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Quantum Calogero models and the degenerate double affine Hecke algebra

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Abstract

Algebraic structure of Calogero-Sutherland-type models is reviewed. Each of the models is related to representations of a degenerate version of the double affine Hecke algebra. From this viewpoint, wavefunctions of the models can be treated in unified manner.

1 Introduction

There are intimate relations between quantum mechanics and special functions. Wavefunctions for some models can explicitly be written in terms of suitable special functions. Recent studies on integrable quantum many-particle systems reveal that wavefunctions of some special cases can be written in terms of multivariable analogue of classical orthogonal polynomials. Furthermore, in those cases, properties of the wavefunctions can be treated in unified manner by using their algebraic structure.

An example of such models is the Sutherland model, which describes interacting particle on a unit circle [Su1, Su2, OP]:

$$\mathcal{H}_{\mathrm{S}} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial \theta_j^2} + \frac{1}{2} \sum_{j<k} \frac{\beta(\beta-1)}{\sin^2[(\theta_j - \theta_k)/2]}, \tag{1.1}$$

where β is coupling constant and we assume β is a non-negative positive integer. In this case, wavefunctions can be written by the so-called Jack polynomials.

Another example is the quantum Calogero model confined in harmonic potential [Ca1, Ca2, Su1, OP]:

$$\mathcal{H}_A = \frac{1}{2} \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 \right) + \sum_{j<k} \frac{\beta(\beta-1)}{(x_j - x_k)^2}, \tag{1.2}$$

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The subscripts "A" signify that this Hamiltonian is invariant under the action of the Weyl group of A_{N-1}-type. There also exists the model invariant under the action of the B_N-type Weyl group [OP, Y]:

$$\mathcal{H}_B = \frac{1}{2}\sum_{j=1}^{N}\left\{-\frac{\partial^2}{\partial z_j^2} + z_j^2 + \frac{\gamma(\gamma-1)}{z_j^2}\right\} + \sum_{j<k}\left\{\frac{\beta(\beta-1)}{(z_j-z_k)^2} + \frac{\beta(\beta-1)}{(z_j+z_k)^2}\right\}. \tag{1.3}$$

(For the latter convenience, we use the letter "z" as the coordinates of the B_N-type model.) We remark that the model associated with the C_N-type Weyl group is equivalent to the B_N case, and D_N-type model is obtained by setting $\gamma = 0$.

In these cases, polynomial part of wavefunctions can be regarded as multivariable generalization of the Hermite (A_{N-1} case) and Laguerre (B_N case) polynomials and has been studied by several authors [BF1, BF2, BF3, vD, Ka1, Ka2, So, UW]. In fact, these three models share the same algebraic structure, the degenerate double affine Hecke algebra. From this viewpoint, each of the models corresponds to individual representation of the degenerate double affine Hecke algebra. Furthermore, by using the intertwining operators between representations of the degenerate double affine Hecke algebra, several known results on the Jack polynomials can be mapped directly to those of the multivariable Hermite and Laguerre polynomials. As applications, we will construct raising operators and shift operators for such polynomials.

2 Dunkl-type operators and multivariable orthogonal polynomials

2.1 Jack polynomials and the Sutherland model

In this subsection, we define our notation and review the theory of symmetric and non-symmetric Jack polynomials [M1, O, KS]. There are several ways to characterize the Jack polynomials; Here we define them as eigenfunctions of some operators. We note that we restrict ourselves to the case associated with the A_{N-1}-type Weyl group since we only use such case.

In the paper [Du1], Dunkl has introduced differential-exchange operators, now called "Dunkl operators", which are associated with root systems. For the A_{N-1}-type root system, the operators are defined as

$$D_j^A = \frac{\partial}{\partial x_j} + \beta \sum_{k(\neq j)} \frac{1-s_{jk}}{x_j - x_k} \qquad (j = 1, \dots, N),$$

where s_{ij} are elements of the symmetric group $\mathfrak{S}_N$. An element s_{ij} acts on functions of x_1, ..., x_N as an operator which permutes arguments x_i and x_j. We remark that the operators D_j preserve the space of polynomials of variables $x_1, \dots, x_N$ which we denote $\mathbb{C}[x]$. These operators satisfy the following properties:

$$\begin{aligned}
[D_i^A, D_j^A] &= 0, \qquad s_{ij}D_j^A = D_i^A s_{ij}, \qquad s_{ij}D_k^A = D_k^A s_{ij} \quad (k \neq i, j), \\
[D_i^A, x_j] &= \delta_{ij}\left(1 + \beta\sum_{k(\neq i)} s_{ik}\right) - (1-\delta_{ij})\beta s_{ij}.
\end{aligned}$$

Heckman introduced "global" Dunkl operators [He1], which are written as $x_j D_j^A$ in our notation. Heckman's operators do not commute each other. Cherednik introduced another version of Dunkl operators that mutually commute [Ch1] (see also [BGHP, KS]):

$$\hat{D}_j^A = x_j D_j^A + \beta \sum_{k(<j)} s_{jk}.$$

The algebra generated by the elements $x_j^{\pm 1}$, $\hat{D}_j^A$ and s_{ij} is isomorphic to the degenerate double affine Hecke algebra $\mathfrak{H}'$ associated with the A_{N-1}-type root system [Ch1, Ch2]. We remark that the elements $x_j^{\pm 1}$, D_j^A and s_{ij} also generate $\mathfrak{H}'$ since D_j^A and $\hat{D}_j^A$ are related through (2.1).

We denote by $\mathfrak{H}_0'$ subalgebra of $\mathfrak{H}'$ generated by $\hat{D}_j^A$ and s_{ij}, which is isomorphic to the degenerate affine Hecke algebra. We further denote by $\tilde{\mathfrak{H}}'$ subalgebra of $\mathfrak{H}'$ generated by x_j, $\hat{D}_j^A$ and s_{ij}. In terms of generators, the defining relations are

$$\begin{aligned}
&[\hat{D}_i^A, \hat{D}_j^A] = [x_i, x_j] = 0, \qquad x_i s_{ij} = s_{ij} x_j, \qquad x_i s_{jk} = s_{jk} x_i \quad (i \neq j, k),\\
&s_j^2 = 1, \qquad s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}, \qquad [s_i, s_j] = 0 \quad (|i-j| \neq 1),\\
&\hat{D}_{j+1}^A s_j - s_j \hat{D}_j^A = \beta, \qquad s_j \hat{D}_{j+1}^A - \hat{D}_j^A s_j = \beta, \qquad [s_i, \hat{D}_j^A] = 0 \quad (j \neq i, i+1),\\
&[\hat{D}_i^A, x_j] = \begin{cases} -\beta x_j s_{ij} & (i > j),\\ x_i + \beta \left(\sum_{k(<i)} x_k s_{ik} + \sum_{k(>i)} x_i s_{ik} \right) & (i = j),\\ -\beta x_i s_{ij} & (i < j), \end{cases}
\end{aligned}$$

where $s_j = s_{j,j+1}$ $(j = 1, \cdots, n-1)$ are the simple transpositions.

Since the operators $\hat{D}_j^A$ commute each other, they can be diagonalized simultaneously by suitable choice of bases of $\mathbb{C}[x]$ [BGHP, O, KS]. Such basis is called *non-symmetric Jack polynomials.* To define the non-symmetric Jack polynomials, we first introduce the ordering $\prec$; For two pairs (λ, w), (μ, w') where λ, μ are partitions and $w, w' \in \mathfrak{S}_N$, we define the ordering $\prec$ as follows:

$$(\mu, w') \prec (\lambda, w) \iff \begin{cases} \text{(i)} & \mu <_{\mathrm{D}} \lambda,\\ \text{(ii)} & \text{if } \mu = \lambda \text{ then } w' <_{\mathrm{B}} w, \end{cases}$$

where $<_{\mathrm{D}}$ is the dominance ordering for partitions [M1], and $<_{\mathrm{B}}$ is the Bruhat ordering for the elements of $\mathfrak{S}_N$ (see, for example, [Hu]).

Definition 2.1 ([BGHP, O, KS]) *An non-symmetric Jack polynomial $E_w^\lambda(x)$, labeled with the partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ and the element $w \in \mathfrak{S}_N$, is characterized by the following properties:*

(i) $E_w^\lambda(x) = x_w^\lambda + \sum_{(\mu,w') \prec (\lambda,w)} u_{ww'}^{\lambda\mu} x_{w'}^\mu$,

(ii) $E_w^\lambda(x)$ *is joint eigenfunction for the operators* $\hat{D}_j^A$,

where we have used the notation $x_w^\lambda = x_{w(1)}^{\lambda_1} \cdots x_{w(N)}^{\lambda_N}$.

We note that our definition of the non-symmetric Jack polynomials is slightly different from the one in the references cited above.

Since the action of $\hat{D}_j^A$ on monomials x_w^λ are given by

$$\hat{D}_j^A x_w^\lambda = (w(\lambda+\beta\delta))_j x_w^\lambda + \sum_{(\mu,w')\prec(\lambda,w)} u_{ww'}^{\lambda\mu} x_{w'}^\mu, \tag{2.1}$$

with $\delta = (N-1, N-2, \cdots, 0)$, the action of $\hat{D}_j^A$ on the non-symmetric Jack polynomials can be evaluated as follows [BGHP, O, KS]:

$$\hat{D}_j^A E_w^\lambda(x) = (w(\lambda+\beta\delta))_j E_w^\lambda(x). \tag{2.2}$$

From physical viewpoint, the operators $\hat{D}_j^A$ are related to the Sutherland model (1.1). To see the relation, we introduce "gauge-transformed" Hamiltonian $\widetilde{\mathcal{H}}_{\mathrm{S}}$:

$$\begin{aligned}\widetilde{\mathcal{H}}_{\mathrm{S}} &= \mathrm{Res}\left(\sum_{j=1}^N \left\{\hat{D}_j^A - \frac{\beta}{2}(N-1)\right\}^2\right)\\ &= \sum_{j=1}^N \left(x_j\frac{\partial}{\partial x_j}\right)^2 + \beta\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}\left(x_j\frac{\partial}{\partial x_j} - x_k\frac{\partial}{\partial x_k}\right) + \frac{\beta^2}{12}N(N^2-1),\end{aligned}$$

where Res X means that action of X is restricted to symmetric functions of the variables $x_1, \ldots, x_N$. If we make a kind of gauge transformation and a change of variables $x_j = \exp(i\theta_j)$, $\widetilde{\mathcal{H}}_{\mathrm{S}}$ reduces to the Sutherland Hamiltonian (1.1):

$$\phi_{\mathrm{S}}^{(\beta)} \circ \widetilde{\mathcal{H}}_{\mathrm{S}} \circ (\phi_{\mathrm{S}}^{(\beta)})^{-1} = \sum_{j=1}^N \left(x_j\frac{\partial}{\partial x_j}\right)^2 - \beta(\beta-1)\sum_{j<k}\frac{2x_jx_k}{(x_j-x_k)^2} = \mathcal{H}_{\mathrm{S}},$$

where $\phi_{\mathrm{S}}^{(\beta)}(x) = \prod_{j<k}|x_j-x_k|^\beta \prod_{j=1}^N x_j^{-\beta(N-1)/2}$ is the ground state wavefunction of the model. The symmetric Jack polynomials appear as polynomial part of wavefunctions for excited states.

