1/2}\rho_t^{1/2}\psi_k\|^2\right)$$
$$= \sum\left(\|\overline{\rho_s^{1/2}\Lambda^{1/2}}\psi_k\|^2 - \|\overline{\rho_t^{1/2}\Lambda^{1/2}}\psi_k\|^2\right)$$

converges uniformly in s,t: $|\delta_{s,t}| \leq \left(e^{cs}+e^{ct}\right)\mathrm{Tr}\,\sigma^*\sigma$, for any $\sigma = \Lambda^{1/2}\rho^{1/2}$ because of the estimate (4.1). Hence,

$$\lim_{t-s\to+0}|\delta_{s,t}| \leq \sum \lim_{t-s\to+0}\left(\|\overline{\rho_s^{1/2}\Lambda^{1/2}}\psi_k\|^2 - \|\overline{\rho_t^{1/2}\Lambda^{1/2}}\psi_k\|^2\right).$$

Now let $\{\varphi_k(\delta)\}$ be a family of elements from $\operatorname{dom}\Lambda^{1/2}$ such that $\|\psi_k - \varphi_k(\delta)\| \leq \delta k^{-2}$. Then, by the inequality $\Lambda_*[h] - \Lambda_*[v] \leq \{\Lambda_*[h]^{1/2} + \Lambda_*[v]^{1/2}\}\,\Lambda_*[h-v]^{1/2}$ we have

$$|\delta_{s,t}| \leq \delta\left(e^{cs}+e^{ct}\right)\mathrm{Tr}\{\sigma^*\sigma\}\sum k^{-2} + \sum\left(\|\rho_s^{1/2}\Lambda^{1/2}\varphi_k\|^2 - \|\rho_t^{1/2}\Lambda^{1/2}\varphi_k\|^2\right).$$

Since the strong continuity of the bounded family of positive operators ρ_t follows from its weak or trace norm continuity and $\Lambda^{1/2}\varphi_k \in \mathcal{H}$, $\|\rho_s^{1/2}\Lambda^{1/2}\varphi_k\|^2 - \|\rho_t^{1/2}\Lambda^{1/2}\varphi_k\|^2 \to 0$ as $t-s\to 0$. Hence, $|\delta_{s,t}| \to 0$ as $t-s\to 0$ because δ can be chosen arbitrary small. This proves assertion 1) of the theorem.

2) The estimate of the last statement of the theorem follows from (4.3) and from the inequality

$$\mathrm{Tr}_*\{\rho P_t^{min}(A)\} = \sup_{\varepsilon\to 0}\mathrm{Tr}\{\rho_t^{1/2}A_\varepsilon\rho_t^{1/2}\} \leq$$
$$c_A\sup_{\varepsilon\to 0}\mathrm{Tr}\{\rho_t^{1/2}\Lambda_\varepsilon\rho_t^{1/2}\} = c_A\,\mathrm{Tr}_*\{\rho P_t^{min}(\Lambda)\} \leq c_A e^{ct}\,\mathrm{Tr}_*\{\rho\Lambda\}.$$

Note that the inequality $A \leq c_a\Lambda$ implies $\|A_0^{1/2}\psi\| \leq c_A^{1/2}\|\Lambda^{1/2}\psi\|$ for any self-adjoint extension A_0 of A. Hence, the operators $\alpha = A_0^{1/2}\Lambda^{-1/2}$ and $\alpha^* = \overline{A_0^{1/2}\Lambda^{-1/2}A_0^{1/2}}$ are bounded: $\|\alpha^*\alpha\| \leq c_A$. Now the proof of the continuity of A_t is a revision f the previous proof in the Hilbert space $\mathcal{H}_0$ with the inner product $(\psi,h)_{\mathcal{H}_0} = (\alpha\psi,\alpha h)_{\mathcal{H}_0}$. This completes the proof.

