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On Some Integration Formulae in Stochastic Analysis*

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Abstract

We consider a finite dimensional version of Evans-Perkins type stochastic integral formula in the theory of measure-valued processes. The principal role of the formula consists in rewriting a product of historical functionals of a specific class and stochastic integral relative to the orthogonal martingale measure in the Walsh sense into a certain expression involved with stochastic integral with respect to a Dawson-Perkins historical process associated with a reference Hunt process. This naturally leads to a variant of stochastic integration by parts formula in stochastic analysis. Our result is an extension of the Evans-Perkins lemma(1995).

§0. Introduction

The purpose of this article consists in a generalization of the Evans-Perkins stochastic integral formula. There are two reasons why this integration formula is so important. For one thing, it can provides with a new formula of transformations of stochastic integrals closely connected with the so-called historical processes. In fact the establishment of the formula asserts that a product of historical functionals of a specific class and stochastic integral relative to the orthogonal martingale measure in the Walsh sense is, in its mathematical expectation form, equivalent to a certain expression of integration that is involved with stochastic integral with respect to a Dawson-Perkins historical process associated with a reference Hunt process. In addition, it also allows us to interpret that the formula is nothing but a variant of stochastic integration by parts in an abstract level, that is very useful as a theoretical tool of stochastic calculus in the theory of measure-valued processes. For

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another, it has an extremely remarkable meaning on an applicational basis. By making use of the formula S.N. Evans and E.A. Perkins have succeeded in deriving a kind of Itô-Wiener chaos expansion for functionals of superprocesses[EP95].

To make sure its importance for the latter case, let us take a quick historical review on the matter. The Itô-Wiener chaos expansion theorem was originally proved by K. Iô (1951). It asserts that every L^2 functional can be described as a constant C_0 plus a sum of multiple stochastic integrals (actually, multiple Wiener integrals) $I_n(f_n)$ with respect to a standard d-dimensional Brownian motion B with L^2 symmetric functions f_n's. In the case of the Brownian motion, an even stronger result is true, that is to say, every L^2 functional has an orthogonal expansion in terms of multiple stochastic integrals with deterministic integrands. As a matter of fact, any two multiple Wiener integrals of different orders are orthogonal. That is why it is called Itô's L^2 orthogonal decomposition theorem as well. One may find the orthogonality properties so useful and powerful in many success stories of this famous theorem applied to various sorts of theories, such as analysis in the Wiener space, Malliavin calculus, white noise analysis, etc. This result has been extended over the past four decades to a wide variety of processes. However, it is a task of extreme difficulty to extend the result to a more general class of processes. So many attempts have been made by plenty of probabilists in the game of extending Itô's theorem. Among them, the work acquired by J. Jacod (1979) has exposed a positive aspect in this direction of researches. That is, he proved the very most general theorem that the existence of such a stochastic integral representation for functionals of a certain process is thoroughly equivalent to the well-posedness of a martingale problem for the underlying process. However, while existence is now generally known, it is not always clear precisely how to write the representation. In most cases it would be hard to get explicit representations for the process functionals even in the sufficiently general setting. On the contrary, a negative aspect of studies in line with this generalization has been brought by E.B. Dynkin (1988). Of course, he gave a similar type expression, and also showed the example that even the definition of multiple stochastic integrals can be difficult, and two stochastic integrals of different orders are no longer orthogonal. For instance, Dynkin's counterexample of lack of orthogonality suggests the criterion, i.e., if the quadratic variational process $\langle M \rangle$ of the integrator martingale M is random (= not deterministic), then two stochastic integrals with respect to dM of distinct multiplicity do fail to be orthoganal. This explains why Itô's theorem can be beautifully perfect, because the quadratic variation of the Brownian motion is deterministic, say, $\langle B \rangle_t = t$. As is easily imagined, it would be certain that the research activities of this direction have become less popular since the discovery of Dynkin's counterexample(1988). And yet S.N. Evans and E.A. Perkins (1991) have showed that any L^2 functional of superprocess may be represented as a constant C_0 plus a stochastic integral with respect to the associated orthogonal martingale measure M. Recently they have obtained the explicit representations involving multiple stochastic integrals for a quite general functional of the so-called Dawson-Watanabe superprocesses. Actually, the results are obtained in the setting of the *historical process* associated with the superprocess. It is this way that suddenly coming up is the historical process in this field. Based upon the previous results(1991), they derived partial analogue of the Itô-Wiener chaos expansion in superprocess setting by taking advantage of the "stochastic integral formula" in question.

Now we shall give a rough idea what the integration formula is like and try to explain precisely the notation appearing in the expression, but in the form as simple as possible. The rigorous definition will be given in the succeeding section. First of all, let us consider the functional $F(H)$ of a historical process H with branching mechanism $\Phi(\alpha,\beta,\gamma,\delta)$ for a real valued function F on $C([0,\infty); M_F(D))$ with the space D of E-valued cadlag paths. Actually, this F should lie in a suitable admissible subspace $U(M(D))$ of $C(C([0,\infty); M_F(D)); \mathbf{R})$. Next consider a stochastic integral $J(\Xi; M) = \int\int \Xi(s,y)dM$ of a bounded predictable function Ξ relative to the orthogonal martingale measure M in the Walsh sense(1986). Then we make a product $F(H)\cdot J(\Xi; M)$. On the other hand, consider the integral of another type $J(F,\Xi; H) = \int\int I^*[F]\Xi(s,y)dH_s ds$ for some predictable function $I^*[F]$ which is determined by the functional $F(H)$ given. Thus we attain the integration formula if we take the mathematical expectation of those terms, i.e., $\mathbf{E}\,[F(H)\cdot J(\Xi; M)] = \mathbf{E}\,[J(F,\Xi; H)]$.