Definition 2.2 ([M1]) *The symmetric Jack polynomials $J_\lambda^{(\beta)}(x)$ are characterized by the following properties:*

(i) $J_\lambda(x) = m_\lambda(x) + \sum_{\mu(<_{\mathrm{D}}\lambda)} u_{\lambda\mu} m_\mu(x)$,

(ii) $J_\lambda(x)$ *are eigenfunctions of the transformed Hamiltonian $\widetilde{\mathcal{H}}_S$,*

where m_λ are the monomial symmetric functions.

The symmetric Jack polynomials are obtained by symmetrizing E_w^λ, i.e.,

$$J_\lambda(x) = \frac{1}{\#\mathfrak{S}_N^\lambda}\prod_{v\in\mathfrak{S}_N} v(E_w^\lambda),$$

where $\mathfrak{S}_N^\lambda$ is a subgroup of $\mathfrak{S}_N$ that preserve λ. This relation follows form the fact that the right hand side satisfies both of the defining properties of the Jack polynomials.

As wavefunctions of the Hamiltonian $\mathcal{H}_{\mathrm{S}}$, the following scalar product is naturally introduced:

$$\langle f(x), g(x)\rangle_{\mathrm{J}}^{(\beta)} = \oint \cdots \oint f(x) g(x^{-1}) \phi_{\mathrm{S}}^{(\beta)}(x) \phi_{\mathrm{S}}^{(\beta)}(x^{-1}) \frac{\mathrm{d}x_1}{2\pi\mathrm{i}x_1} \cdots \frac{\mathrm{d}x_N}{2\pi\mathrm{i}x_N},$$

where the integration contour is the unit circle in the complex plane. This scalar product can alternatively be written as

$$\langle f(x), g(x)\rangle_{\mathrm{J}}^{(\beta)} = (-1)^{\beta N(N-1)/2} \left[f\bar{g}(\phi_{\mathrm{S}}^{(\beta)})^2 \right]_0, \tag{2.3}$$

where $[\,\cdot\,]_0$ stands for the constant term and $\bar{g} = g(x^{-1})$. By a direct calculation, we see that the operators $\hat{D}_j^A$ are self-adjoint with respect to the scalar product (2.3).

Proposition 2.3 ([M1]) *The Jack polynomials $J_\lambda(x)$ are pairwise orthogonal with respect to the scalar product (2.3).*

Proof. We first introduce generating function of symmetric commuting operators [BGHP, Ka1]:

$$\hat{\Delta}_{\mathrm{J}}(u) = \prod_{j=1}^{N} (u + \hat{D}_j^A).$$

If we expand $\hat{\Delta}_{\mathrm{J}}(u)$ as polynomial in u, the coefficients form a set of symmetric commuting operators which contains $\widetilde{\mathcal{H}}_{\mathrm{S}}$. Using (2.2), we can evaluate the action of $\hat{\Delta}_{\mathrm{J}}(u)$ on the Jack polynomials:

$$\hat{\Delta}_{\mathrm{J}}(u) J_\lambda^{(\beta)}(x) = \prod_{j=1}^{N} \{u + \lambda_{N-j+1} + \beta(j-1)\} J_\lambda^{(\beta)}(x). \tag{2.4}$$

Since all the eigenvalues of $\hat{\Delta}_{\mathrm{J}}(u)$ are distinct and the operator $\hat{\Delta}_{\mathrm{J}}(u)$ is self-adjoint, we conclude that the Jack polynomials $J_\lambda(x)$ are pairwise orthogonal with respect to the scalar product (2.3). □

The property below follows from the fact that the Jack polynomials form an orthogonal basis of the space of symmetric polynomials $\mathbb{C}[x]^{\mathfrak{S}_N}$:

$$\langle J_\lambda^{(\beta)}(x), m_\mu(x)\rangle_{\mathrm{J}}^{(\beta)} = 0 \quad \text{for all } \mu <_{\mathrm{D}} \lambda.$$

One can use this relation instead of the second property of Definition 2.2.

2.2 Multivariable Hermite polynomials and A_{N-1}-type Calogero model

We introduce an analogue of the creation and annihilation operators:

$$a_j^\dagger = \frac{1}{\sqrt{2}}(-D_j^A + x_j), \qquad a_j = \frac{1}{\sqrt{2}}(D_j^A + x_j).$$

The commutation relations of these operators are the same as those of x_j and D_j^A by construction. We then make a gauge transformation on $a_j^\dagger$ and a_j:

$$\begin{aligned}\tilde{a}_j^\dagger &= \tilde{\phi}_A^{-1} \circ a_j^\dagger \circ \tilde{\phi}_A = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_j} + 2x_j - \beta \sum_{k(\neq j)} \frac{1 - s_{jk}}{x_j - x_k}\right), \\ \tilde{a}_j &= \tilde{\phi}_A^{-1} \circ a_j \circ \tilde{\phi}_A = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_j} + \beta \sum_{k(\neq j)} \frac{1 - s_{jk}}{x_j - x_k}\right),\end{aligned}$$

with $\tilde{\phi}_A = \prod_{k=1}^N \exp(-x_k^2/2)$. Since this transformation leaves the commutation relations unchanged, we can introduce the following isomorphism:

$$\rho^A(x_j) = \tilde{a}_j^\dagger, \qquad \rho^A(D_j) = \tilde{a}_j, \qquad \rho^A(s_{ij}) = s_{ij}.$$

It should be remarked that this mapping has already been appeared implicitly in [UW], however, treated only as an isomorphism of the algebra. To construct eigenstates of $\widetilde{\mathcal{H}}_A$, we should introduce intertwiner between two representations which will be discussed in the followings.

We can obtain a set of commuting operators by applying ρ^A to $\hat{D}_j^A$:

$$\tilde{h}_j^A = \rho^A(\hat{D}_j^A) = \tilde{a}_j^\dagger \tilde{a}_j + \beta \sum_{k(<j)} s_{jk}.$$

The mapping ρ^A gives another representation of $\tilde{\mathfrak{H}}'$ on $\mathbb{C}[x]$. We then introduce intertwining operator σ^A, which is a linear operator on $\mathbb{C}[x]$ such that

$$\sigma^A\left(f(x_1, \ldots, x_N)\right) = f(\tilde{a}_1^\dagger, \ldots, \tilde{a}_N^\dagger) \cdot 1 \quad \text{for all} \quad f(x_1, \ldots, x_N) \in \mathbb{C}[x].$$

The intertwiner σ^A enjoys the following property.

Theorem 2.4 *$\sigma^A(Qf(x)) = \rho^A(Q)\sigma^A(f(x))$ for all $Q \in \mathfrak{H}_0'$, $f(x) \in \mathbb{C}[x]$.*

Proof. Since both Q and $f(x)$ are elements of $\tilde{\mathfrak{H}}'$, it suffices to prove $\sigma^A(P \cdot 1) = \rho^A(P) \cdot 1$ for all $P \in \tilde{\mathfrak{H}}'$. We then note that every element P of $\tilde{\mathfrak{H}}'$ can be represented in the following form:

$$P = \sum_{\mathbf{n}} \sum_{w \in \mathfrak{S}_N} p_{\mathbf{n},w}(x)(\hat{D}^A)^{n_1} \cdots (\hat{D}_N^A)^{n_N} w, \tag{2.5}$$

where $p_{\mathbf{n},w}(x)$ are some polynomials. Considering the action of (2.5) on 1, we have

$$P \cdot 1 = \sum_{\mathbf{n}(n_1=0)} \sum_{w \in \mathfrak{S}_N} p_{\mathbf{n},w}(x)\beta^{n_2} \cdots ((N-1)\beta)^{n_N},$$

since $w \cdot 1 = 1$ for all $w \in \mathfrak{S}_N$ and $\hat{D}_j^A \cdot 1 = \beta(j-1)$ for all j. On the other hand, applying ρ^A to (2.5), we have

$$\rho^A(P) = \sum_{\mathbf{n}} \sum_{w \in \mathfrak{S}_N} p_{\mathbf{n},w}(\tilde{a}^\dagger)(\tilde{h}^A)^{n_1} \cdots (\tilde{h}_N^A)^{n_N} w.$$

Since $\tilde{h}_j^A \cdot 1 = \beta(j-1)$ for all j, we conclude that $\sigma^A(P \cdot 1) = \rho^A(P) \cdot 1$ for all $P \in \mathfrak{H}'$. □

The representation ρ^A is related to the A_{N-1}-type Calogero model. If we define $\widetilde{\mathcal{H}}_A$ as

$$\begin{aligned}\widetilde{\mathcal{H}}_A &= \mathrm{Res}\left(\sum_{j=1}^{N}\tilde{h}_j^A\right) - \frac{\beta}{2}N(N-1)\\ &= \frac{1}{2}\sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial x_j^2} + 2x_j\frac{\partial}{\partial x_j}\right) - \beta\sum_{j<k}\frac{1}{x_j - x_k}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}\right),\end{aligned}$$

we can obtain the A_{N-1}-type Calogero Hamiltonian (1.2) via gauge transformation:

$$\mathcal{H}_A = \phi_A^{(\beta)} \circ \widetilde{\mathcal{H}}_A \circ (\phi_A^{(\beta)})^{-1} + \frac{N}{2} + \frac{\beta}{2}N(N-1),$$

with $\phi_A^{(\beta)} = \prod_{j<k}|x_j - x_k|^\beta \prod_{j=1}^N \exp(-x_j^2/2)$ ground state wavefunction.

We then introduce scalar product for this case:

$$\langle f, g\rangle_{\mathrm{H}}^{(\beta)} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x)g(x)(\phi_A^{(\beta)})^2 \mathrm{d}x_1\cdots\mathrm{d}x_N \tag{2.6}$$

By a direct calculation, we see that the operator $\tilde{a}_j^\dagger$ is adjoint of $\tilde{a}_j$ with respect to the scalar product (2.6) for all $j = 1, \ldots, N$. Note that x_j $(= (\rho^A)^{-1}(\tilde{a}_j^\dagger))$ is not adjoint of $D_j(= (\rho^A)^{-1}(\tilde{a}_j))$ for the Jack case.

Multivariable Hermite polynomials are defined by using this scalar product [BF1, vD]. In fact, the definition in [BF1] and that in [vD] are slightly different. Here we shall follow [vD]:

Definition 2.5 ([vD]) *Multivariable Hermite polynomials $H_\lambda^{(\beta)}(x)$ are characterized by the following properties:*

(i) $H_\lambda^{(\beta)}(x) = m_\lambda(x) + \sum_{\mu(<_{\mathrm{D}}\lambda)} u_{\lambda\mu}^A m_\mu(x),$

(ii) $\langle H_\lambda^{(\beta)}(x), m_\mu(x)\rangle_{\mathrm{H}}^{(\beta)} = 0$ *for all* $\mu <_{\mathrm{D}} \lambda$.