§5. Completion of the set of regular infinitesimal maps with piecewise constant coefficients

Consider the Markov evolution equation

$$\partial_t P_{s,t}(B) = \mathcal{L}_t(P_{s,t}(B)), \qquad P|_{t=s}(B) = B, \quad 0 \le s \le t, \tag{5.1}$$

with infinitesimal map $\mathcal{L}_t(\cdot)$ that is a *simple* function of time, i.e., takes constant values on time intervals; the total number of these intervals is assumed finite in any bounded subset of $\mathbb{R}_+$:

$$\mathcal{L}_t(\cdot) = \mathcal{L}_i(\cdot), \quad \Phi_t(\cdot) = \Phi_i(\cdot), \quad H_t = H_i \quad \forall t \in (t_{i-1}, t_i], \quad t_0 = 0, \quad i > 0.$$

The solution $P_{t,s}(\cdot)$ of Eq. (5.1) are constructed as the composition of the solutions of equations with constant coefficients on each interval. It has the characteristic property of a left cocycle: $P_{t,\tau}P_{\tau,s} = P_{t,s}$, $s < \tau < t$; in what follows, $P_{t,s}(\cdot)$ is referred to as a *Markov cocycle*: $P_{t,s}(\cdot) = P^{(n)}_{t-t_n}(\cdots P^{(k(s))}_{t_{k(s)}-s}(\cdot))$, where $P^{(k)}_t(\cdot) = e^{t\mathcal{L}_k}(\cdot)$ is the minimal dynamical semigroup with constant generator $\mathcal{L}_k(\cdot)$, on the half-interval $(t_{k-1}, t_k]$, $k(s) = \{\min_k t_k > s\}$, $t_n = \{\max t_j : t_j < t\}$.

If there exists a sequence of completely positive conservative Markov cocycles converging at every $t \in \mathbb{R}_+$ in the ultraweak or weak sense, then, clearly, the limit is a completely positive and conservative Markov cocycle. First, we consider a priori estimates for the solution of the time-dependent Lindblad equation with simple coefficients.

Let the generator G_t be a *simple* strongly measurable function of $\mathbb{R}^+$ and let G_t have a joint invariant core $\mathcal{D} \subseteq \operatorname{dom} G_t^N \subseteq \operatorname{dom}\Lambda$ for some $N \ge 2$. In sequel we suppose that the assumptions (3.2)–(3.5) be fulfilled at each moment $t \in \mathbb{R}_+$ and $D_t : \mathcal{H}_1 \to \mathcal{H}_{-1}$ is a strongly measurable family of bounded operators such that

$$|(\Lambda)^{-1/2}\varphi, D_t(\Lambda)^{-1/2}\varphi| \le c_D(t)\|\varphi\|^2 \tag{5.2}$$

$$\mathcal{L}_t(\Lambda)_*[\varphi] \le c_\Lambda(t)\Lambda_*[\varphi], \tag{5.3}$$

where $c_D(t), c_\Lambda(t) \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$.

Lemma 5.1. *If conditions* (5.2–5.3), *are satisfied at each point in every time-interval* $(t_{k-1}, t_k]$, *then the minimal solution of the Lindblad equation*

$$\partial_t P_t = \mathcal{L}_t(P_t) + D_t, \qquad P|_{t=0} = 0,$$

satisfies the estimate

$$|(P_t^{\min})_*[\varphi]| \le \Lambda_*[\varphi]\int_0^t ds \exp\left\{\int_s^t c_\Lambda(r)\,dr\right\}c_D(s) \quad \forall\varphi \in \operatorname{dom}\Lambda_*$$

where $c_D(s) = \sum |c_k(s)|$.