For the rest of this section, we observe that any two multiple integrals of different orders as for $J(\Xi; M)$ are not orthogonal any longer. Let us take a look at this in the following because it is easy. Let $\mathcal{P}$ denote the $(\mathcal{G}_t)_{t\geq\tau}$-predictable σ-field of functions on $(\tau,\infty)\times\Omega$ and $(U,\mathcal{U})$ is a measurable space. The following is a well-known fact(Stricker-Yor(1978)). Suppose the following two conditions:

(C.1) $\varphi : U\times(\tau,\infty)\times D \to \mathbf{R}$ is bounded and $\mathcal{U}\times\mathcal{B}((\tau,\infty))\times\mathcal{D}$-measurable.

(C.2) $\psi : U\times[\tau,\infty)\times\Omega \to \mathbf{R}$ is a $\mathcal{U}\times\mathcal{P}$-measurable function, satisfying

$$\sup_{u\in U}\sup_{\tau\leq t\leq\theta} \mathbf{P}\,[|\psi(u,t)|^p] < \infty, \qquad \forall\theta>\tau,\ p\geq 1.$$

Then we have

(a) The stochastic integral

$$\int_{\tau+}^{t}\int_D \varphi(u,s,y)\psi(u,s)dM(s,y)$$

is well-defined for any $u\in U$, $\forall t>\tau$.

(b) It satisfies

$$\sup_{u\in U}\sup_{\tau\leq t\leq\theta} \mathbf{P}\left|\int_{\tau+}^{t}\int_D \varphi(u,s,y)\psi(u,s)dM(s,y)\right|^p < \infty \qquad \forall\theta>\tau, \forall p\geq 1.$$

(c) Moreover, there exists a $\mathcal{U}\times\mathcal{P}$-measurable mapping α such that the random set

$$\left\{\alpha(u,\cdot,\cdot) \neq \int_{\tau+}^{\cdot}\int_D \varphi(u,s,y)\psi(u,s)dM(s,y)\right\}$$

is evanescent for all $u \in U$.

Remark. The above assertion implies that for almost all paths the map α and the stochastic integral $\int\int \varphi\psi dM$ are the same. More precisely, if α and $\int\int \varphi\psi dM$ are indistinguishable, then one has

$$\alpha(u,t) = \int_{\tau+}^{t}\int_{D} \varphi(u,s)\psi(u,s)dM(s,y), \quad a.s.$$

for all $t \in (\tau,\infty)$, $\forall u \in U$. However, notice that the converse is not true.

We may apply the above-mentioned fact to get the absence of orthogonality for the multiple stochastic integrals.

1. Suppose that for any $m \geq 1$

$$\varphi_i \in b(\{\mathcal{B}((\tau,\infty))\}^{m-i+1} \times \mathcal{D}), \qquad i = 1,2,\cdots,m.$$

Applying the above result we can construct a $\mathcal{B}((\tau,\infty))^{m-1} \times \mathcal{P}$-measurable function

$$\alpha_1 : (\tau,\infty)^{m-1} \times (\tau,\infty) \times \Omega \to \mathbf{R}$$

such that $\alpha_1(s_2,\cdots,s_m;\cdot)$ is indistinguishable from

$$\int_{\tau+}^{\cdot}\int_{D} \varphi_1(s_1,\cdots,s_m;y_1)dM(s_1,y_1)$$

for any $s_2,s_3,\cdots,s_m$.

2. Since the mapping: $(s_3,\cdots,s_m;s_2;\omega) \mapsto \alpha_1(s_2,s_3,\cdots,s_m;s_2;\omega)$ is $\mathcal{B}((\tau,\infty))^{m-2} \times \mathcal{P}$-measurable, we can apply the above result again to construct

$$\alpha_2 : (\tau,\infty)^{m-2} \times (\tau,\infty) \times \Omega \to \mathbf{R}\,,$$

where α_2 is a $\mathcal{B}((\tau,\infty))^{m-2} \times \mathcal{P}$-measurable function such that $\alpha_2(s_3,\cdots,s_m;\cdot;\cdot)$ is indistinguishable from

$$\int_{\tau+}^{\cdot}\int_{D} \varphi_2(s2,\cdots,s_m;y_2)\alpha_1(s_2,\cdots,s_m;s_2)dM(s_2,y_2)$$

for any $s_2,\cdots,s_m$.

3. Continuing in this way, we can construct successively $\alpha_3,\cdots,\alpha_m$. We write $I_m(\varphi_1,\cdots,\varphi_m;t) = \alpha_m(t)$. For $m \geq 1$, $\mathcal{I}_m$ denotes the set of all random variables of the form $I_m(\varphi_1,\cdots,\varphi_m;t)$ for $\varphi_1,\cdots,\varphi_m$, and $t > \tau$. Put

$$\mathcal{I} := \mathbf{R} \bigcup \left\{ \bigcup_{m=1}^{\infty} \mathcal{I}_m \right\}.$$

4. This is nothing but an attempt at giving meaning to multiple stochastic integrals with respect to M. We can regard $I_m(\varphi_1,\cdots,\varphi_m;t)$ as an interpretation of the notation:

$$\underset{\substack{\tau<s_1<\cdots<s_m<t \\ D\times\cdots(m)\cdots\times D}}{\int\int\cdots(m)\cdots\int\int} \prod_{i=1}^{m} \varphi_i(s_i,\cdots,s_m;y_i)dM(s_1,y_1)\cdots dM(s_m,y_m)$$

Thus, the linear span of $\mathcal{I}_m$ is analogous to the m-th Wiener chaos.

5. As a matter of fact, this analogy cannot be complete, because $\mathcal{I}_k$ is not orthogonal to $\mathcal{I}_l$ in $L^2(\mathbf{P})$ if $k \neq l$.

6. To see this, let us take $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$, for example. In addition, assume that φ_1, φ_2 are both constant functions taking the value 1, for simplicity. Then we readily get

$$\mathbf{P}\,[I_1(\varphi_1;t)\cdot I_2(\varphi_1,\varphi_2;t)] = \mathbf{P}\left[\int_{\tau+}^{t} H_s(D)\{H_s(D)-H_\tau(D)\}ds\right] \neq 0.$$

The last inequality is due to the moment estimate by E.B. Dynkin(1988). So that, we cannot hope for a full analogue of the Itô-Wiener chaos expansion for this generalized stochastic integrals.

