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ROBIN FUNCTIONS FOR COMPLEX MANIFOLDS AND APPLICATIONS

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0. Introduction. In [Y] and later in [LY] the problem of the second variation of the Robin function for a smooth variation of domains in $\mathbf{C}^n$ for $n \geq 2$ was studied. Precisely, let $\mathcal{D} = \cup_{t\in B}(t, D(t)) \subset B \times \mathbf{C}^n$ be a variation of domains $D(t)$ in $\mathbf{C}^n$ each containing a fixed point z_0 and with $\partial D(t)$ of class C^∞ for $t \in B := \{t \in \mathbf{C} : |t| < \rho\}$. We let $g(t,z)$ for $t \in B$ and $z \in \overline{D(t)}$ be the $\mathbf{R}^{2n}$-Green function for the domain $D(t)$ with pole at z_0; i.e., $g(t,z)$ is harmonic in $D(t)\setminus\{z_0\}$, $g(t,z) = 0$ for $z \in \partial D(t)$, and $g(t,z) - \frac{1}{||z-z_0||^{2n-2}}$ is harmonic near z_0. We call

$$\lambda(t) := \lim_{z\to z_0}[g(t,z) - \frac{1}{||z-z_0||^{2n-2}}]$$

the *Robin constant* for $(D(t), z_0)$. Then

$$\frac{\partial^2\lambda}{\partial t\partial\bar{t}}(t) = -c_n\int_{\partial D(t)} k_2(t,z)||\nabla_z g||^2 d\sigma_z - 4c_n\int\int_{D(t)}\sum_{a=1}^n |\frac{\partial^2 g}{\partial t\partial\bar{z}_a}|^2 dV_z. \tag{1}$$

Here, c_n is a positive dimensional constant and

$$k_2(t,z) := ||\nabla_z\psi||^{-3}[\frac{\partial^2\psi}{\partial t\partial\bar{t}}||\nabla_z\psi||^2 - 2\Re\{\frac{\partial\psi}{\partial t}\sum_{a=1}^n\frac{\partial\psi}{\partial\bar{z}_a}\frac{\partial^2\psi}{\partial\bar{t}\partial z_a}\} + |\frac{\partial\psi}{\partial t}|^2\Delta_z\psi],$$

where $\psi(t,z)$ is a defining function for $\mathcal{D}$, is the so-called *Levi-curvature* of $\partial\mathcal{D}$ at (t,z); the numerator is the sum of the Levi-form of ψ applied to the n complex tangent vectors $(-\frac{\partial\psi}{\partial z_j}, 0, ..., \frac{\partial\psi}{\partial t}, 0, ...)$. In particular, if $\mathcal{D}$ is pseudoconvex (strictly pseudoconvex) at a point (t,z) with $z \in \partial D(t)$, it follows that $k_2(t,z) \geq 0$ ($k_2(t,z) > 0$) so that $-\lambda(t)$ is subharmonic in B. Given D a bounded domain in $\mathbf{C}^n$, we let $\Lambda(z)$ be the Robin constant for (D,z). If we fix a point $\zeta_0 \in D$, for $\rho > 0$ sufficiently small and $a \in \mathbf{C}^n$, the disk $\{\zeta = \zeta_0 + at,\ |t| < \rho\} := \zeta_0 + aB$ is contained in D. Under the biholomorphic mapping $T(t,z) = (t, z - at)$ of $B \times D$, we get the variation of domains $\mathcal{D} = T(B \times D)$ where each domain $D(t) := T(t,D) = D - at$ contains ζ_0. Letting $\lambda(t) = \Lambda(\zeta_0 + at)$ denote the Robin constant for $(D(t), \zeta_0)$ and using (1) yields part of the following result, which was proved in [Y] and [LY].

Theorem. *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ with C^2 boundary. Then $\log(-\Lambda(z))$ and $-\Lambda(z)$ are real-analytic, strictly plurisubharmonic exhaustion functions for D.*

In this note, we study a generalization of the second variation formula (1) to complex manifolds. We use our new formula to develop a "rigidity lemma" which allows us to construct, in certain cases, strictly plurisubharmonic exhaustion functions for Levi-pseudoconvex subdomains D of complex manifolds; i.e., we use the Robin function to verify that D is Stein. We remark that when we use the term *pseudoconvex* in describing certain complex manifolds or domains in complex manifolds, we always mean *Levi-pseudoconvex*.

1. The variation formula. Our general set-up is this: let M be an $n-$dimensional complex manifold (compact or not) equipped with a Hermitian metric

$$ds^2 = \sum_{a,b=1}^n g_{a\bar{b}}dz_a \otimes d\bar{z}_b$$

and let $\omega := i\sum_{a,b=1}^n g_{a\bar{b}}dz_a \wedge d\bar{z}_b$ be the associated (real) $(1,1)-$form. As in the introduction, we take $n \geq 2$. We write $g^{\bar{a}b} := (g_{a\bar{b}})^{-1}$ for the elements of the inverse matrix to $(g_{a\bar{b}})$. Given the standard operators $*, \partial, \overline{\partial}, d = \partial + \overline{\partial}, \delta := - * \partial *$, we get the Laplacian operator

$$\Delta = \delta\overline{\partial} + \overline{\partial}\delta + \overline{\delta}\partial + \partial\overline{\delta}$$

which, in local coordinates acting on functions has the form

$$\Delta u = -2\{\sum_{a,b=1}^n g^{\bar{b}a}\frac{\partial^2 u}{\partial\bar{z}_b\partial z_a} + \frac{1}{2}\sum_{a,b=1}^n(\frac{1}{G}\frac{\partial(Gg^{\bar{b}a})}{\partial z_a}\frac{\partial u}{\partial\bar{z}_b} + \frac{1}{G}\frac{\partial(Gg^{\bar{a}b})}{\partial\bar{z}_a}\frac{\partial u}{\partial z_b})\}$$

where $G := \det(g_{a\bar{b}})$. We remark that if ds^2 is Kähler, i.e., if $d\omega = 0$, then $\Delta u = -2\sum_{a,b=1}^{n} g^{\bar{b}a}\frac{\partial^2 u}{\partial\bar{z}_b\partial z_a}$.