Proof. In this case, the Duhamel equation reads as follows: $(P_t^{\min})_*[\varphi] =$

$$= \int_0^t P^{(n)}_{t-t_n}\cdots P^{(k(s))}_{t_{k(s)}-s}(D_s)_*[\varphi]ds = \int_0^t \mathrm{Tr}_*\{T^{(k(s))}_{t_{k(s)}-s}\cdots T^{(n)}_{t-t_n}(\rho_\varphi)D_s\}ds$$

Hence,

$$|(P_t^{\min})_*[\varphi]| \le \int_0^t ds\ \mathrm{Tr}_*\{T^{(k(s))}_{t_{k(s)}-s}\dots T^{(n)}_{t-t_n}(\rho_\varphi)\Lambda\}c_D(s),$$

where the semigroup $T_t^{(k)}(\cdot)$ in $\mathcal{T}(\mathcal{H})$ is predual of $P_t^{(k)}(\cdot)$ and $\rho_\varphi = |\varphi\rangle\langle\varphi| \in \mathcal{T}_\Lambda(\mathcal{H})$. Hence, Theorem 4.4 implies the estimate:

$$|(P_t^{\min})_*[\varphi]| \le \mathrm{Tr}_*(\rho_\varphi\Lambda)\int_0^t ds\, e^{(t-t_n)c_\Lambda(t_n)+\dots+(t_{k(s)}-s)c_\Lambda(t_{k-1})}c_D(s)$$

$$= \Lambda_*[\varphi]\int_0^t ds\ \exp\left\{\int_s^t c_\Lambda(\tau)\,d\tau\right\}c_D(s) \qquad \forall\varphi\in \mathrm{dom}\ \Lambda.$$

This completes the proof.

Now let $\mathcal{L}_t(\cdot)$ be a function of the parameter t continuous in the sense (1.3)–(1.4). Consider the sequence of partitions $\{t_i(N)\}$ of the semi axis R_+ such that $0 \le t_{k(s)}(N) - s \le 2^{-N}$ for every $s \in I\!\!R_+$. We assign the sequence of simple infinitesimal operators $\mathcal{L}_\tau^{(N)}(\cdot) = \mathcal{L}_{t_{k(\tau)}}(\cdot)$ to the infinitesimal operator $\mathcal{L}_t(\cdot)$. Let $P_{t,s}^{(N)}$ be the composition of the minimal solutions of the equations with piecewise constant generators $\mathcal{L}_t^{(N)}(\cdot)$. Consider the difference $\delta_t^{(M,N} = P_{t,0}^{(N)}(B) - P_{t,0}^{(M)}(B)$, which is uniformly bounded and satisfies the following nonhomogeneous equation with trivial initial condition:

$$\partial_t\delta_t^{(M,N)} = \mathcal{L}_t^{(N)}(\delta_t^{(M,N)}) + D_t^{(M,N)}, \qquad \delta_t^{(M,N)}\big|_{t=0} = 0,$$

where $D_t^{(M,N)} = \mathcal{L}_t^{(N)}(P_{t,0}^{(M)}(B)) - \mathcal{L}_t^{(M)}(P_{t,0}^{(M)}(B))$, that is $|\,\mathrm{Tr}_*\{\rho D_t^{(M,N)}\}| \le 2K_t\,\mathrm{Tr}_*(\rho\Lambda)$. Since the difference $\delta_t^{(M,N)}$ is bounded in norm, it suffices to prove the ultraweak convergence $\delta_t^{(M,N)} \to 0$ as $M,N\to\infty$ on subset of $\mathcal{T}_\Lambda(\mathcal{H})$. We use the linear span of the set of pure states $\rho_\varphi = |\varphi\rangle\langle\varphi|$, $\varphi\in\mathrm{dom}\,\Lambda_*$, as such a subset.