§1. Notation and the Result

Let $(E,\mathcal{E})$ be a Polish space and let $D(\mathbf{R}_+;E)$ be the space of E-valued cadlag paths on $[0,\infty)$ and we sometimes write this space as D or $D(E)$ for simplicity. Note that D is a Polish space as well (cf. §2.1, p.13 in [DP91]). We denote by $M(D)$ or $M_F(D)$ the space of finite measures on D with the topology of weak convergence. $\langle\mu,f\rangle$ or sometimes $\mu(f)$ denotes the integral $\int f d\mu$ when μ is a measure and f is a suitable μ-integrable function. Set $T_s = [s,\infty)$, and in particular $T_0 = [\tau,\infty)$. Define $C(M(D)) := C(T_0;M(D))$, and we write $D(t) = (\tau,t]\times D$ for the integral domain. When $\mathcal{F}$ is the σ-field or the usual filtration, then $f\in\mathcal{F}$ indicates that the function f is $\mathcal{F}$-measurable and $\mathcal{P}(\mathcal{F})$ is the totality of $(\mathcal{F})$-predictable functions, and $b\mathcal{P}(\mathcal{F})$ denotes the whole space of functions that are all bounded elements of $\mathcal{P}(\mathcal{F})$. We use the symbol $U(M(D))$ for an admissible subset of the space $C(C(M(D));\mathbf{R})$; more precisely $U(M(D))$ is the totality of real valued continuous functions F on $C(M(D))$ such that for some compactly supported finite measure $L(dt)$ on T_0, the estimate

$$|\Delta F(h,g)| \leq \langle L, g(\cdot,D)\rangle$$

holds for all $f,g\in C(M(D))$, where we define $\Delta f(x,y) := F(x+y)-F(x)$.

$Y = (D,\mathcal{D},\mathcal{D}_{t+},\theta_t,Y_t,P_x)$ is the canonical realization of the Hunt process. Let $\Phi(x,\lambda)$ denote the branching mechanism for the corresponding superprocess, namely,

$$\Phi(x,\lambda) = -\alpha(x)\lambda - \gamma(x)\lambda^{1+\beta} + \delta\int_0^\infty (1-e^{-\lambda u})\nu(x,du), \quad \lambda\in\mathbf{R}; \quad \beta\in(0,1]$$

with a measurable kernel $\nu(x,du)$ from $(E,\mathcal{E})$ to $(\mathbf{R}^+,\mathcal{B}(\mathbf{R}^+))$ such that

$$\sup_x \int_0^\infty u\wedge u^2\,\nu(x,du) < \infty.$$

Let $H = (\Omega,\mathcal{G},\mathcal{G}_{t+}, H_t,Q_m)$ denote a (Y,Φ)-historical process in the sense of Dawson-Perkins(1991) (cf. [DP91]) with $\Omega = D(M(D))$ (see also [F88], [DIP89]). We may call it a $(\alpha,\beta,\gamma,\delta)$-historical process as well in what follows. On the other hand, suggested by

[DkTn98] (see also [P95]) on a filtered space $\bar{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq\tau}, \mathbf{P})$ we introduce a generalized $\{\gamma, a, b, g\}$-historical process $K = \{K_t, t \geq \tau\}$ with generator A of the corresponding path-valued process Y^s(cf.[P92]). $L^2(H)$ denotes the L^2 space of $(\mathcal{D} \times \mathcal{G}_t)_{t\geq\tau}$-predictable functions $f : (\tau, \infty) \times D \times \Omega \to \mathbf{R}$ with respect to $ds\otimes\ dH_s\otimes\ dQ_m$. Moreover, $L^2(K)$ denotes the totality of $(\mathcal{F}_t \times \mathcal{C})_{t\geq\tau}$-predictable functions $f : (\tau, \infty) \times \Omega \times C \to \mathbf{R}$ such that

$$\mathbf{P} \int_\tau^t \int \gamma(s,y) f(s,y)^2 K_s(dy) ds < \infty, \quad \text{for } t \geq \tau.$$

It is well known that there exists an orthogonal martingale measure $\tilde{M}$ in the sense of Walsh(1986) [W86] such that the stochastic integral with respect to $\tilde{M}$

$$\int_{\tau+}^t \int_C f(s,y) d\tilde{M}(s,y)$$

is well-defined and belongs to the class $M_c^{2,loc}(\mathcal{F}_t)$ of square integrable continuous $(\mathcal{F}_t)$-local martingales under the measure $\mathbf{P}$ for each element f of $L^2(K)$. We denote by M the corresponding martingale measure for the element of $L^2(H)$, and $\int\int f dM$ is contained in $M_c^2(\mathcal{G}_t)$. Then notice that

$$\left\langle \int_{\tau+}^{\cdot} \int_C f(s,y) d\tilde{M}(s,y) \right\rangle_t = \int_{\tau+}^t \int_C \gamma(s,\omega,y) f(s,\omega,y)^2 K_s(dy) ds \quad \forall t, \quad a.s. \tag{1}$$

holds, where γ is a $(\mathcal{C}_t \times \mathcal{F}_t)$-predictable process such that $(\exists)\gamma^{-1}$ is locally bounded.