Using the intertwiner σ^A, we can construct an operator representation of $H_\lambda^{(\beta)}(x)$.

Proposition 2.6 ([Kal, UW]) *Multivariable Hermite polynomials $H_\lambda^{(\beta)}(x)$ are related to the Jack polynomials as follows:*

$$H_\lambda^{(\beta)}(x) = 2^{-|\lambda|/2}\sigma^A(J_\lambda^{(\beta)}(x)) = 2^{-|\lambda|/2}J_\lambda^{(\beta)}(\tilde{a}^\dagger)\cdot 1.$$

Proof. We can easily see that $2^{|\lambda|/2}J_\lambda(\tilde{a}_1^\dagger, \ldots, \tilde{a}_N^\dagger)\cdot 1$ satisfy the condition (i) of Definition 2.5. Hence it suffices to show (ii). Applying σ^A to (2.4), we have

$$\hat{\Delta}_{\mathrm{H}}(u)J_\lambda^{(\beta)}(\tilde{a}_1^\dagger, \ldots, \tilde{a}_N^\dagger)\cdot 1 = \prod_{j=1}^{N}\{u + \lambda_{N-j+1} + \beta(j-1)\}\, J_\lambda^{(\beta)}(\tilde{a}_1^\dagger, \ldots, \tilde{a}_N^\dagger)\cdot 1,$$

where we denote $\hat{\Delta}_{\mathrm{H}}(u) = \rho^A(\hat{\Delta}_{\mathrm{J}}(u)) = \prod_{j=1}^N(u+\tilde{h}_j^A)$. Since all the eigenvalues of $\hat{\Delta}_{\mathrm{H}}(u)$ are distinct and the operator $\hat{\Delta}_{\mathrm{H}}(u)$ are self-adjoint with respect to the scalar product (2.6), we conclude that the polynomials $J_\lambda^{(\beta)}(\tilde{a}^\dagger)\cdot 1$ are orthogonal with respect to the scalar product

(2.6). On the other hand, one may know that the polynomials $J_\lambda^{(\beta)}(\tilde{a}^\dagger)\cdot 1$ form an orthogonal basis of $\mathbb{C}[x]^{\mathfrak{S}_N}$ by considering the leading term. It follows that $\langle J_\lambda^{(\beta)}(\tilde{a}^\dagger)\cdot 1, m_\mu\rangle_{\mathrm{J}}^{(\beta)} = 0$ for all $\mu <_{\mathrm{D}} \lambda$, which proves the theorem. □

It should be noted that Ujino and Wadati [UW] have shown that $J_\lambda^{(\beta)}(\tilde{a}^\dagger)\cdot 1$ diagonalize the first two of the family of commuting operators that contains $\widetilde{\mathcal{H}}_A$. The proof given here is essentially the same as that given in [Ka1].

The scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}^{(\beta)}$ induces another scalar product on $\mathbb{C}[x]$:

$$\langle\langle f(x), g(x)\rangle\rangle_A = \langle f(\tilde{a}^\dagger)\cdot 1, g(\tilde{a}^\dagger)\cdot 1\rangle_{\mathrm{H}}^{(\beta)}.$$

This gives another example of scalar product which makes the Jack polynomials orthogonal. On the other hand, Dunkl [Du2] introduced the scalar product $\left[f(\hat{D}^A)g(x)\right]_0$. These scalar products coincide up to a constant factor:

$$\begin{aligned}\langle\langle f(x), g(x)\rangle\rangle_A &= \langle 1, f(\tilde{a})g(\tilde{a}^\dagger)\cdot 1\rangle_{\mathrm{H}}^{(\beta)}\\ &= \langle 1,1\rangle_{\mathrm{H}}^{(\beta)}\left[f(\tilde{a})g(\tilde{a}^\dagger)\cdot 1\right]_0 = \langle 1,1\rangle_{\mathrm{H}}^{(\beta)}\left[f(\hat{D}^A)g(x)\right]_0.\end{aligned}$$

We shall evaluate the value $\langle 1,1\rangle_{\mathrm{H}}^{(\beta)}$ in section 4.2. (See Proposition 4.8 below.)

2.3 Multivariable Laguerre polynomials and B_N-type Calogero model

Dunkl operators associated with the B_N-type root system are defined as follows [Du1, Y]:

$$D_j^B = \frac{\partial}{\partial z_j} + \beta\sum_{k(\neq j)}\left(\frac{1-s_{jk}}{z_j - z_k} + \frac{1 - t_j t_k s_{jk}}{z_j + z_k}\right) + \gamma\frac{1-t_j}{z_j}, \tag{2.7}$$

where s_{jk} and t_j are elements of the B_N-type Weyl group. An element s_{ij} acts as same as in the A_{N-1}-case and t_j acts as sign-change, i.e. replaces the coordinate z_j by $-z_j$. The commutation relations of the B_N-type Dunkl operators are

$$\begin{aligned}&[D_i^B, D_j^B] = 0, \qquad s_{ij}D_j^B = D_i^B s_{ij}, \qquad s_{ij}D_k^B = D_k^B s_{ij} \quad (k\neq i,j),\\ &t_j D_j^B = -D_i^B t_j, \qquad t_j D_k^B = D_k^B t_j \quad (k\neq j),\\ &[D_i^B, z_j] = \delta_{ij}\left\{1 + \beta\sum_{k(\neq i)}(s_{ik} + t_i t_k s_{ik}) + 2\gamma t_j\right\} - (1-\delta_{ij})\beta(s_{ij} - t_i t_k s_{ik}).\end{aligned}$$

We then define Cherednik-type commuting operators associated with (2.7):

$$\hat{D}_j^B = z_j D_j^B + \beta\sum_{k(<j)}(s_{jk} + t_j t_k s_{ik}).$$

Note that the operators $\hat{D}_j^B$ do *not* coincide with the Cherednik operators associated with the B_N-type Weyl group.

Lemma 2.7 *All of the operators $\hat{D}_j^B$, s_{ij}, t_j and z_j^2 preserve $\mathbb{C}[z_1^2,\ldots,z_N^2]$.*

Proof. Only $\hat{D}_j^B$ need to prove. We introduce the notation $\mathrm{Res}^{(t)}(X)$ which means the action of the operator X is restricted to the functions with the symmetry $t_j f(z) = f(z)$. Under this restriction, $\mathrm{Res}^{(t)}(\hat{D}_j^B)$ is equivalent to $2\hat{D}_j^A$ if we make a change of the variables $x_j = z_j^2/2$. Since $\hat{D}_j^A$ preserve $\mathbb{C}[x]$, the operators $\hat{D}_j^B$ preserve $\mathbb{C}[z^2]$. □

From these facts, we can define representation ι of $\tilde{\mathfrak{H}}'$ on $\mathbb{C}[z^2]$:

$$\iota(x_j) = \frac{1}{2}z_j^2, \qquad \iota(\hat{D}_j^A) = \frac{1}{2}\hat{D}_j^B, \qquad \iota(s_{ij}) = s_{ij}.$$

We now introduce creation and annihilation operators for the B_N case:

$$b_j^\dagger = \frac{1}{\sqrt{2}}(-D_j^B + z_j), \qquad b_j = \frac{1}{\sqrt{2}}(D_j^B + z_j).$$

The commutation relations of these operators are the same as those of z_j and D_j^B by construction. We then make a gauge transformation on $b_j^\dagger$ and b_j:

$$\begin{aligned}
\tilde{b}_j^\dagger &= \tilde{\phi}_B^{-1} \circ b_j^\dagger \circ \tilde{\phi}_B = \frac{1}{\sqrt{2}}\left\{-\frac{\partial}{\partial z_j} + 2z_j - \beta \sum_{k(\neq j)} \left(\frac{1-s_{jk}}{z_j - z_k} + \frac{1-t_j t_k s_{jk}}{z_j + z_k}\right) + \gamma\frac{1-t_j}{z_j}\right\}, \\
\tilde{b}_j &= \tilde{\phi}_B^{-1} \circ b_j \circ \tilde{\phi}_B = \frac{1}{\sqrt{2}}\left\{\frac{\partial}{\partial z_j} + \beta \sum_{k(\neq j)} \left(\frac{1-s_{jk}}{z_j - z_k} + \frac{1-t_j t_k s_{jk}}{z_j + z_k}\right) + \gamma\frac{1-t_j}{z_j}\right\},
\end{aligned}$$

with $\tilde{\phi}_B = \prod_{k=1}^N \exp(-z_k^2/2)$. Since this transformation leaves the commutation relations unchanged, we can define the following algebra isomorphism:

$$\kappa(x_j) = \tilde{b}_j^\dagger, \qquad \kappa(D_j^B) = \tilde{b}_j, \qquad \kappa(s_{ij}) = s_{ij}, \qquad \kappa(t_j) = t_j.$$

We then define the operators $\tilde{h}_j^B$ as follows:

$$\tilde{h}_j^B = \kappa(\hat{D}_j^B) = \tilde{b}_j^\dagger \tilde{b}_j + \beta \sum_{k(<j)} (s_{jk} + t_j t_k s_{ik}).$$

Lemma 2.8 *The operators $\tilde{h}_j^B$ and $(\tilde{b}_j^\dagger)^2$ preserve $\mathbb{C}[z_1^2, \ldots, z_N^2]$.*

Proof. Since the operators D_j^B preserve $\mathbb{C}[z_1, \ldots, z_N]$, it is clear that both $\tilde{h}_j^B$ and $(\tilde{b}_j^\dagger)^2$ also preserve $\mathbb{C}[z_1, \ldots, z_N]$. Then it suffices to prove $[t_i, \tilde{h}_j^B] = [t_i, (\tilde{b}_j^\dagger)^2] = 0$ for all i, j, which can be proved by a direct calculation. □

Using both ι and κ, we introduce another representation of $\tilde{\mathfrak{H}}'$ on $\mathbb{C}[z^2]$:

$$\rho^B(x_j) = \kappa(\iota(x_j)) = \frac{1}{2}(\tilde{b}_j^\dagger)^2, \qquad \rho^B(\hat{D}_j^A) = \kappa(\iota(\hat{D}_j^A)) = \frac{1}{2}\tilde{h}_j^B, \qquad \rho^B(s_{ij}) = s_{ij}.$$

We introduce a linear map of $\mathbb{C}[x]$ to $\mathbb{C}[z^2]$ by using ρ^B:

$$\sigma^B(f(x_1, \ldots, x_N)) = f((\tilde{b}_1^\dagger)^2/2, \ldots, (\tilde{b}_N^\dagger)^2/2) \cdot 1 \quad \text{for all} \quad f(x_1, \ldots, x_N) \in \mathbb{C}[x].$$

As in the A_{N-1}-case, the intertwiner σ^B enjoys the following property.