Let us set

$$\rho_{t,s}^{(M,N)} = T^{(k(s))}_{t_{k(s)}-s}\dots T^{(n)}_{t-t_n}(\rho_\varphi) \in \mathcal{T}_\Lambda(\mathcal{H}).$$

Then Lemma 5.1 ensures the following estimate for $\mathrm{Tr}\{\rho_\varphi\delta_t^{(M,N)}\}$:

$$\left|\mathrm{Tr}\{\rho_\varphi\delta_t^{(M,N)}\}\right| \le \left|\int_0^t ds\ \mathrm{Tr}_*\{\rho_{t,s}^{(M,N)}D_s^{(M,N)}\}\right|$$

$$\le ||B||\int_0^t ds\ \exp\left\{\int_s^t c_\Lambda(s)\,d\tau\right\}\sup_{||Y||\le 1}||\mathcal{L}_s^{(N)}(Y) - \mathcal{L}_s^{(M)}(Y)||_{\mathcal{B}_\Lambda}$$

uniformly in M and N. By assumptions (1.3)–(1.4), the supremum $\delta_{s,t_{k(s)}}^{(M,N)} = \sup_{||Y||\le 1}||\mathcal{L}_s^{(N)}(Y) - \mathcal{L}_s^{(M)}(Y)||_{\mathcal{B}_\Lambda}$ is a uniformly continuous function in s, M, N. Hence, there exists a uniform upper bound in $\mathcal{L}_1^{\mathrm{loc}}(I\!\!R_+)$ for the integrand, and we can use the dominated convergence theorem, which gives

$$\lim_{M,N\to\infty}\left|\mathrm{Tr}\{\rho\delta_t^{(M,N)}\}\right| = \lim_{M,N}||B||\int_0^t ds\ \exp\left\{\int_s^t c_\Lambda(\tau)\,d\tau\right\}\delta_{s,t_{k(s)}}^{(M,N)} \to 0$$

since $h = s - t_{k(s)} \to 0$.

Finally, we obtain the following result.

Theorem 5.2. *Assumptions* (3.1–3.5), (5.2–5.4) *are sufficient for the sequence* $\{P_{t,s}^{(N)}(B)\}$ *to be ultraweak fundamental and converging to the conservative completely positive cocycle* $P_{t,s}(B)$.

Consider the class $\mathcal{S}_\Lambda$ of *simple* infinitesimal maps $\mathcal{L}_t(\,\cdot\,)$, satisfying the assumptions of Theorem 5.1.

Definition 5.1. The sequence $\{\mathcal{L}_t^{(N)}(\,\cdot\,)\}$ from $\mathcal{S}_\Lambda$ is said to be Tr_Λ-*fundamental* if it converges in the locally convex topology generated by the system of seminorms

$$\sigma_T(\mathcal{L}_t^{(N)}) = \int_0^T dt \sup_{\|B\|\le 1} \|\mathcal{L}_t^{(N)}(B)\|_{\mathcal{B}_\Lambda} \exp\Big\{\int_s^t d\tau\, c_\Lambda(\tau)\Big\},$$

for all $T > 0$. The infinitesimal map with variable coefficients $\mathcal{L}_t(\,\cdot\,)$ is said to be $\mathcal{S}_\Lambda$-*measurable* if the sequence

$$\mathcal{L}_t^{(N)}(\,\cdot\,) = \mathcal{L}_{t_k}(\,\cdot\,), \qquad t \in (t_k, t_{k+1}], \quad t_k - t_{k-1} = 2^{-N},$$

converges to $\mathcal{L}_t(\,\cdot\,)$ with respect to the seminorms $\sigma_{\rho,T}$.

Theorem 5.2 shows that $\mathcal{S}_\Lambda$-measurable infinitesimal maps form a natural class of generators of conservative Markov cocycles.