For stopped paths and related measures, we adopt the same notational system and terminology as in [P95]. For $y \in D$, we define $y^{t-}(s)$ as $y(s)$ itself if $s < t$ and as $y(t-)$ if $s \geq t$. $Q(s,y)$ is a σ-finite measure on $C(M(D))$ such that

$$Q\big(s, y^{s-};\ \{h \in C(M(D));\ \ \tau \leq \exists t \leq s,\ \ h(t) \neq 0\}\big) = 0,$$

which can be defined by the canonical measure $R(\tau, t, y; d\zeta)$ associated with the law of $K_t = K(t)$ and the path restriction mapping π (cf. §2, pp.1781-1782 in [EP95]) together with a discussion involved with the Dawson-Perkins theory(1991) (e.g. Theorem 2.2.3(pp.27-28) and Proposition 3.3(pp.38-39) in [DP91]). Let F be a real valued Borel function on $C(M(D))$. Assume that

$$I[F](s,y,h) := \int_{C(M(D))} \Delta F(h,g) Q(s, y^{s-}; dg) \tag{2}$$

is well-defined and bounded below for all $s > \tau$, $y \in D$, and $h \in C(M(D))$. For a bounded $(\mathcal{F}_t)$-stopping time T, we define the Campbell measure P_T associated with $K(t)$ by

$$P_T(A \times B) := \mathbf{P}\ (K(T,A) \cdot \mathbf{I}_B\{K(T)\})/m(C) \tag{3}$$

for any $A\times B \in (D\times\Omega, \mathcal{D}\times\mathcal{F})$ (cf. [P95], p.21; or [DP91], p.62). Notice that $K_\tau = m$. Since the mapping $(s,y,\omega) \mapsto I[F](s,y,K(\omega))$ is bounded below and measurable with respect to the product of the predictable σ-field associated with the filtration $(\mathcal{D}_t)$ and the σ-field $\mathcal{F}$, we can apply Lemma 2.2(p.1783) [EP95] together with the projection operation argument and the predictable section theorem (e.g. Theorem 2.14(p.19) or Theorem 2.28(p.23), [JS87]; see also [E82], pp.50-52), to deduce that there exists a $(\mathcal{D}_t \times \mathcal{F}_t)_{t\geq\tau}$-predictable function $I^{\#}[F](s,y,\omega) : (\tau,\infty) \times D \times \Omega \to \mathbf{R}$ such that

$$P_T\{I[F](T)/(\mathcal{D}\times\mathcal{F})_T\} = I^{\#}[F](T)$$

holds P_T-a.s. for all bounded $(\mathcal{F}_t)$-predictable stopping times $T > s$. We denote by $I^*[F]$ the $(\mathcal{D}_t \times \mathcal{G}_t)$-predictable function constructed by the same procedure, with the σ-finite measure $Q(s,y)$ based upon a (Y,Φ)-historical process H. It is quite interesting to note that in particular

$$\mathbf{P}\int_D I[F](T,y)K(T,dy) = \mathbf{P}\int_D I^{\#}[F](T,y)K(T,dy).$$

We shall introduce an approximation map. For each $l \in \mathbf{N}$, let us choose a partition $\Delta(l) = \{t^{(l)}(j);\ 1 \leq j \leq k[l]\}$ such that $\tau = t^{(l)}(0) < t^{(l)}(1) < \cdots < t^{(l)}(k[l]) < \infty$,

$$\lim_{l\to\infty}\{\sup_k \Delta t[l;k]\} = 0 \quad \text{and} \quad \lim_{l\to\infty} t^{(l)}(k[l]) = +\infty.$$

The approximation map $W[l]$ from $C(M(D))$ into $C(M(D))$ is defined by

$$W[l](g)(t) := \{Sb(t^{(l)}(i+1))\cdot g(t^{(l)}(i)) - Sb(t^{(l)}(i))\cdot g(t^{(l)}(i+1))\}\Delta t[l;i]^{-1}$$

if $t \in [t^{(l)}(i), t^{(l)}(i+1))$, and $:= g(t^{(l)}(k[l]))$ if $t \geq t^{(l)}(k[l])$, for any element g of $C(M(D))$ with $Sb(k) = k - t$. Immediately we get

Lemma 1.(cf. Lemma 4 [DK98a]) *Let F be an element of $C(C(M(D));\ \mathbf{R})$. Then for all $g \in C(M(D))$*

$$\lim_{l\to\infty}(F\circ W[l])(g) = F(g).$$

We are now in a position to state Evans-Perkins' stochastic integral formulae, which provide with the proto-type of our extended result. The following theorem asserts that a finite-dimensional version of stochastic integration by parts formula holds when one rewrites an expression of historical functionals of a specific class and stochastic integral relative to the orthogonal martingale measure M in the Walsh sense [W86] into a certain stochastic integral with respect to a Dawson-Perkins historical process H associated with a reference Hunt process Y.

Theorem 1.([EP95]) *Assume that $\Phi : C(M(D)) \to \mathbf{R}$ is a cylinder function with representing function $\varphi : [M(D)]^k \to \mathbf{R}$ and base $\tau < t(1) < \cdots < t(k)$, such that*

$$|\Delta\varphi(\alpha,\beta)| \leq C\sum_j \beta_j(D)$$

for some positive constant C, for all α, $\beta = (\beta_j) \in [M(D)]^k$. For all $t > \tau$ we have the following integration formula

$$\mathbf{P}\left\{\Phi(H) \int\int_{D(t)} \Psi(s,y)\, dM(s,y)\right\} = \mathbf{P} \int\int_{D(t)} I^*[\Phi](s,y)\Psi(s,y)\, H_s(dy)ds$$

with $\Phi = F \circ W[l]$, if Ψ is a bounded $\mathcal{D}_t \times \mathcal{H}_t$-predictable function.

Theorem 2. (Evans-Perkins' Formula(1995)) *Let $F \in U(M(D))$. If Ξ is an element of $b\mathcal{P}(\mathcal{D}_t \times \mathcal{G}_t)$, then for all $t > s$,*

$$Q_m\left\{F(H) \int\int_{D(t)} \Xi(s,y) \quad dM(s,y)\right\} = Q_m \int\int_{D(t)} I^*[F](s,y)\Xi(s,y)H_s(dy)ds. \qquad (4)$$

The following is our main result in this paper. It is a finite dimensional version of Evans-Perkins type stochastic integral formula. It is also quite interesting to note that this formula can be naturally regarded as a variant of stochastic integration by parts formula in stochastic analysis for measure-valued processes. Note that K is a predictable measure-valued process whose law is specified by a general martingale problem (MP)$[\tau, K_\tau, \gamma, a, b, g]$ (cf. [DkTn98],[P92]; see also [Dk98a]). We postpone explanation of assumptions (A.1)-(A.5) in the following theorem until §4.