Theorem 2.9 *$\sigma^B(Qf(x)) = \rho^B(Q)\sigma^B(f(x)) \cdot 1$ for all $Q \in \mathfrak{H}'_0$, $f(x_1,\ldots,x_N) \in \mathbb{C}[x]$.*

Proof is given in the same fashion as Theorem 2.4, so we omit details.

The operators $\widetilde{b}_j^\dagger$ and $\widetilde{b}_j$ are related to the B_N-type Calogero Hamiltonian (1.3); If we define $\widetilde{\mathcal{H}}_B$ as

$$\begin{aligned}\widetilde{\mathcal{H}}_B &= \operatorname{Res}\left(\sum_{j=1}^{N} \widetilde{h}_j^B\right) - \beta N(N-1) \\ &= \frac{1}{2}\sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial z_j^2} + 2z_j\frac{\partial}{\partial z_j} - \frac{2\gamma}{z_j}\frac{\partial}{\partial z_j}\right) - 2\beta\sum_{j<k}\frac{1}{z_j^2 - z_k^2}\left(z_j\frac{\partial}{\partial z_j} - z_k\frac{\partial}{\partial z_k}\right),\end{aligned}$$

we can obtain the Hamiltonian (1.3) via gauge transformation:

$$\mathcal{H}_B = \phi_B^{(\beta)} \circ \widetilde{\mathcal{H}}_B \circ (\phi_B^{(\beta)})^{-1} + \left(\frac{1}{2} + \gamma\right) N + \beta N(N-1),$$

with $\phi_B^{(\beta)} = \prod_{j<k} |z_j^2 - z_k^2|^\beta \prod_{j=1}^N |z_j|^\gamma \exp(-z_j^2/2)$ ground state wavefunction of (1.3).

Scalar product associated with this model is

$$\langle f(z), g(z)\rangle_{\mathrm{L}}^{(\beta)} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z)g(z)(\phi_B^{(\beta)})^2 \mathrm{d}z_1 \cdots \mathrm{d}z_N. \tag{2.8}$$

By a direct calculation, we can show that the operator $\widetilde{b}_j^\dagger$ is adjoint of $\widetilde{b}_j$ with respect to the scalar product (2.8), and hence the operator $\widetilde{h}_j^B$ is self-adjoint for all $j = 1,\ldots,N$.

Now we define multivariable Laguerre polynomials [vD].

Definition 2.10 ([vD]) *Multivariable Laguerre polynomials $L_\lambda^{(\beta)}(z)$ are characterized by the following properties:*

(i) $L_\lambda^{(\beta)}(z) = m_\lambda(z^2) + \sum_{\mu(<_{\mathrm{D}}\lambda)} u_{\lambda\mu} m_\mu(z^2)$,

(ii) $\langle L_\lambda^{(\beta)}(z), m_\mu(z^2)\rangle_{\mathrm{L}}^{(\beta)} = 0 \quad$ *for all* $\mu <_{\mathrm{D}} \lambda$.

We can construct an operator representation of $L_\lambda^{(\beta)}(z)$ by using the intertwiner σ^B.

Proposition 2.11 ([Ka2]) *Multivariable Laguerre polynomials $L_\lambda^{(\beta)}(z)$ are related to the Jack polynomials as follows:*

$$L_\lambda^{(\beta)}(z) = \sigma^B(J_\lambda^{(\beta)}(x)) = J_\lambda^{(\beta)}((\widetilde{b}^\dagger)^2/2) \cdot 1.$$

One can prove this statement in the same way as Proposition 2.6, so we omit details.

3 Construction of raising operators

As is shown in the previous section, the multivariable Hermite and Laguerre polynomials are expressed in terms of the Jack polynomials whose arguments are Dunkl-type operators. Some properties of the multivariable Hermite and Laguerre polynomials are obtained directly from those of the Jack polynomials simply by applying ρ^A or ρ^B. As an example, we will construct raising operators for the polynomials.

Lapointe and Vinet constructed raising operators for the Jack polynomials [LV]. Using their raising operators, they obtained Rodorigues-type formula for the Jack polynomials. Raising operators for the multivariable Hermite polynomials have been constructed by Ujino and Wadati [UW]. The raising operators above are constructed by the use of Dunkl operators of Heckman-type (non-commutative). In those cases, relation to the degenerate double affine Hecke algebra is still unclear.

On the other hand, Kirillov and Noumi gave another expression of raising operators by using Cherednik operators [KN]. In our notation, their raising operators are expressed as the following form:

$$\begin{aligned} B_m^{\mathrm{J}} &= \sum_{k_1<\cdots<k_m} x_{k_1}x_{k_2}\cdots x_{k_m}(\hat{D}_{k_1}^A+\beta(2-k_1)) \\ &\qquad \times(\hat{D}_{k_2}^A+\beta(3-k_2))\cdots(\hat{D}_{k_m}^A+\beta(m-k_m+1)). \end{aligned}$$

We recall a important property of these operators.

Theorem 3.1 ([KN]) *Action of the operators $B_m^{\mathrm{J}} \in \tilde{\mathfrak{H}}'$ on the Jack polynomials are given by*

$$B_m^{\mathrm{J}} J_\lambda^{(\beta)}(x) = \prod_{j=1}^m (\lambda_j+\beta(m-j+1)) J_{\lambda+(1^m)}^{(\beta)}(x),$$

where $\lambda+(1^m)=(\lambda_1+1,\ldots,\lambda_N+1)$.

Applying σ^A or σ^B to B_m^{J}, we obtain raising operators for the Hermite-case or the Laguerre-case respectively:

$$\begin{aligned} B_m^{\mathrm{H}} &= \sum_{k_1<\cdots<k_m} \tilde{a}_{k_1}^\dagger \tilde{a}_{k_2}^\dagger\cdots \tilde{a}_{k_m}^\dagger(\tilde{h}_{k_1}^A+\beta(2-k_1)) \\ &\qquad \times(\tilde{h}_{k_2}^A+\beta(3-k_2))\cdots(\tilde{h}_{k_m}^A+\beta(m-k_m+1)), \\ B_m^{\mathrm{L}} &= \sum_{k_1<\cdots<k_m} \tilde{b}_{k_1}^\dagger \tilde{b}_{k_2}^\dagger\cdots \tilde{b}_{k_m}^\dagger(\tilde{h}_{k_1}^B+\beta(2-k_1)) \\ &\qquad \times(\tilde{h}_{k_2}^B+\beta(3-k_2))\cdots(\tilde{h}_{k_m}^B+\beta(m-k_m+1)). \end{aligned}$$

Form the theorem 3.1 and the propositions 2.6, 2.11, it immediately follows that:

Proposition 3.2 (i) $B_m^{\mathrm{H}} H_\lambda^{(\beta)}(x) = 2^{-m/2}\prod_{j=1}^m(\lambda_j+\beta(m-j+1))H_{\lambda+(1^m)}^{(\beta)}(x)$,

(ii) $B_m^{\mathrm{L}} L_\lambda^{(\beta)}(z) = \prod_{j=1}^m(\lambda_j+\beta(m-j+1))L_{\lambda+(1^m)}^{(\beta)}(z)$.

Applying the raising operators repeatedly, one can obtain Rodorigues-type formulas for the multivariable Hermite and Laguerre polynomials:

$$\begin{aligned} H_\lambda^{(\beta)}(x) &= 2^{|\lambda|/2}\prod_{(i,j)\in\lambda}(\lambda_i-j+\beta(\lambda_j'-i+1))^{-1}(B_N^{\mathrm{H}})^{\lambda_N}(B_{N-1}^{\mathrm{H}})^{\lambda_{N-1}-\lambda_N}\cdots(B_1^{\mathrm{H}})^{\lambda_1-\lambda_2}\cdot 1, \\ L_\lambda^{(\beta)}(z) &= \prod_{(i,j)\in\lambda}(\lambda_i-j+\beta(\lambda_j'-i+1))^{-1}(B_N^{\mathrm{L}})^{\lambda_N}(B_{N-1}^{\mathrm{L}})^{\lambda_{N-1}-\lambda_N}\cdots(B_1^{\mathrm{L}})^{\lambda_1-\lambda_2}\cdot 1, \end{aligned}$$

where $\lambda'=(\lambda_1',\lambda_2',\ldots)$ is the conjugate partition to λ.

4 Construction of shift operators

In this section, we construct shift operators for the multivariable Hermite and Laguerre polynomials. Each of such shift operators are related to one of the scalar products (2.3), (2.6), (2.8) (see the theorems 4.2, 4.6 and 4.9 below). However properties related to scalar product cannot be obtained simply by applying ρ^A or ρ^B. It require a little more effort to construct shift operators.

It should be noted that the notion of hypergeometric shift operators were introduced originally by Opdam, and Heckman gave an expression by using the Dunkl operators [He2, He1]. Since our construction of shift operators are based on the Cherednik operators, relation to the degenerate double affine Hecke algebra becomes clear.

4.1 Shift operators for the Jack polynomials

In this subsection, we review the method of constructing shift operators for the Jack polynomials by the use of the Cherednik operators. Our method is based on the lecture note of Kirillov Jr. [Ki]; All results given in this section can be obtained by taking limiting procedure on those of [Ki]. However, all proofs given here are algebraic and we avoid using limiting procedure so that we can apply the results to the Hermite and Laguerre cases.

Consider elements of $\mathfrak{H}_0'$ that have the following forms:

$$\mathcal{X} = \prod_{i<j}(x_i - x_j), \qquad \mathcal{Y}_{\mathrm{J}} = \prod_{i<j}(\beta - \hat{D}_i^A + \hat{D}_j^A), \qquad \hat{\mathcal{Y}}_{\mathrm{J}} = \prod_{i<j}(-\beta - \hat{D}_i^A + \hat{D}_j^A). \tag{4.1}$$

We note that these operators preserve $\mathbb{C}[x]$. From the defining relations of $\mathfrak{H}_0'$, we know that

$$\begin{aligned}
(s_j+1)(-\beta - \hat{D}_j^A + \hat{D}_{j+1}^A) &= (-\beta - \hat{D}_{j+1}^A + \hat{D}_j^A)(s_j - 1), \\
(s_j-1)(\beta - \hat{D}_j^A + \hat{D}_{j+1}^A) &= (\beta - \hat{D}_{j+1}^A + \hat{D}_j^A)(s_j + 1), \\
s_j(c - \hat{D}_j^A + \hat{D}_k^A)(c - \hat{D}_{j+1}^A + \hat{D}_k^A) &= (c - \hat{D}_j^A + \hat{D}_k^A)(c - \hat{D}_{j+1}^A + \hat{D}_k^A)s_j,
\end{aligned} \tag{4.2}$$

with c arbitrary constant and $k \neq j, j+1$. Then, if we define $\mathbb{C}[x]^{\mathfrak{S}_N}$ and $\mathbb{C}[x]^{-\mathfrak{S}_N}$ as

$$\begin{aligned}
\mathbb{C}[x]^{\mathfrak{S}_N} &= \{f(x) \in \mathbb{C}[x] \mid (s_j - 1)f(x) = 0\}, \\
\mathbb{C}[x]^{-\mathfrak{S}_N} &= \{f(x) \in \mathbb{C}[x] \mid (s_j + 1)f(x) = 0\},
\end{aligned}$$

we see that

$$\mathcal{X} \in \mathbb{C}[x]^{-\mathfrak{S}_N}, \qquad \mathcal{Y}_{\mathrm{J}}\left(\mathbb{C}[x]^{\mathfrak{S}_N}\right) = \mathbb{C}[x]^{-\mathfrak{S}_N}, \qquad \hat{\mathcal{Y}}_{\mathrm{J}}\left(\mathbb{C}[x]^{-\mathfrak{S}_N}\right) = \mathbb{C}[x]^{\mathfrak{S}_N}. \tag{4.3}$$

We now introduce shift operators for the Jack polynomials as $G_{\mathrm{J}} = \mathcal{X}^{-1}\mathcal{Y}_{\mathrm{J}}$, $\hat{G}_{\mathrm{J}} = \hat{\mathcal{Y}}_{\mathrm{J}}\mathcal{X}$. The operators G_{J} and $\hat{G}_{\mathrm{J}}$ enjoy the following properties:

Lemma 4.1 ([Ki]) (i) $G_{\mathrm{J}} J_{\lambda+\delta}^{(\beta)},\ \hat{G}_{\mathrm{J}} J_\lambda^{(\beta+1)} \in \mathbb{C}[x]^{\mathfrak{S}_N}$.