§6. Examples

1. Let $\mathcal{H} = L_2(\mathbb{R})$, and let the coefficients of the map $\mathcal{L}_t(\,\cdot\,)$ be

$$H_t = \Omega(x,t)x^2, \qquad \Phi_t(B) = -\partial_x a(x,t) B a(x,t)\partial_x,$$

where a and Ω are smooth bounded functions such that

$$\sup_x a(x,t) < \infty, \quad \sup_x |\Omega(x,t)| < \infty, \quad \inf_x a(x,t) > 0 \qquad \forall t \in [0,T].$$

We set $A(x,t) = a^2(x,t)$. The operator $L_t = -\frac{1}{2}\Phi_t(I) = \frac{1}{2}A(\cdot,t)\partial_x^2 + aa'_x\partial_x$ is the generator of a diffusion process ξ_τ, satisfying the stochastic differential equation

$$d\xi_\tau = a(\xi_\tau,\tau)\big(dw_\tau + a(\xi_\tau,\tau)\, a'_x(\xi_\tau,\tau)\, d\tau\big), \qquad \xi_t = x,$$

where w_τ is the standard Wiener process (see [24]). The two-parameter family $W_{s,t}$ of contraction operators with the generator $-G_t$, $G_t = \frac{1}{2}\Phi_t(I) + iH_t$, can be represented as the conditional expectation

$$W_{s,t}\psi(x) = \mathbb{M}_{x,t}\psi(\xi_s)\exp\Big\{-i\int_s^t d\tau\,\Omega(\xi_\tau,\tau)\xi_\tau^2\Big\}.$$

Let us check the conservativity conditions (1.2) and the continuity criteria (1.3) and (1.4) with respect to the operator

$$\Lambda = \lambda(-\partial_x^2 + x^2 + I), \qquad \lambda = \sup_{x\in\mathbb{R},\ t\in(0,T)} \{|\Omega(x,t)| + A(x,t)\}.$$

First, we have the following identity for the commutator: $i[H_t,\Lambda] = i[\Omega x^2, \partial_x^2] = -i\{b\partial_x + \partial_x b\}$, where $b = \partial_x(\Omega(x,t)x^2)$. For every $\varepsilon \in \mathbb{R}_+$, the well known inequality

$$\pm(A^*B + B^*A) \leq A^*\varepsilon A + B^*\varepsilon^{-1}B \tag{6.1}$$

holds, and in the case $A = i\partial_x$, $B = b = \partial_x(\Omega x^2)$, $\varepsilon = 1$, we have $i[H_t,\Lambda] \leq (b^2 - \partial_x^2) \leq \lambda\Lambda$ if

$$\sup_{t\in(0,T]} (1+|x|)^{-1}|\partial_x \Omega x^2| \leq \lambda. \tag{6.2}$$

Straightforward computations imply the identities

$$\Phi_t(x^2) - \Phi_t(I)\circ x^2 = -2a(x,t)a'_x(x,t)x - a^2(x,t),$$

$$\Phi_t(\partial_x^2) - \Phi_t(I)\circ \partial_x^2 = -\left(a'_x(x,t)\right)^2 \partial_x^2.$$

Therefore, condition (1.2) is satisfied if

$$\sup_{x,t}(1+|x|)\left(|\Omega'_x(x,t)| + |\Omega'_x(x,t)|\right) < \infty, \qquad \sup_{x,t}|a'_x(x,t)| < \infty. \tag{6.3}$$

Let us check the continuity assumptions.

Consider $\rho \in \mathcal{T}_{\Lambda^2}(\mathcal{H})$, that is $\rho = \Lambda^{-\frac{1}{2}}\sigma\Lambda^{-\frac{1}{2}}$, $\sigma \in \mathcal{T}(\mathcal{H})$, $\sigma_1 = x\rho x \in \mathcal{T}(\mathcal{H})$, $\sigma_2 = \partial_x\,\rho\,\partial_x \in \mathcal{T}(\mathcal{H})$, by definition of the reference operator Λ. For any sequence of operators A_n such that $\sup_n \|A_n\| < \infty$ and $A_n \to 0$ in the strong sense, we have $|\operatorname{Tr} A_n\sigma| \to 0$ $\forall\sigma \in \mathcal{T}(\mathcal{H})$. Therefore,