Theorem 3.(Stochastic Integration Formula) *Let Φ be the same cylinder function with representing function φ as in* Theorem 1. *Then under assumptions* (A.1)-(A.5), *for $t > \tau$*

$$\mathbf{P}\left\{\Phi(K) \int\int_{D(t)} \Psi(s,y)d\tilde{M}(s,y)\right\} = \mathbf{P} \int\int_{D(t)} I^{\#}[\Phi](s,y)\Psi(s,y)\gamma(s,y)K_s(dy)ds$$

holds where Ψ is a bounded $(\mathcal{D}_t \times \mathcal{F}_t)_{t\geq\tau}$-predictable function, K_t is a generalized historical process, and $I^{\#}[\Phi]$ is a predictable function determined in accordance with the given Φ.

Remark 1. The assertion of the above theorem is quite similar to Theorem 2.4(p.1785, §2, [EP95]). However, our stochastic integration by parts formula is valid for a more general historical process K, while Evans-Perkins showed the formula(Theorem 1 and Theorem 2) just for a $(Y, -\lambda^2/2)$ historical process H, say, for a simple case of $(\alpha, \beta, \gamma, \delta) = (0, 1, 1/2, 0)$.
Remark 2. Note that it is not hard to extend the assertion in Theorem 3 to the case of a more general functional $F(K)$, just as described in Theorem 2 for the special process H. As a matter of fact, once the integral formula as given in Theorem 3 is established, it is a kind of routine work to generalize it(cf. §3, [Dk98a]). We shall refer to the matter in §3.

§2. Preliminaries

Set $I = [0,1]$, $E^* = D \times I$ and $D^* = D(\mathbf{R}_+, E^*)$, and let $\mathcal{D}^*$ (resp. $\mathcal{D}_t^*$) be the Borel σ-field (resp. the canonical filtration) of D^*. Now $X = (D^*, \mathcal{D}^*, \mathcal{D}_t^*, X_t, P_x^*)$ denotes the inhomogeneous Borel strong Markov process (IBSMP) [DP91, p.22] with cadlag paths and $x = (y, n) \in E^*$. Let G be an (X^s, A^*) historical process starting at (τ, μ), defined on the stochastic basis $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbf{P}^*)$. Suppose that $\varphi : (\tau, \infty) \times D \times \Omega \to I$ be an element of $\mathcal{P}(\mathcal{D}_t \times \mathcal{H}_t)$. Given any cadlag function $n : \mathbf{R}_+ \to I$, we can construct a σ-finite counting measure n^* on $\mathbf{R}_+ \times I$ by assigning an atom of mass one to each point (s, z) such that $n(s) - n(s-) = z \neq 0$. Put

$$A(t,x,\omega) := n^*(\{(s,z) \in [\tau, t) \times I; \quad \varphi(s,y,\omega) > z\}) \tag{5}$$

and $B(t,x,\omega) = \mathbf{I}\{A(t,x,\omega) = 0\}$. Then we can define an $M_F(D)$-valued process $K[\varphi](t)$ by

$$K[\varphi; J](t) := \int_{D^*} \mathbf{I}\{J\}(y) B(t,x) G_t(dx). \tag{6}$$

Put

$$I_1(\varphi, N) = \int \int_{D^*(t)} \varphi(s,y) dN(s,x)$$

and

$$I_2(\varphi, G) = \int \int_{D^*(t)} \gamma(s,y) \varphi(s,y)^2 G_s(dx) ds$$

with $D^*(t) = (\tau, t] \times D^*$. Then we define

$$\Lambda[\varphi](t) := \exp\Big\{ I_1(\varphi, N) - \frac{1}{2} I_2(\varphi, G) \Big\}. \tag{7}$$

Note that $\Lambda[\varphi](t)$ is a $\mathcal{H}_t$-martingale [EP95, p.1798]. The new probability space $(\Omega, \mathcal{H}, \mathbf{P}^*[\varphi])$ is defined by $\mathbf{P}^*[\varphi]\{F\} := \mathbf{P}^*\{F \cdot \Lambda[\varphi](t)\}$ (cf. [Dk98a]; see also [Dk97b]) for any $F \in b\mathcal{H}_t$ with

$$\mathcal{H} := \bigvee_{t \geq \tau} \mathcal{H}_t \tag{8}$$

(see Theorem 2.1(pp.125-126) and Theorem 2.3b(p.127), [EP94]). According to [DP91] and suggested by [EP95], we introduce the following notation. For an IBSM process X, we can define $P_{\tau,\mu}^* \equiv P^*[\tau, \mu]$ as

$$P^*[\tau,\mu](A) := \int_{D^*} P_x^*\{(x/\tau/X) \in A\} \mu(dx) \tag{9}$$

for any $A \in \mathcal{D}^*$, $\mu \in M(D^*)^\tau$. Generally,

$$M(C)^t := \{m \in M_F(C); \ y = y^t, \quad m - a.s. \ \ y \in C\}, \quad \forall t.$$