(ii) $G_{\mathrm{J}} J_{\lambda+\delta}^{(\beta)} = c_\lambda^{(\beta+1)} m_\lambda +$ *"lower terms" with respect to* $<_{\mathrm{D}}$, *with* $c_\lambda^{(\beta+1)} = \prod_{i<j}\{\lambda_{N-j+1} - \lambda_{N-i+1} + j - i + \beta(j-i-1)\}$.

(iii) $\hat{G}_{\mathrm{J}} J_\lambda^{(\beta+1)} = \tilde{c}_\lambda^{(\beta+1)} m_{\lambda+\delta} +$ *"lower terms" with respect to* $<_{\mathrm{D}}$, *with* $\tilde{c}_\lambda^{(\beta+1)} = \prod_{i<j} \{\lambda_{N-j+1} - \lambda_{N-i+1} + j - i + \beta(j-i+1)\}$.

Proof. (i) Follows from (4.3).

(ii) For the longest element w_0 of $\mathfrak{S}_N$, i.e. $w_0(j) = N - j + 1$, equation (2.1) reduces to

$$\hat{D}_j^A x_{w_0}^\lambda = (\lambda_{N-j+1} + \beta(j-1)) x_{w_0}^\lambda + \sum_{(\mu,w')\prec(\lambda,w_0)} u_{w_0 w'}^{\lambda\mu} x_{w'}^\mu.$$

Using this relation, we can calculate the action of $\mathcal{Y}_{\mathrm{J}}$:

$$\begin{aligned}
\mathcal{Y}_{\mathrm{J}} J_{\lambda+\delta}^{(\beta)} &= \prod_{i<j} (\beta - \hat{D}_i^A + \hat{D}_j^A)(x_{w_0}^{\lambda+\delta} + \text{"lower terms" with respect to } \prec) \\
&= c_\lambda^{(\beta+1)} x_{w_0}^{\lambda+\delta} + \text{"lower terms" with respect to } \prec. \qquad (4.4)
\end{aligned}$$

On the other hand, (4.3) implies that $\mathcal{Y}_{\mathrm{J}} J_{\lambda+\delta}^{(\beta)}$ is divisible by $\mathcal{X}$. Together with (i), this concludes the proof.

(iii) Can also be proved in similar way. □

The following theorem implies that $\hat{G}_{\mathrm{J}}$ is, in a sense, adjoint of G_{J}:

Theorem 4.2 **([Ki])** *For* $f, g \in \mathbb{C}[x]^{\mathfrak{S}_N}$, $\langle G_{\mathrm{J}} f, g\rangle_{\mathrm{J}}^{(\beta+1)} = \langle f, \hat{G}_{\mathrm{J}} g\rangle_{\mathrm{J}}^{(\beta)}$.

To prove this theorem, we introduce symmetrizer $\mathcal{P}_+$ and anti-symmetrizer $\mathcal{P}_-$ as

$$\mathcal{P}_+ = \frac{1}{\#\mathfrak{S}_N} \sum_{w\in\mathfrak{S}_N} w, \qquad \mathcal{P}_- = \frac{1}{\#\mathfrak{S}_N} \sum_{w\in\mathfrak{S}_N} (-1)^{l(w)} w,$$

where $l(w)$ is the length of the element w. We further prepare a lemma.

Lemma 4.3 **([Ki])** $\mathcal{P}_-(\mathcal{Y}_{\mathrm{J}} - \hat{\mathcal{Y}}_{\mathrm{J}}) = \sum_j \hat{g}_j(\hat{D}_1^A, \ldots, \hat{D}_N^A)(s_j - 1)$ *for some* $\hat{g}_j(x_1, \cdots, x_N) \in \mathbb{C}[x]$.

It should be remarked that this lemma is a degenerate version of a lemma given in [Ki]. We will give a proof in Appendix A for reader's convenience.

Now we go back to the proof of Theorem 4.2.

Proof of Theorem 4.2. It is clear that $\mathcal{P}_+$ does not affect constant term of polynomials. Then, for $f, g \in \mathbb{C}[x]^{\mathfrak{S}_N}$, we know that

$$\begin{aligned}
\langle G_{\mathrm{J}} f, g\rangle_{\mathrm{J}}^{(\beta+1)} &= (-1)^{(\beta+1)N(N-1)/2} \left[(\mathcal{X}^{-1}\mathcal{Y}_{\mathrm{J}} f)\bar{g}(\phi_{\mathrm{S}}^{(\beta+1)})^2\right]_0 \\
&= (-1)^{(\beta+1)N(N-1)/2} \left[\mathcal{P}_+\left((\mathcal{Y}_{\mathrm{J}} f)\bar{g}\bar{\mathcal{X}}(\phi_{\mathrm{S}}^{(\beta)})^2\right)\right]_0 \\
&= (-1)^{\beta N(N-1)/2} \left[\mathcal{P}_-(\mathcal{Y}_{\mathrm{J}} f)\bar{g}\bar{\mathcal{X}}(\phi_{\mathrm{S}}^{(\beta)})^2\right]_0.
\end{aligned}$$

From Lemma 4.3, we see that $\mathcal{P}_-(\mathcal{Y}_{\mathrm{J}} - \hat{\mathcal{Y}}_{\mathrm{J}}) f = 0$ for all $f \in \mathbb{C}[x]^{\mathfrak{S}_N}$. Hence we can replace $\mathcal{Y}_{\mathrm{J}}$ by $\hat{\mathcal{Y}}_{\mathrm{J}}$:

$$\langle G_{\mathrm{J}} f, g\rangle_{\mathrm{J}}^{(\beta+1)} = (-1)^{\beta N(N-1)/2} \left[\mathcal{P}_-(\hat{\mathcal{Y}}_{\mathrm{J}} f)\bar{g}\bar{\mathcal{X}}(\phi_{\mathrm{S}}^{(\beta)})^2\right]_0 = \langle \hat{\mathcal{Y}}_{\mathrm{J}} f, \mathcal{X} g\rangle_{\mathrm{J}}^{(\beta)} = \langle f, \hat{G}_{\mathrm{J}} g\rangle_{\mathrm{J}}^{(\beta)}.$$

In the last equality, we have used the self-adjointness of $\hat{D}_j^A$. □

Using Theorem 4.2, we can evaluate the action of G_{J} and $\hat{G}_{\mathrm{J}}$ on the Jack polynomials.

Proposition 4.4 ([Ki]) $G_{\mathrm{J}} J^{(\beta)}_{\lambda+\delta} = c^{(\beta+1)}_{\lambda} J^{(\beta+1)}_{\lambda}$, $\hat{G}_{\mathrm{J}} J^{(\beta+1)}_{\lambda} = \tilde{c}^{(\beta+1)}_{\lambda} J^{(\beta)}_{\lambda+\delta}$, *where the constants* $c^{(\beta)}_{\lambda}$ *and* $\tilde{c}^{(\beta)}_{\lambda}$ *are defined in Lemma 4.1.*

Proof. Assume $\mu <_{\mathrm{D}} \lambda$. Then we have

$$\langle G_{\mathrm{J}} J^{(\beta)}_{\lambda+\delta}, m_{\mu}\rangle^{(\beta)}_{\mathrm{J}} = \langle J^{(\beta)}_{\lambda+\delta}, \hat{G}_{\mathrm{J}} m_{\mu}\rangle^{(\beta)}_{\mathrm{J}} = \langle J^{(\beta)}_{\lambda+\delta}, m_{\mu+\delta} + \text{ lower terms }\rangle^{(\beta)}_{\mathrm{J}} = 0,$$

where we have used Lemma 4.1 and Theorem 4.2. From this fact, along with Lemma 4.1 (ii), we see that $G_{\mathrm{J}} J^{(\beta)}_{\lambda+\delta}$ coincides with $c^{(\beta+1)}_{\lambda} J^{(\beta+1)}_{\lambda}$. The latter can be proved in similar way. □

With these preliminaries, it is possible to derive the following result.

Proposition 4.5 ([M1])

$$\langle J^{(\beta)}_{\lambda}, J^{(\beta)}_{\lambda}\rangle^{(\beta)}_{\mathrm{J}} = N! \prod_{k=1}^{\beta} \prod_{i<j} \frac{\lambda_i - \lambda_j - k + \beta(j-i+1)}{\lambda_i - \lambda_j + k + \beta(j-i-1)}. \tag{4.5}$$

Proof. From Proposition 4.4, it follows that

$$\begin{aligned}\langle J^{(\beta+1)}_{\lambda}, J^{(\beta+1)}_{\lambda}\rangle^{(\beta+1)}_{\mathrm{J}} &= \frac{1}{c^{(\beta+1)}_{\lambda}} \langle G J^{(\beta)}_{\lambda+\delta}, J^{(\beta+1)}_{\lambda}\rangle^{(\beta+1)}_{\mathrm{J}} \\ = \frac{1}{c^{(\beta+1)}_{\lambda}} \langle J^{(\beta)}_{\lambda+\delta}, \hat{G}_{\mathrm{J}} J^{(\beta+1)}_{\lambda}\rangle^{(\beta)}_{\mathrm{J}} &= \frac{\tilde{c}^{(\beta+1)}_{\lambda}}{c^{(\beta+1)}_{\lambda}} \langle J^{(\beta)}_{\lambda+\delta}, J^{(\beta)}_{\lambda+\delta}\rangle^{(\beta)}_{\mathrm{J}}.\end{aligned}$$

Applying this relation repeatedly, we have

$$\langle J^{(\beta)}_{\lambda}, J^{(\beta)}_{\lambda}\rangle^{(\beta)}_{\mathrm{J}} = \prod_{k=0}^{\beta-1} \frac{\tilde{c}^{(\beta-k)}_{\lambda+k\delta}}{c^{(\beta-k)}_{\lambda+k\delta}} \langle J^{(\beta=0)}_{\lambda+\beta\delta}, J^{(\beta=0)}_{\lambda+\beta\delta}\rangle^{(\beta=0)}_{\mathrm{S}},$$

which gives the desired result. □

The norm formula (4.5) can be rewritten into the following form:

$$\langle J^{(\beta)}_{\lambda}, J^{(\beta)}_{\lambda}\rangle^{(\beta)}_{\mathrm{J}} = \frac{(N\beta)!}{(\beta!)^N} \prod_{(i,j)\in\lambda} \frac{j-1+\beta(N-i+1)}{j+\beta(N-i)} \cdot \frac{\lambda_i - j + 1 + \beta(\lambda'_j - i)}{\lambda_i - j + \beta(\lambda'_j - i + 1)}. \tag{4.6}$$

A proof of the equivalence between (4.5) and (4.6) is given in Appendix B.