$$\operatorname{Tr}_*\{B[H_t - H_s, \rho]\} = \operatorname{Tr}\{B(x\Delta\Omega_{s,t}x\Lambda^{-\frac{1}{2}}\sigma\Lambda^{-\frac{1}{2}} - \Lambda^{-\frac{1}{2}}\sigma\Lambda^{-\frac{1}{2}}x\Delta\Omega_{s,t}x)\} \to 0$$

where $\Delta\Omega_{s,t}(x) = \Omega_t(x) - \Omega_s(x)$, because the function $x\Omega(x,t)$ is uniformly bounded by the assumption (6.2), the operator $x\Lambda^{-\frac{1}{2}}$ is bounded in $\mathcal{H}$, and $|\Omega(x,t) - \Omega(x,s)| \to 0$ everywhere in $\mathbb{R}$ as $t - s \to 0$. Furthermore, from inequality (6.1) we have the following estimate for the difference of completely positive maps:

$$\Phi_t^\dagger(\rho) - \Phi_s^\dagger(\rho) \leq \frac{\varepsilon}{2}\left(\Phi_t^\dagger(\rho) + \Phi_s^\dagger(\rho)\right) + \varepsilon^{-1}\left(a(x,t) - a(x,s)\right)\sigma_2\left(a(x,t) - a(x,s)\right).$$

Therefore, by the same arguments and inequality (6.1), we obtain

$$\left|\operatorname{Tr}_*\left(B\left(\Phi_t^\dagger(\rho) - \Phi_s^\dagger(\rho)\right)\right)\right| \leq \lambda\varepsilon\|B\|\operatorname{Tr}_*(\rho\Lambda) + \varepsilon^{-1}\operatorname{Tr}_*\left\{\left(a(\cdot,t) - a(\cdot,s)\right)^2\sigma_2\right\} \to 0,$$

because $|a(x,t)|$ is a bounded function of x and t, and $|a(x,t) - a(x,s)| \to 0$ as $t - s \to 0$ for all $x \in \mathbb{R}$.

The following inequality can be proved in a similar way:

$$\operatorname{Tr}_*\{B(\Phi_s^\dagger(I) - \Phi_t^\dagger(I))\circ\rho\} \leq \|B\|\operatorname{Tr}\{\sigma\varphi(s,t)\},$$

where $\varphi(s,t) = \Lambda^{-\frac{1}{2}}\left(\Phi_s^\dagger(I) - \Phi_t^\dagger(I)\right)\Lambda^{-\frac{1}{2}}$, $\rho = \Lambda^{-1}\sigma\Lambda^{-1} \in \mathcal{T}_{\Lambda^2}(\mathcal{H})$.

The operator family $\varphi(s,t)$ is uniformly bounded. Note that $a\partial_x^2 = \partial_x^2 a - (\partial_x a'_x + a'_x\partial_x)$, and consequently (6.1) gives the following inequality:

$$\|\varphi(s,t)\psi\| \leq \left\|\left(|a^2(\cdot,t) - a^2(\cdot,s)| + \varepsilon^{-1}|a'_x(\cdot,t) - a'_x(\cdot,s)|^2\right)\partial_x^2\Lambda^{-1}\psi\right\| + \varepsilon\|\partial_x^2\Lambda^{-1}\psi\|$$

for arbitrary small ε. Thus, $\|\varphi(s,t)\psi\| \to 0$ since the differences $|a(x,t) - a(x,s)|$ and $|a'_x(x,t) - a'_x(x,s)|$ are uniformly bounded in x by (6.2), and

$$|a(x,t) - a(x,s)| \to 0, \qquad |a'_x(x,t) - a'_x(x,s)| \to 0.$$

as $t - s \to 0$ for all $x \in \mathbb{R}$. Hence, the limit Markov cocycle $P_{s,t}(\cdot) = w^* - \lim P_{s,t}^{(N)}(\cdot)$ exists and is conservative if the continuity assumptions (6.2), (6.3) are satisfied for the uniformly bounded coefficients $a(x,t)$ and $\Omega(x,t)$.