Also we define

$$\bar{\mathcal{D}}_t^* := \mathcal{D}_{t+}^* \bigvee \{P^*[\tau, \mu] - \text{null subsets in } D^*\},$$

and F^* denotes the space of $f \in b(\mathcal{B}(T_0) \times \mathcal{D}^*)$ such that $f(t,x) = f(t,x^t)$, for all $t \geq \tau$ and the mapping: $T_0 \in t \mapsto f(t,X) \in \mathbf{R}$ is $P^*[\tau,\mu]$- a.s. right continuous for all $t \geq \tau$. Let A^* denote the totality of $(\varphi, \psi) \in F^* \times F^*$ such that

$$\Phi(t,X) := \varphi(t,X) - \varphi(\tau,X) - \int_{\tau+}^{t} \psi(s,X)ds$$

is a $\bar{\mathcal{D}}_t^*$-martingale under $P^*[\tau,\mu]$ for all $t \geq \tau$. Furthermore we assume from now on that the (X^s, A^*) historical process G (resp. the (Y^s, A) historical process K) are defined on the filtered probability space $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbf{P})$ (resp. $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$) and also that

$$\mathcal{F}_t := \bigcap_{s>t} \sigma\{K_r;\ \ \tau \leq r \leq s\} \bigvee \sigma\{\mathbf{P} \ - \text{null sets}\},$$

for $t \geq \tau$, and $\mathcal{F} = \bigvee\{\mathcal{F}_t;\ t \geq \tau\}$ denotes the minimal σ-algebra generated by $\{\mathcal{F}_t;\ t \geq \tau\}$ and

$$K(t,A) = G(t, \{x = (y,n) \in D^*;\ \ y \in A\}),$$

holds for all $A \in \mathcal{D}$. We may further assume that K is also a (Y^s, A) historical process on $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbf{P})$. Then we have

Proposition 4.(cf. Lemma 4.5, p.1794, [EP95]) *Let T be a $(\mathcal{D}_t^* \times \mathcal{H}_t)$-stopping time such that*

$$[T] \subset \bigcup_m [U_m] \times \Omega$$

where $\{U_m\}$ is a countable collection of $\mathcal{D}_{t+}^*$-stopping times.

(a) *Then*

$$\int_{D^*} \Phi(t \wedge T(x), x) G_t(dx) = \int \int_{D^*(t)} \Phi(s \wedge T(x), x) dN(s,x)$$

holds $\mathbf{P}$ -a.s. *for any $t \geq \tau$.*

(b) *The both sides in the above equality belong to M_c^2.*

It is easy to show the following proposition if we apply Proposition 4 by making use of Dawson's Girsanov theorem [D93] (see also [P95]).

Proposition 5.(cf. Theorem 5.1, p.1798, [EP95]) *The law of $K[\varphi]$ under $\mathbf{P}$ $[\varphi]$ is equivalent to the law of K under $\mathbf{P}$.*

§3. Generalization of the Cylinder Function Case

As mentioned in Remark 2 of §1, the essential part of an extension of the Evans-Perkins type integration formula is compressed into the study on its finite dimensional case, namely, Theorem 3. The general case easily follows from a kind of routine work. So we shall only take a short excursion to this topics in accordance with [Dk98a]. We define a real valued function L^* on $C(M(D))$ by

$$L^*[g] := \int_{T_0} g(t,D)L(dt) = \langle L, g(\cdot, D)\rangle. \tag{10}$$

If $F \in U(M(D))$, then from definition (cf. §1) notice that $|\Delta F(f,g)| \leq L^*[g]$ holds for all $f, g \in C(M(D))$. By assumption of Theorem 2 we can easily obtain

Lemma 2. *For all $f, g \in C(M(D))$, we have $|\Delta(F \circ W[l])(f,g)| \leq (L^* \circ W[l])[g]$.*

In connection with the measure L (see §1), we introduce the finite measure $L(l) \equiv L(l, dt)$ which concentrates its mass on $\{t^{(l)}(j);\ 0 \leq j \leq k[l]\}$ (cf. [Dk98a, p.5]). We have $(L^* \circ W[l])[g] = \langle L(l), g(\cdot, D)\rangle$ for $g \in C(M(D))$. Moreover,

$$\lim_{l\to\infty} L(l, (s, +\infty)) = L((s, +\infty)), \qquad \lim_{l\to\infty} (L^* \circ W[l])[f] = L^*[f]$$

holds for all but countably many $s > \tau$ and for any $f \in C(M(D))$. Recall that the following lemma holds with ease for $s < t$ from Lemma 3.4(pp.41-43), [DP91].

Lemma 3. *The following relations hold:*

$$\int g(t, D)\, Q(s, y; dg) = \int \xi(D)\, R(s, t, y; d\xi) = 1.$$

The following Lemma 4 is a companion result (with the similar type equality in §1) directly derived from Lemma 2.2 [EP95] if we repeat the same argument of projection operation and predictable section theorem, which has been stated in §1. That is to say,

Lemma 4. *There exists a bounded $\mathcal{D}_t \times \mathcal{F}_t$-predictable function $I^*[F \circ W[l]](s, y, \omega)$ such that*

$$\mathbf{P} \int_D I[F \circ W[l]](T, y)\, K(T, dy) = \mathbf{P} \int_D I^{\#}[F \circ W[l]](T, y)\, K(T, dy)$$

holds for all bounded $\mathcal{F}_t$-predictable stopping times T.

We introduce now the first important result in derivation for the general case.

Proposition 6. *The equality*

$$\mathbf{P} \int\int_{D(t)} \{Q(s, y^{s-}) L^*[g]\} \qquad K_s(dy) ds$$
$$= \lim_{l\to\infty} \mathbf{P} \int\int_{D(t)} \{Q(s, y^{s-})(L^* \circ W[l])[g]\}\, K_s(dy) ds$$

holds with $g \in C(M(D))$ for all $t > \tau$.

It is quite interesting to note that Dawson's Girsanov type theory stated in §2 remains even valid if we replace φ by Ψ appearing in the statement of Theorem 3. As a matter of fact, an $M_F(D)$-valued process $K[\Psi](t)$ is well-defined, $\Lambda[\Psi](t)$ is a $\mathcal{H}_t$-martingale, and the probability measure $\mathbf{P}\ [\Psi](\cdot)$ is well-defined as well. In addition, note that Proposition 5 in §2 says that the law of $K[\Psi]$ under $\mathbf{P}\ [\Psi]$ is the same as that of K under $\mathbf{P}$. The next proposition is one of the most important assertions in this section.