4.2 Shift operators for the multivariable Hermite polynomials

In this section, we will construct shift operators for the multivariable Hermite polynomials. It should be noted that Heckman has constructed shift operators for the Hamiltonian H_A without harmonic potential [He2]. However, for the application to norm formulas, it is needed to compute actions of the shift shift operators on polynomials explicitly. Our method gives an unified and straightforward way to compute such actions.

To construct shift operators, we first introduce $\mathcal{Y}_{\mathrm{H}}$ and $\hat{\mathcal{Y}}_{\mathrm{H}}$ as follows:

$$\mathcal{Y}_{\mathrm{H}} = \rho^A(\mathcal{Y}_{\mathrm{J}}) = \prod_{i<j}(\beta - \tilde{h}^A_i + \tilde{h}^A_j), \qquad \hat{\mathcal{Y}}_{\mathrm{H}} = \rho^A(\hat{\mathcal{Y}}_{\mathrm{J}}) = \prod_{i<j}(-\beta - \tilde{h}^A_i + \tilde{h}^A_j).$$

Using these operators, we define shift operators for the multivariable Hermite polynomials as $G_{\mathrm{H}} = \mathcal{X}^{-1}\mathcal{Y}_{\mathrm{H}}$, $\hat{G}_{\mathrm{H}} = \hat{\mathcal{Y}}_{\mathrm{H}}\mathcal{X}$. We stress that we have used same $\mathcal{X}$ as (4.1), and therefore $G_{\mathrm{H}} \neq \rho^A(G_{\mathrm{J}})$, $\hat{G}_{\mathrm{H}} \neq \rho^A(\hat{G}_{\mathrm{J}})$. This reflects the characteristics of the scalar products (2.3), (2.6).

If we apply ρ^A to (4.2), we have

$$\begin{aligned}
(s_j+1)(-\beta-\widetilde{h}_j^A+\widetilde{h}_{j+1}^A) &= (-\beta-\widetilde{h}_{j+1}^A+\widetilde{h}_j^A)(s_j-1),\\
(s_j-1)(\beta-\widetilde{h}_j^A+\widetilde{h}_{j+1}^A) &= (\beta-\widetilde{h}_{j+1}^A+\widetilde{h}_j^A)(s_j+1),\\
s_j(c-\widetilde{h}_j^A+\widetilde{h}_k^A)(c-\widetilde{h}_{j+1}^A+\widetilde{h}_k^A) &= (c-\widetilde{h}_j^A+\widetilde{h}_k^A)(c-\widetilde{h}_{j+1}^A+\widetilde{h}_k^A)s_j,
\end{aligned}$$

with c arbitrary constant and $k \neq j, j+1$. These relations imply that

$$\mathcal{Y}_{\mathrm{H}}\left(\mathbb{C}[x]^{\mathfrak{S}_N}\right) = \mathbb{C}[x]^{-\mathfrak{S}_N}, \qquad \hat{\mathcal{Y}}_{\mathrm{H}}\left(\mathbb{C}[x]^{-\mathfrak{S}_N}\right) = \mathbb{C}[x]^{\mathfrak{S}_N}. \tag{4.7}$$

Furthermore, if we apply σ^A to (4.4), we see that

$$\mathcal{Y}_{\mathrm{H}} H_{\lambda+\delta}^{(\beta)} = c_\lambda^{(\beta+1)} x_{w_0}^{\lambda+\delta} + \text{“lower terms” with respect to } \prec. \tag{4.8}$$

For the proof of the shift relations for the Jack polynomials (Proposition 4.4), Theorem 4.2 played a crucial role. Here we state analogous result for Hermite case:

Theorem 4.6 *For $f, g \in \mathbb{C}[x]^{\mathfrak{S}_N}$, $\langle G_{\mathrm{H}} f, g\rangle_{\mathrm{H}}^{(\beta+1)} = \langle f, \hat{G}_{\mathrm{H}} g\rangle_{\mathrm{H}}^{(\beta)}$.*

Proof. The proof is similar to that of Theorem 4.2. For $f, g \in \mathbb{C}[x]^{\mathfrak{S}_N}$, we know that

$$\langle G_{\mathrm{H}} f, g\rangle_{\mathrm{H}}^{(\beta+1)} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathcal{P}_-(\mathcal{Y}_{\mathrm{H}} f) g \mathcal{X}(\phi_A^{(\beta)})^2 \mathrm{d}x_1\cdots\mathrm{d}x_N.$$

On the other hand, applying ρ^A to Lemma 4.3, we obtain

$$\mathcal{P}_-(\mathcal{Y}_{\mathrm{H}} - \hat{\mathcal{Y}}_{\mathrm{H}}) = \sum_j \hat{g}_j(\tilde{a}_1^\dagger, \ldots, \tilde{a}_N^\dagger)(s_j - 1) \quad \text{for some } \hat{g}_j(x_1, \cdots, x_N) \in \mathbb{C}[x].$$

From this relation, we find that $\mathcal{P}_-(\mathcal{Y}_{\mathrm{H}} - \hat{\mathcal{Y}}_{\mathrm{H}})f = 0$ for all $f \in \mathbb{C}[x]^{\mathfrak{S}_N}$. Hence we can replace $\mathcal{Y}_{\mathrm{H}}$ by $\hat{\mathcal{Y}}_{\mathrm{H}}$:

$$\begin{aligned}
\langle G_{\mathrm{H}} f, g\rangle_{\mathrm{H}}^{(\beta+1)} &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathcal{P}_-(\hat{\mathcal{Y}}_{\mathrm{H}} f) g \mathcal{X}(\phi_A^{(\beta)})^2 \mathrm{d}x_1\cdots\mathrm{d}x_N\\
&= \langle \hat{\mathcal{Y}}_{\mathrm{H}} f, \mathcal{X} g\rangle_{\mathrm{H}}^{(\beta)} = \langle f, \hat{G}_{\mathrm{H}} g\rangle_{\mathrm{H}}^{(\beta)}
\end{aligned}$$

In the last equality, we have used the self-adjointness of the operator $\widetilde{h}_j^A$. □

Now we are in position to state that:

Proposition 4.7 $G_{\mathrm{H}} H_{\lambda+\delta}^{(\beta)} = c_\lambda^{(\beta+1)} H_\lambda^{(\beta+1)}$, $\hat{G}_{\mathrm{H}} H_\lambda^{(\beta+1)} = \tilde{c}_\lambda^{(\beta+1)} H_{\lambda+\delta}^{(\beta)}$, *where the constants $c_\lambda^{(\beta)}$ and $\tilde{c}_\lambda^{(\beta)}$ are defined in Lemma 4.1.*

Proof. From (4.7) and (4.8), we know that $(c_\lambda^{(\beta+1)})^{-1} G_{\mathrm{H}} H_{\lambda+\delta}^{(\beta)}$ satisfies the first condition of Definition 2.5. So it suffice to prove the orthogonality which can be shown in the same way as Proposition 4.4. The second equation can be proved in similar way. □

Using Proposition 4.7 and Theorem 4.6, we can prove the norm formula for $H_\lambda^{(\beta)}$.

Proposition 4.8 ([BF1, vD])

$$\langle H_\lambda^{(\beta)}, H_\lambda^{(\beta)}\rangle_{\mathrm{H}}^{(\beta)} = \frac{\pi^{N/2} N!}{2^{|\lambda|+\beta N(N-1)/2}}$$
$$\times \prod_{j=1}^{N} (\lambda_j + \beta(N-j))! \prod_{k=1}^{\beta} \prod_{i<j} \frac{\lambda_i - \lambda_j - k + \beta(j-i+1)}{\lambda_i - \lambda_j + k + \beta(j-i-1)}, \tag{4.9}$$

where $|\lambda| = \sum_j \lambda_j$.

Proof. Using Proposition 4.7 and Theorem 4.6, we see that

$$\langle H_\lambda^{(\beta+1)}, H_\lambda^{(\beta+1)}\rangle_{\mathrm{H}}^{(\beta+1)} = \frac{\tilde{c}_\lambda^{(\beta+1)}}{c_\lambda^{(\beta+1)}} \langle H_{\lambda+\delta}^{(\beta)}, H_{\lambda+\delta}^{(\beta)}\rangle_{\mathrm{H}}^{(\beta)}.$$

On the other hand, since $H_\lambda^{(\beta=0)}(x)$ is direct product of the (one-variable) Hermite polynomials, one can evaluate the norm easily:

$$\langle H_\lambda^{(\beta=0)}, H_\lambda^{(\beta=0)}\rangle_{\mathrm{H}}^{(\beta=0)} = \frac{\pi^{N/2}\cdot \#\mathfrak{S}_N^\lambda}{2^{|\lambda|}} \prod_{j=1}^{N} \lambda_j!.$$

Using these relations, one arrives at the formula above. □

The norm formula (4.9) can be rewritten into the following form [BF1]:

$$\langle H_\lambda^{(\beta)}, H_\lambda^{(\beta)}\rangle_{\mathrm{H}}^{(\beta)} = \frac{\pi^{N/2}}{2^{|\lambda|+\beta N(N-1)/2}} \cdot \frac{\prod_{j=1}^{N}(j\beta)!}{(\beta!)^N}$$
$$\times \prod_{(i,j)\in\lambda} \frac{\{j-1+\beta(N-i+1)\}\{\lambda_i - j + 1 + \beta(\lambda'_j - i)\}}{\lambda_i - j + \beta(\lambda'_j - i + 1)}.$$

It should be remarked that other proofs of these formulas have been given via limiting procedure [BF1, vD].