2. Consider the infinitesimal operator $\mathcal{L}_t(\cdot)$ of the master equation with time-dependent coefficients

$$\Phi_t(B) = (L + \overline{f}(t)W)^* B(L + f(t)W), \quad f(t) \in \mathcal{L}_\infty(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R}),$$
$$H_t = H_0 + (f(t)W^*L - \overline{f}(t)L^*W)/2i, \tag{6.5}$$

where L is a closed operator, W is a unitary operator, H_0 is a symmetric operator in the Hilbert space $\mathcal{H}$.

If the unital property of the minimal solution of the master equation with the generator $\mathcal{L}_t(\cdot)$ holds for all $f(\cdot) \in \mathbb{C}(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R})$, $||f|| \leq c$ for some $c > 0$ then the solution of the corresponding quantum stochastic differential equation is unique and isometric [25]. This remark explains our interest in unital property of time-dependent master equations.

Assume that the operator $L^*L = \Lambda \geq I$ is essentially self-adjoint and $W : \mathcal{H}_k \to \mathcal{H}_k$ is a bounded operator for $k = 1, 2$. Then the densely defined operators $\Lambda^{-\frac{1}{2}}L^*$, $L\Lambda^{-\frac{1}{2}}$ are contractions in $\mathcal{H}$. Note that the difference $\delta_{s,t}(\cdot) = \mathcal{L}_t^\dagger(\cdot) - \mathcal{L}_s^\dagger(\cdot)$ does not contain the operator H_0 and the terms which are bilinear in L^*, L:

$$\delta_{s,t}(\rho) = (|f(t)|^2 - |f(s)|^2)W[\rho, W^*] + \overline{(f(t) - f(s))}[L, \rho]W + (f(t) - f(s))W^*[\rho, L].$$

For $\rho \in \mathcal{T}_\Lambda(\mathcal{H})$, the operator $L\rho$ is bounded both in $\mathcal{T}(\mathcal{H})$ and in $\mathcal{B}(\mathcal{H}_\Lambda)$. Indeed,

$$||L\rho||_{\mathcal{B}} \leq ||L\Lambda^{-\frac{1}{2}}||_{\mathcal{B}}\, ||\sigma||_{\mathcal{B}}\, ||\Lambda^{-\frac{1}{2}}||_{\mathcal{B}} \leq ||\sigma||_{\mathcal{B}} \leq ||\sigma||_{\mathcal{T}} = ||\rho||_{\mathcal{T}_\Lambda},$$
$$||L\rho||_{\mathcal{T}} = \sup_{U:\, U^*U = UU^* = I} \operatorname{Tr}\{UL\rho\} = \sup \operatorname{Tr}_*\{UL\Lambda^{-\frac{1}{2}}\sigma\Lambda^{-\frac{1}{2}}\}$$
$$\leq ||\Lambda^{-\frac{1}{2}}UL\Lambda^{-\frac{1}{2}}||_{\mathcal{B}} \operatorname{Tr}|\sigma| \leq ||\sigma||_{\mathcal{T}} = ||\rho||_{\mathcal{T}_\Lambda}.$$

Therefore, $\delta_{s,t}(\cdot)$ is a bounded mapping from $\mathcal{T}(\mathcal{H}_\Lambda)$ to $\mathcal{T}(\mathcal{H})$. Thus,

$$\left|\operatorname{Tr}_*\{B(\mathcal{L}_t^\dagger(\rho) - \mathcal{L}_s^\dagger(\rho)\})\right| \leq \left|\operatorname{Tr}\{B\delta_{s,t}(\rho)\}\right| \leq 6\big(|f(t) - f(s)|\big)\left(1 + ||f||_\infty\right)||\rho||_{\mathcal{T}_\Lambda}\, ||B||_{\mathcal{B}}$$

and the assumptions (1.3)-(1.4) are clearly satisfied.