Proposition 7. *For all $t > \tau$, if $Z \in \mathcal{P}(\mathcal{D}_t \times \mathcal{F}_t)$, then*

$$\mathbf{P} \int\int_{D(t)} I^{\#}[F](s, y) Z(s, y) \qquad K_s(dy) ds$$
$$= \lim_{l\to\infty} \mathbf{P} \int\int_{D(t)} I^{\#}[F \circ W[l]](s, y) Z(s, y)\, K_s(dy) ds. \tag{11}$$

In order to get the above, we have only to apply the previous result Proposition 6 together with Lemma 4 by employing the Fubini theorem and a variant of dominated convergence theorem of Lebesgue type.

It is well known that for each $n \geq 1$, $\mathbf{P}\ \{K_t(D)^n\}$ is uniformly bounded on compact intervals. On this account, we can deduce the next assertion by taking it into consideration that L has a compact support, i.e.,

Lemma 5. *For each* $n \geq 1$, $\mathbf{P}\ \{(L^* \circ W[l])[K]^n\}$ *is bounded in* l.

Another direct result by the aforementioned well known fact is: for all $t > \tau$, the stochastic integral

$$\int\int_{D(t)} \varphi(s,y)\, d\tilde{M}(s,y)$$

has moments of all orders if $\varphi \in b\mathcal{P}(\mathcal{D}_t \times \mathcal{F}_t)$. The following assertion is a simple result from the aforementioned result and Lemma 5. That is, for all $t > \tau$

$$\mathbf{P}\left|\big(L^* \circ W[l]\big)[K] \int\int_{D(t)} \Psi(s,y)\, d\tilde{M}(s,y)\right|^2 \tag{12}$$

is bounded in l. As we have $\left|\big(F \circ W[l]\big)(K)\right| \leq |F(0)| + \big(L^* \circ W[l]\big)[K]$ from Lemma 2 (with setting $g = 0$) and $\lim_{l\to\infty}\big(F \circ W[l]\big)(K) = F(K)$ from Lemma 1, a uniform integrability argument (e.g. [E82],[JS87]) together with (12) shows:

Proposition 8. *For all* $t > \tau$

$$\mathbf{P}\left\{F(K) \int\int_{D(t)} \Psi(s,y)\, d\tilde{M}(s,y)\right\} = \lim_{l\to\infty} \mathbf{P}\left\{\big(F \circ W[l]\big)(K) \int\int_{D(t)} \Psi(s,y)\, d\tilde{M}(s,y)\right\}.$$

To complete the extension discussion in this section we have only to observe that $F \circ W[l]$ satisfies all the conditions of the main result in this article, say, Theorem 3 (cf. Lemma 22, pp.9-10, [Dk98a]). Thus we have a finite dimensional special case of stochastic integration by parts formula related to historical processes as far as Proposition 4 and Proposition 5 stated in §2 are both valid. Hence, an application of Theorem 3 with Proposition 8 leads to

$$\begin{aligned}
\mathbf{P}\left\{F(K) \int\int_{D(t)} \Psi(s,y)\, d\tilde{M}\right\} &= \lim_{l\to\infty} \mathbf{P}\left\{\big(F \circ W[l]\big)(K) \int\int_{D(t)} \Psi(s,y)\, d\tilde{M}\right\} \\
&= \lim_{l\to\infty} \mathbf{P} \int\int_{D(t)} I^{\#}[F \circ W[l]]\ \gamma(s,y)\Psi(s,y)\, K_s(dy)ds \\
&= \mathbf{P} \int\int_{D(t)} I^{\#}[F](s,y)\ \gamma(s,y)\Psi(s,y)\, K_s(dy)ds, \qquad (13)
\end{aligned}$$

because in the last equality we employed Proposition 7. Thus we attain

Theorem 9. $\mathbf{P}\ [F(K) \int\int \Psi(s,y)d\tilde{M}] = \mathbf{P}\ \int\int I^{\#}[F]\gamma(s,y)\Psi(s,y)K_s(dy)ds.$

§4. Assumptions and Sketch of Proof of the Main Theorem

We shall give a rough sketch of the proof of our main result (Theorem 3) which is stated in §1. Since the space is limited, computation in details will be sacrificed for the sake of simplicity and clearness. D^t denotes the image of D under the map: $y \mapsto y^t$. We define a measure $K^*[s,t]$ on D^s by $K^*[s,t](F) := K_t(\{y : y^s \in F\})$. Then the measure $K^*[s,t]$ is atomic with a finite set of atoms, and we write $L[s,t](\subset D^s)$ for the locations of these atoms. For $s \in (a,b]$, let $\lambda_s[\varphi]$ be the random measure on D that places mass $\varphi(s,y)$ at each point y in $(L[b,c])^s = L[s,c]$. On the other hand, let $\{T_N\}$ be a reducing sequence [Tn98](see also [P95]). With some localization arguments in stochastic calculus, the Perkins-Girsanov theorem of Dawson type guarantees the existence of a probability measure $\mathbf{Q}_N$ on $(\Omega, \mathcal{F})$ such that

$$\left.\frac{d\,\mathbf{Q}_N}{d\,\mathbf{P}}\right|_{\mathcal{F}_t} = \exp\left\{ \int_\tau^{t\wedge T_N}\int g\gamma^{-1}(s)\ \mathbf{I}\ (g(s)\neq 0)d\tilde{M}(s,y) - \frac{1}{2}\int_\tau^{t\wedge T_N}\int g^2\gamma^{-1}(s)\ \mathbf{I}\ (g(s)\neq 0)K_s(dy)ds\right\}.$$

For brevity's sake we rather write $\mathcal{E}(t\wedge T_N)$ than the above. On this account, $K_{\cdot\wedge T_N}$ satisfies the martingale problem (MP)$[\gamma_N, a_N, b_N, 0]$ instead of (MP)$[\gamma, a, b, g]$, where we set $f_N := f\cdot\ \mathbf{I}\ (\tau < t \le T_N)$. Moreover, for $s \in (a,b]$, $y \in D^s$, the symbol $\mathcal{M}[s,y]$ denotes the mapping of the set of functions $\{m : (\tau,\infty) \to M_F(D)\}$ into itself and is defined as follows: i.e., $\{\mathcal{M}[s,y]m\}_t(F)$ is equal to $m_t(F)$ if $t < s$, or is equal to $m_t(\{y' \in F : (y')^s \neq y\})$ if $t \ge s$.