4.3 Shift operators for the multivariable Laguerre polynomials

We first define $\mathcal{X}_{\mathrm{L}}$, $\mathcal{Y}_{\mathrm{L}}$ and $\hat{\mathcal{Y}}_{\mathrm{L}}$ as follows:

$$\mathcal{X}_{\mathrm{L}} = \prod_{i<j}(z_i^2 - z_j^2), \qquad \mathcal{Y}_{\mathrm{L}} = \prod_{i<j}(\beta - \tilde{h}_i^B/2 + \tilde{h}_j^B/2), \qquad \hat{\mathcal{Y}}_{\mathrm{L}} = \prod_{i<j}(-\beta - \tilde{h}_i^B/2 + \tilde{h}_j^B/2). \tag{4.10}$$

After same discussion as in the previous subsection, we see that

$$\mathcal{Y}_{\mathrm{L}}\left(\mathbb{C}[z^2]^{\mathfrak{S}_N}\right) = \mathbb{C}[z^2]^{-\mathfrak{S}_N}, \qquad \hat{\mathcal{Y}}_{\mathrm{L}}\left(\mathbb{C}[z^2]^{-\mathfrak{S}_N}\right) = \mathbb{C}[z^2]^{\mathfrak{S}_N}, \tag{4.11}$$

and

$$\mathcal{Y}_{\mathrm{H}} L_{\lambda+\delta}^{(\beta)} = c_\lambda^{(\beta+1)} z_{w_0}^{2(\lambda+\delta)} + \text{“lower terms” with respect to } \prec. \tag{4.12}$$

Now we define the shift operators for Laguerre case as $G_{\mathrm{L}} = \mathcal{X}_{\mathrm{L}}^{-1}\mathcal{Y}_{\mathrm{L}}$, $\hat{G}_{\mathrm{L}} = \hat{\mathcal{Y}}_{\mathrm{L}}\mathcal{X}_{\mathrm{L}}$. These operators enjoy the following properties:

Theorem 4.9 *For* $f, g \in \mathbb{C}[z^2]^{\mathfrak{S}_N}$, $\langle G_{\mathrm{L}} f, g\rangle_{\mathrm{L}}^{(\beta+1)} = \langle f, \hat{G}_{\mathrm{L}} g\rangle_{\mathrm{L}}^{(\beta)}$.

Proof. The proof for this case is also similar to that of Theorem 4.2. For $f, g \in \mathbb{C}[z^2]^{\mathfrak{S}_N}$, we know that

$$\langle G_{\mathrm{L}} f, g\rangle_{\mathrm{L}}^{(\beta+1)} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathcal{P}_-(\mathcal{Y}_{\mathrm{L}} f) g \mathcal{X}_{\mathrm{L}} (\phi_B^{(\beta)})^2 \mathrm{d}z_1 \cdots \mathrm{d}z_N.$$

On the other hand, applying ρ^B to Lemma 4.3, we find that

$$\mathcal{P}_-(\mathcal{Y}_{\mathrm{L}} - \hat{\mathcal{Y}}_{\mathrm{L}}) = \sum_j \hat{g}_j(\tilde{b}_1^\dagger, \ldots, \tilde{b}_N^\dagger)(s_j - 1) \quad \text{for some } \hat{g}_j(x_1, \cdots, x_N) \in \mathbb{C}[x].$$

From this relation, we see that $\mathcal{P}_-(\mathcal{Y}_{\mathrm{L}} - \hat{\mathcal{Y}}_{\mathrm{L}})f = 0$ for all $f \in \mathbb{C}[z^2]^{\mathfrak{S}_N}$. Hence we can replace $\mathcal{Y}_{\mathrm{L}}$ by $\hat{\mathcal{Y}}_{\mathrm{L}}$:

$$\begin{aligned}\langle G_{\mathrm{H}} f, g\rangle_{\mathrm{L}}^{(\beta+1)} &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathcal{P}_-(\hat{\mathcal{Y}}_{\mathrm{L}} f) g \mathcal{X}_{\mathrm{L}} (\phi_A^{(\beta)})^2 \mathrm{d}z_1 \cdots \mathrm{d}z_N \\ &= \langle \hat{\mathcal{Y}}_{\mathrm{L}} f, \mathcal{X}_{\mathrm{L}} g\rangle_{\mathrm{L}}^{(\beta)} = \langle f, \hat{G}_{\mathrm{L}} g\rangle_{\mathrm{L}}^{(\beta)}\end{aligned}$$

In the last equality, we have used the self-adjointness of the operator $\tilde{h}_j^B$. □

We then state the following results:

Proposition 4.10 $G_{\mathrm{L}} L_{\lambda+\delta}^{(\beta)} = c_\lambda^{(\beta+1)} L_\lambda^{(\beta+1)}$, $\hat{G}_{\mathrm{L}} L_\lambda^{(\beta+1)} = \tilde{c}_\lambda^{(\beta+1)} L_{\lambda+\delta}^{(\beta)}$, *where the constants* $c_\lambda^{(\beta)}$ *and* $\tilde{c}_\lambda^{(\beta)}$ *are defined in Lemma 4.1.*

Proof. From (4.11) and (4.12), we know that $(c_\lambda^{(\beta+1)})^{-1} G_{\mathrm{L}} L_{\lambda+\delta}^{(\beta)}$ satisfies the first condition of Definition 2.10 up to a constant factor. So it suffice to prove the orthogonality which can be shown in the same way as Proposition 4.4. The second equation can be proved in similar way. □

Using Proposition 4.10 and Theorem 4.9, we can prove the norm formula for $L_\lambda^{(\beta)}$.

Proposition 4.11 ([BF1, vD])

$$\begin{aligned}\langle L_\lambda^{(\beta)}, L_\lambda^{(\beta)}\rangle_{\mathrm{L}}^{(\beta)} &= N! \prod_{j=1}^N (\lambda_j + \beta(N-j))! \\ &\times \prod_{j=1}^N \Gamma(\lambda_j + \beta(N-j) + \gamma + 1/2) \prod_{k=1}^{\beta}\prod_{i<j} \frac{\lambda_i - \lambda_j - k + \beta(j-i+1)}{\lambda_i - \lambda_j + k + \beta(j-i-1)} \end{aligned} \qquad (4.13)$$

where $\Gamma(\cdot)$ *denotes the gamma function.*

Proof. The proof of this proposition is similar to the Hermite case. We only note the following formula for the case $\beta = 0$:

$$\langle L_\lambda^{(\beta=0)}, L_\lambda^{(\beta=0)}\rangle_{\mathrm{H}}^{(\beta=0)} = (\#\mathfrak{S}_N^\lambda) \prod_{j=1}^N \{\lambda_j! \cdot \Gamma(\lambda_j + \gamma + 1/2)\},$$

which follows from the norm formula of the one-variable Laguerre polynomials. □

The norm formula (4.13) can be rewritten into the following form [BF1]:

$$\langle L_\lambda^{(\beta)}, L_\lambda^{(\beta)}\rangle_{\mathrm{H}}^{(\beta)} = \frac{\prod_{j=1}^N (j\beta)!}{(\beta!)^N} \prod_{j=1}^N \Gamma(\lambda_j + \beta(N-j) + \gamma + 1/2)$$
$$\times \prod_{(i,j)\in\lambda} \frac{\{j-1+\beta(N-i+1)\}\{\lambda_i - j + 1 + \beta(\lambda_j' - i)\}}{\lambda_i - j + \beta(\lambda_j' - i + 1)}.$$

It should be remarked that other proofs of these formulas have been given via limiting procedure [BF1, vD].

5 Concluding remarks

In this paper, we have reviewed the common algebraic structure of Calogero-Sutherland-type models, i.e. the degenerate double affine Hecke algebra. From this viewpoint, we can construct the intertwining operators that map the Jack polynomials to the multivariable Hermite and Laguerre polynomials.

We restrict ourselves to symmetric polynomials though the operators σ^A and σ^B are applicable to non-symmetric case, i.e. we can obtain the non-symmetric counterparts of the multivariable Hermite and Laguerre polynomials:

$$E_w^{(\mathrm{H})\lambda}(x) = 2^{-|\lambda|/2} E_w^\lambda(\tilde{a}^\dagger)\cdot 1, \qquad E_w^{(\mathrm{L})\lambda}(z) = E_w^\lambda((\tilde{b}^\dagger)^2/2)\cdot 1.$$

Baker and Forrester named these polynomials non-symmetric Hermite and Laguerre polynomials respectively, and studied their properties [BF2, BF3]. We note that some of their results may be obtained directly form the corresponding properties of the Jack polynomials by applying the intertwiners.

Our constructs are based on the degenerate double affine Hecke algebra, so it is expected that the results given here extend to non-degenerate case. Baker and Forrester studied isomorphism between affine Hecke algebras that maps the Macdonald polynomials to the multivariable Al-Salam and Carlitz polynomials [BF4]. On the other hand, van Diejen proposed difference counterpart of the Hamiltonians $\mathcal{H}_A$ and $\mathcal{H}_B$ [vD]. It would be nice to clarify algebraic structure of his models. We hope to report on them in the near future.

Appendices

Appendix A: Proof of Lemma 4.3

In Appendix A, we will give a proof of Lemma 4.3. We remark again that the proof given in this section is a limiting case of [Ki].

We begin with seeing some properties of the anti-symmetrizer.

Lemma A.1 ([Ki]) (i) *The anti-symmetrizer $\mathcal{P}_-$ is divisible by $1+(-1)^{l(w_0)}w_0$ both on the left and on the right.*

(ii) *For all $j = 1,\ldots,N-1$, the anti-symmetrizer $\mathcal{P}_-$ is divisible by $1-s_j$ both on the left and on the right.*

Proof. (i) $\mathfrak{S}_N$ can be divided into pairs (w, ww_0). Then, rewriting into the summation over such pairs, we have

$$\mathcal{P}_- = \sum_{(w,ww_0)} \left\{(-1)^{l(w)}w + (-1)^{l(ww_0)}ww_0\right\} = \sum_{(w,ww_0)} (-1)^{l(w)}w\left\{1 + (-1)^{l(w_0)}w_0\right\}.$$

Divisibility on the left is proved similarly.

(ii) Can also be proved by similar discussion. □

From Lemma A.1 (ii), we know that Ker $\mathcal{P}_- \supset \sum_j$ Ker $(1 - s_j)$. To describe kernel of the anti-symmetrizer, we first investigate kernels of $1 - s_j$ and their union.

Lemma A.2 ([Ki]) (i) *Let V is a representation of $\mathfrak{S}_N$, and denote $V_j =$* Ker $(1-s_j)$, *$V' = \sum_j V_j$. Then V' is $\mathfrak{S}_N$-invariant.*

(ii) *Assume V is a finite-dimensional irreducible representation of $\mathfrak{S}_N$. Then we have*

$$V' = \begin{cases} 0 & (\textit{if } V \textit{ is the sign representation}), \\ V & (\textit{otherwise}). \end{cases}$$

Proof. (i) From the definition of V_j, it follows that $s_j(s_iv) = s_iv$ for all $v \in s_iV_j$. If we introduce $v_\pm = (v \pm s_iv)/2$, we see that $s_j(v_+ - v_-) = v_+ - v_-$ which means $v_+ - v_- \in V_j$. Since $v_+ \in V_i$ by definition, we obtain $v = v_+ + v_- \in V_i + V_j$. This leads to $s_iV_j \subset V_i + V_j$, which concludes the proof.