The family of operators $G_t = \frac{1}{2}L_t^*L_t + iH_t$ can be considered as a perturbation of the generator $G = \frac{1}{2}L^*L + iH_0$ by the operator $g_t = \frac{1}{2}|f(t)|^2 + \overline{f}(t)L^*W$ which is relatively bounded with respect to Λ:

$$||g_t\psi|| \leq \frac{1}{2}||f||^2||\psi|| + ||f||\, ||L\Lambda^{-1}||\, ||\Lambda\psi|| \leq \frac{c^2}{2}||\psi|| + c||\Lambda\psi||$$

with the upper bound $c = \sup ||f(t)||$. Hence, by the semigroup perturbation theory [22], the operator G_t with the domain $\operatorname{dom} G_t = \operatorname{dom} G$ is a generator of the strongly continuous bounded semigroup $W_s = \exp\{-sG_t\}$ with t fixed.

We assume that the operators H_t are symmetric on $\operatorname{dom} G$. Then for any $\psi \in \operatorname{dom} G$ we have

$$\frac{d}{ds}||W_s\psi||^2 = -||(W_s\psi, L_t^*L_t W_s\psi)|| \leq -||W_s\psi||^2.$$

Therefore, W_s is a contractive semigroup.

In sequel we assume for simplicity that $||H_0\psi|| \leq ||\Lambda\psi||$, $\forall\psi \in \operatorname{dom}\Lambda$. Then $\operatorname{dom} G_t = \operatorname{dom}\Lambda$ is an invariant joint domain of the generators G_t. Consider the trace-form $\mathrm{Tr}_*\{\Lambda\mathcal{L}_t^\dagger(\rho)\}$ for $\rho \in \mathcal{T}_{\Lambda^2}(\mathcal{H})$. Using the algebraic identity

$$\mathcal{L}_t(\Lambda) = L^*[L^*, L]L + i[H_0, \Lambda] + |f(t)|^2 W^*[\Lambda, W]$$
$$+f(t)\left([W^*, \Lambda]L + \frac{1}{2}W^*[L^*, L]L\right) + \overline{f}(t)\left(L^*[\Lambda, W] + \frac{1}{2}L^*[L^*, L]W\right) \quad (6.5)$$

and assuming that the densely defined operators $[L^*, L]$ can be extended to $\mathcal{H}$ as a bounded operator, $||[\Lambda, W]\psi||^2 \leq c_0\Lambda_*[\psi]$ and $||[H_0, \Lambda]\psi||^2 \leq c\Lambda_*[\psi]$ on $\operatorname{dom}\Lambda_*$, we obtain from (6.1) and (6.5) the desired inequality (1.2)

$$\mathcal{L}_t(\Lambda)_* \leq c_t\Lambda_* \text{ on } \operatorname{dom}\Lambda \text{ and hence } \mathrm{Tr}_*\{\Lambda\mathcal{L}_t^\dagger(\rho)\} \leq c_t||\rho||_{\mathcal{T}_\Lambda}$$

for some uniformly bounded function c_t. Thus, the conditions

$$[L^*, L] \in \mathcal{B}(\mathcal{H}); \quad ||H_0\psi|| \leq ||\Lambda\psi||, \ \forall\psi \in \operatorname{dom}\Lambda;$$
$$||[\Lambda, W]\psi||^2 \leq c_0\Lambda_*[\psi], \quad ||[H_0, \Lambda]\psi||^2 \leq c\Lambda_*[\psi] \quad \forall\psi \in \operatorname{dom}\Lambda^{1/2},$$
$$W : \mathcal{H}_k \to \mathcal{H}_k, \text{ is a bounded operator for } k = 1, 2$$

imply the existence of the unital cocycle which has the formal generator of the master equation with time-dependent coefficients (6.5).

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