Let us now introduce our assumptions for the principal result, say, Theorem 3.

(A.1) $g : [\tau,\infty)\times\Omega\times C \to \mathbf{R}$ is a $(\mathcal{F}_t\times\mathcal{C}_t)^*$-predictable process such that $g\gamma^{-1}\cdot\ \mathbf{I}\ (g\neq 0)$ is locally bounded.

(A.2) For any predictable function f on $[\tau,\infty)\times I\times D^*\times\Omega$, the counting measure n^* satisfies

$$\int_{D^*}\int\int_{(\tau,t]\times I} f(s,z,x)n^*(ds\otimes dz)G_t(dx)$$
$$= \int_{\tau+}^t\int_{D^*}\left(\int_I f(s,z,x)dz\right)G_s(dx)ds + \int\int_{D^*(t)}\left(\int\int_{(\tau,s]\times I} f(u,z,x)n^*(du\otimes dz)\right)dN(s,x).$$

(A.3) There exists a random measure Λ_φ on $(\tau,\infty)\times D$ such that

$$\int\int_{D(\infty)} f(s,y)\Lambda_\varphi(ds\otimes dy) = \int_{a+}^b\int_D f(s,y)\lambda_s[\varphi](dy)ds$$

holds for any suitable predictable function f.

(A.4) $\Psi(s,y)\mathcal{E}(t\wedge T_N)^{-1}$ is uniformly bounded in s, K_s-a.e. y, $\mathbf{Q}_N$-a.s.

(A.5) For $\varepsilon > 0$ we have

$$\mathbf{Q}_N[F(K[\varepsilon\varphi]) - F(K)\ /\ \mathcal{F}]$$
$$= \varepsilon\cdot e^{-\varepsilon\Lambda_\varphi((\tau,\infty)\times D)}\int\int_{D(\infty)}\{F(\mathcal{M}[s,y]K) - F(K)\}\Lambda_\varphi(ds\otimes dy) + R(\varepsilon, F, \varphi)$$

where the residue function R satisfies $|R(\varepsilon, F, \varphi)| \leq o(\varepsilon)$.

Thanks to (A.1) we can resort to the Perkins-Girsanov theorem to reduce it to a simpler case. That is, it is sufficient to verify the integral formula for a special $\{\gamma_N, a_N, b_n, 0\}$-historical process $K_{\cdot\wedge T_N}$ under $\mathbf{Q}_N$ instead of the generalized K with $\mathbf{P}$. Indeed what we have to show is

$$\mathbf{Q}_N\left\{\Phi(K_{\cdot\wedge T_N})\int\int \Psi\cdot\mathcal{E}(s\wedge T_N)^{-1}d\tilde{M}\right\} = \mathbf{Q}_N\int\int I^{\#}[\Phi]\gamma\Psi\cdot\mathcal{E}(s\wedge T_N)^{-1}dK_s ds.$$

Both sides above are well-defined by virtue of (A.4). Furthermore, $\varphi = \Psi\cdot\mathcal{E}^{-1}$ is applicable to (5)-(7) in §2. Hence, by the auguments in §2, $\Lambda[\Psi\cdot\mathcal{E}^{-1}](t)$ is a $\mathcal{H}_t$-martingale and the measure $\mathbf{Q}_N[\Psi\cdot\mathcal{E}^{-1}]$ is given by $\mathbf{Q}_N[\{\cdot\}\Lambda[\Psi\cdot\mathcal{E}^{-1}]]$. Then it follows from Proposition 5 that the law of $K_{\cdot\wedge T_N}[\Psi\cdot\mathcal{E}^{-1}]$ under $\mathbf{Q}_N[\Psi\cdot\mathcal{E}^{-1}]$ is equivalent to that of $K_{\cdot\wedge T_N}$ under $\mathbf{Q}_N$, which implies that

$$\mathbf{Q}_N\{\Phi(K_{\cdot\wedge T_N})\} = \mathbf{Q}_N[\varepsilon\Psi\mathcal{E}^{-1}]\Big\{\Phi(K_{\cdot\wedge T_N}[\varepsilon\Psi\mathcal{E}^{-1}])\Big].$$

With an application of elementary stochastic calculus, this enables us to acquire further reduction, namely, a simple computation of limit. As a matter of fact, we have only to compute

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\,\mathbf{Q}_N\Big\{\Phi(K_{\cdot\wedge T_N}[\varepsilon\Psi\mathcal{E}^{-1}]) - \Phi(K_{\cdot\wedge T_N})\Big\}. \tag{14}$$

Clearly (A.5) works nicely for this calculation. While, knowledge of Campbell measure and cluster random measure, especially understanding of Poisson cluster representation is really helpful in proceeding the computation of (14), together with (A.2), (A.3) and (A.5). In fact, after longsome calculation and a little elaborate consideration of measure transformation, we observe

$$\begin{aligned}\mathbf{Q}_N\int\int\Big\{\Phi(\mathcal{M}[s,y]K_{\cdot\wedge T_N}) \;&-\; \Phi(K_{\cdot\wedge T_N})\Big\}\Lambda_{\Psi\cdot\mathcal{E}^{-1}}(ds\otimes dy)\\ &= \;-\,\mathbf{Q}_N\int\int I^{\#}[\Phi]\gamma\cdot\Psi\mathcal{E}^{-1}dK_{s\wedge T_N}ds.\end{aligned}$$

Consequently the integral formula is established by a limit procedure of another term:

$$\varepsilon^{-1}\,\mathbf{Q}_N[\Phi(K_{\cdot\wedge T_N})\cdot(\Lambda[\varepsilon\Psi\mathcal{E}^{-1}](t) - 1)].$$

The last computation requires uniform boundedness argument as well as convergence discussion of stochastic integral, which can be muddled through by assumption.

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