(ii) From (i), it follows that V' is a subrepresentation. Due to the irreducibility, V' can be either 0 or V. If $V' = 0$, then we have $V_j = 0$ for all j. This means that $1 - s_j$ is invertible, i.e. for all $v \in V$, there exists u such that $v = (1 - s_j)u$. Then we obtain $s_jv = -v$ for all $v \in V$, i.e. V is the sign representation. □

From Lemma A.2(ii), it immediately follows:

$$\text{Ker } \mathcal{P}_- = \sum_j \text{Ker } (1 - s_j)$$

for any finite-dimensional representation of $\mathfrak{S}_N$. Note this identity also holds for the representation of $\mathfrak{S}_N$ in the space of polynomials $\mathbb{C}[x]$, since this representation is a direct sum of finite-dimensional representations.

We now introduce operators $\hat{s}_j$ as

$$\hat{s}_j = s_j + \beta\frac{s_j - 1}{x_j - x_{j+1}}.$$

Using these operators, we can define another representation of the degenerate affine Hecke algebra $\mathfrak{H}'_0$ on $\mathbb{C}[x]$:

$$\rho'(\hat{D}^A_j) = x_j, \qquad \rho'(s_j) = \hat{s}_j.$$

Using the isomorphism ρ', we introduce deformed anti-symmetrizer $\mathcal{P}^{(\beta)}_-$ as

$$\mathcal{P}^{(\beta)}_- = \rho'(\mathcal{P}_-) = \frac{1}{\#\mathfrak{S}_N} \sum_{w \in \mathfrak{S}_N} (-1)^{l(w)}\hat{w},$$

where $\hat{w} = \rho'(w)$.

Lemma A.3 **([Ki])** Ker $\mathcal{P}_-^{(\beta)} = \text{Ker } \mathcal{P}$ *for the action of* $\mathcal{P}_-^{(\beta)}$ *in* $\mathbb{C}[x]$*:*

Proof. By similar discussion to Proposition A.1, we know that $\mathcal{P}_-^{(\beta)}$ is divisible by $\hat{s}_j - 1$ both on the left and on the right for every $j = 1, \ldots, N-1$. Hence we have

$$\text{Ker } \mathcal{P}_-^{(\beta)} \supset \sum_j \text{Ker } (1 - \hat{s}_j) = \sum_j \text{Ker } (1 - s_j) = \text{Ker } \mathcal{P}, \tag{A.1}$$

and thus $\dim(\text{Ker } \mathcal{P}_-^{(\beta)}) \geq \dim(\text{Ker } \mathcal{P}_-)$. On the other hand, if we denote $\mathbb{C}[x]_n$ as space of polynomials of order n, it is clear that $\mathcal{P}_-^{(\beta)}$ preserves $\mathbb{C}[x]_n$. Since $\dim(\text{Ker } \mathcal{P}_-^{(\beta)})$ can not decrease under specialization, it follows that $\dim(\text{Ker } \mathcal{P}_-^{(\beta)}) \leq \dim(\text{Ker } \mathcal{P}_-^{(\beta=0)}) = \dim(\text{Ker } \mathcal{P}_-)$ and hence we have

$$\dim(\text{Ker } \mathcal{P}_-^{(\beta)}) = \dim(\text{Ker } \mathcal{P}_-). \tag{A.2}$$

Thus it follows from (A.1) and (A.2) that Ker $\mathcal{P}_-^{(\beta)} = \text{Ker } \mathcal{P} = \sum_j \text{Ker } (1 - s_j)$. □

We then define $\mathcal{Y}'$ and $\hat{\mathcal{Y}}'$ as

$$\mathcal{Y}' = \rho'(\mathcal{Y}_J) = \prod_{i<j}(\beta - x_i + x_j), \qquad \hat{\mathcal{Y}}' = \rho'(\hat{\mathcal{Y}}_J) = \prod_{i<j}(-\beta - x_i + x_j).$$

Lemma A.4 **([Ki])**

$$\mathcal{P}_-(\mathcal{Y}' - \hat{\mathcal{Y}}')f = 0 \quad \textit{for all} \quad f \in \mathbb{C}[x]^{\mathfrak{S}_N}.$$

Proof. We can show that

$$(1 + (-1)^{N(N-1)/2} w_0)(\mathcal{Y}' - \hat{\mathcal{Y}}') = (\mathcal{Y}' - \hat{\mathcal{Y}}')(1 - w_0).$$

by the direct calculation. Considering the action on $\mathbb{C}[x]^{\mathfrak{S}_N}$, we have

$$(1 + (-1)^{N(N-1)/2} w_0)(\mathcal{Y}' - \hat{\mathcal{Y}}')f = 0,$$

for all $f \in \mathbb{C}[x]^{\mathfrak{S}_N}$. Using this formula and Proposition A.1 (i), we obtain the desirous result. □

From Lemma A.3 and Lemma A.4, we know that

$$\mathcal{P}_-^{(\beta)}(\mathcal{Y}' - \hat{\mathcal{Y}}')f = 0 \quad \text{for all } f \in \mathbb{C}[x]^{\mathfrak{S}_N}.$$

On the other hand, the following statement can easily be proved:

Lemma A.5 *Let $\hat{A}$ be a operator of the form, $\hat{A} = \sum_{w \in \mathfrak{S}_N} g_w \hat{w}$ with $g_w \in \mathbb{C}[x]$. If $\hat{A}f = 0$ for all $f \in \mathbb{C}[x]^{\mathfrak{S}_N}$, then $\hat{A}$ can be represented in the following form:*

$$\hat{A} = \sum_j \hat{g}_j(\hat{s}_j - 1) \quad \textit{for some} \quad \hat{g}_j \in \mathbb{C}[x]^{\mathfrak{S}_N}.$$

Proof. The operator $\hat{A}$ can be rewritten as

$$\hat{A} = \sum_{j_1,\dots,j_k} \hat{g}'_{j_1,\dots,j_k}(\hat{s}_{j_1}-1)\cdots(\hat{s}_{j_k}-1)+\hat{g}'_0 \quad \text{for some } \hat{g}'_{j_1,\dots,j_k} \in \mathbb{C}[x].$$

Then the assumption of the proposition means $\hat{g}'_0 = 0$, which gives the desirous result. □

Applying Lemma A.5 to Lemma A.4, we conclude that

$$\mathcal{P}^{(\beta)}_-(\mathcal{Y}'-\hat{\mathcal{Y}}') = \sum_j \hat{g}_j(x_1,\cdots,x_N)(\hat{s}_j-1) \quad \text{for some } \hat{g}_j(x_1,\dots,x_N) \in \mathbb{C}[x]. \qquad \text{(A.3)}$$

Applying $(\rho')^{-1}$ completes the proof of Lemma 4.3.

Appendix B: Equivalence of the two expressions for the norm formula

In Appendix B, we will give a proof of equivalence between two expressions of the norm formulas. We first begin with considering the Jack case.

Let λ be a partitions satisfying the following conditions (see Figure 1 below):

$$\begin{aligned}
&\lambda_{p-1} > \lambda_p = \cdots = \lambda_{p+r_1-1} > \lambda_{p+r_1} = \cdots = \lambda_{p+r_1+r_2-2} \\
&\quad > \cdots > \lambda_{p+r_1+\cdots+r_{m-1}} = \cdots = \lambda_{p+r_1+\cdots+r_m-1} > \lambda_{p+r_1+\cdots+r_m} = \cdots = 0, \\
&\lambda'_1 = \cdots = \lambda'_{s_1} > \lambda'_{s_1+1} = \cdots = \lambda'_{s_1+s_2} \\
&\quad > \cdots > \lambda'_{s_1+\cdots+s_{m-1}+1} = \cdots = \lambda'_{s_1+\cdots+s_m} > \lambda'_{s_1+\cdots+s_m} = \cdots = 0.
\end{aligned}$$

We further define μ as $\mu = (\lambda_1,\cdots,\lambda_p+1,\cdots,\lambda_N)$.

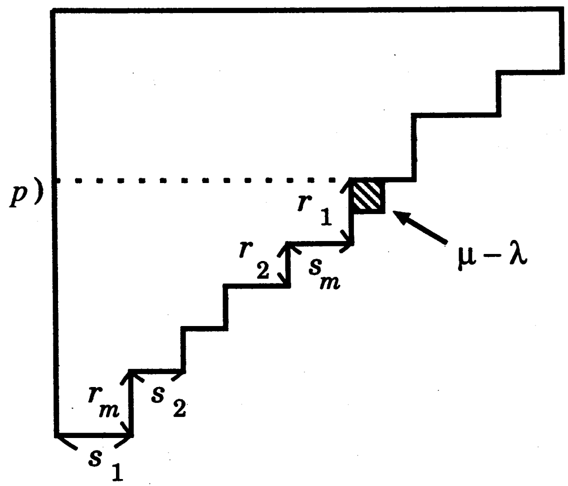

Figure 1: Young diagram of λ and μ

Calculating the ratio $\langle J^{(\beta)}_\mu, J^{(\beta)}_\mu\rangle^{(\beta)}_{\mathrm{J}} / \langle J^{(\beta)}_\lambda, J^{(\beta)}_\lambda\rangle^{(\beta)}_{\mathrm{J}}$ by using (4.5) or (4.6), one can show that both cases reduce to

$$\frac{\langle J^{(\beta)}_\mu, J^{(\beta)}_\mu\rangle^{(\beta)}_{\mathrm{J}}}{\langle J^{(\beta)}_\lambda, J^{(\beta)}_\lambda\rangle^{(\beta)}_{\mathrm{J}}} = \prod_{i=1}^{p-1} \frac{\lambda_i-\lambda_p+\beta(p-i)}{\lambda_i-\lambda_p+\beta(p-i-1)} \cdot \frac{\lambda_i-\lambda_p-1+\beta(p-i)}{\lambda_i-\lambda_p-1+\beta(p-i-1)}$$

$$\times\frac{s_m+1+\beta(r_1-1)}{1+\beta(r_1-1)}\cdots\frac{s_m+\cdots+s_1+1+\beta(r_1+\cdots+r_m-1)}{s_m+\cdots+s_2+1+\beta(r_1+\cdots+r_m-1)}$$
$$\times\frac{\beta r_1}{s_m+\beta r_1}\cdot\frac{s_m+\beta(r_1+r_2)}{s_m+s_{m-1}+\beta(r_1+r_2)}\cdots\frac{s_m+\cdots s_2+\beta(r_1+\cdots+r_m)}{s_m+\cdots+s_1+\beta(r_1+\cdots+r_m)}$$
$$\times\frac{s_m+\cdots s_1+\beta(N-p+1)}{s_m+\cdots+s_1+1+\beta(N-p)}\cdot\frac{1}{\beta}.$$

On the other hand, if we consider the simplest case $\lambda=\phi$, both (4.5) and (4.6) reduce to $\langle 1,1\rangle_J^{(\beta)}=(\beta N)!/(\beta!)^N$. Hence, by induction, we conclude that (4.5) and (4.6) are equivalent for all λ.

The Hermite and Laguerre cases can be proved in the similar fashion.

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