Given a nonnegative C^∞ function $c = c(z)$ on M, we call a C^∞ function u on an open set $D \subset M$ *c−harmonic* on D if $\Delta u + cu = 0$ on D. In particular, if we fix a point $p_0 \in M$ and a coordinate neighborhood U of p_0, we can find a c−harmonic function Q_0 in $U \setminus \{p_0\}$ satisfying

$$\lim_{p\to p_0}\frac{Q_0(p)}{d(p,p_0)^{2n-2}} = 1$$

where $d(p,p_0)$ is the geodesic distance (with respect to the metric ds^2) between p and p_0. We call Q_0 a *fundamental solution* for Δ and c at p_0. Fixing p_0 in a smoothly bounded domain $D \subset\subset M$ and fixing a fundamental solution Q_0, the *c−Green function* g for (D,p_0) is the c−harmonic function in $D\setminus\{p_0\}$ satisfying $g = 0$ on ∂D (g is continuous up to ∂D) and $g(p) - Q_0(p)$ is regular at p_0. We note that, provided $c \not\equiv 0$, the c−Green function *always* exists (cf. [NS]) and is nonnegative on D. Then

$$\lambda := \lim_{p\to p_0}[g(p) - Q_0(p)]$$

is called the *c−Robin constant* for (D,p_0).

Now let $\mathcal{D} = \cup_{t\in B}(t, D(t)) \subset B\times M$ be a variation of domains $D(t)$ in M each containing a fixed point p_0 and with $\partial D(t)$ of class C^∞ for $t \in B$. Let $g(t,z)$ be the c−Green function for $(D(t),p_0)$ and $\lambda(t)$ the corresponding c−Robin constant.

We have

$$\frac{\partial^2\lambda}{\partial t\partial\bar{t}}(t) = -c_n\int_{\partial D(t)} k_2(t,z)\sum_{a,b=1}^{n}(g^{\bar{a}b}\frac{\partial g}{\partial\bar{z}_a}\frac{\partial g}{\partial z_b})d\sigma_z$$

$$-4c_n\{||\bar{\partial}\frac{\partial g}{\partial t}||^2_{D(t)} + \frac{1}{2}||\sqrt{c}\frac{\partial g}{\partial t}||^2_{D(t)} + \int\int_{D(t)}[\Re\{\frac{1}{i}\frac{\partial g}{\partial t}\bar{\partial}\frac{\partial g}{\partial t}\wedge\partial*\omega\} + \frac{1}{2i}|\frac{\partial g}{\partial t}|^2\bar{\partial}\partial*\omega]\}$$

where $||f||^2_{D(t)} = \int_{D(t)} f\wedge *\bar{f} \geq 0$, $d\sigma_z$ is the area element on $\partial D(t)$ with respect to the Hermitian metric, and

$$k_2(t,z) :=$$

$$[\sum_{a,b=1}^{n} g^{\bar{a}b}\frac{\partial\psi}{\partial\bar{z}_a}\frac{\partial\psi}{\partial z_b}]^{-3/2}[\frac{\partial^2\psi}{\partial t\partial\bar{t}}(\sum_{a,b=1}^{n} g^{\bar{a}b}\frac{\partial\psi}{\partial\bar{z}_a}\frac{\partial\psi}{\partial z_b}) - 2\Re\{\frac{\partial\psi}{\partial t}(\sum_{a,b=1}^{n} g^{\bar{a}b}\frac{\partial\psi}{\partial\bar{z}_a}\frac{\partial^2\psi}{\partial z_b\partial\bar{t}})\} + |\frac{\partial\psi}{\partial t}|^2(\sum_{a,b=1}^{n} g^{\bar{a}b}\frac{\partial^2\psi}{\partial\bar{z}_a\partial z_b})],$$

$\psi(t,z)$ being a defining function for $\mathcal{D}$.

Note that if $\mathcal{D}$ is pseudoconvex at a point $(t,z) \in \partial\mathcal{D}$ with $z \in \partial D(t)$, then $k_2(t,z) \geq 0$. This follows since we can always choose local coordinates near a point $z \in M$ so that $g_{a\bar{b}}(z) = \delta_{ab}$. A simple calculation shows that $\partial*\omega = 0$ if ds^2 is a Kähler metric; hence we have the following result.

Corollary 1.1. *Suppose that ds^2 is a Kähler metric on M. Then*

$$\frac{\partial^2\lambda}{\partial t\partial\bar{t}}(t) = -c_n\int_{\partial D(t)} k_2(t,z)\sum_{a,b=1}^{n}(g^{\bar{a}b}\frac{\partial g}{\partial\bar{z}_a}\frac{\partial g}{\partial z_b})d\sigma_z - 4c_n\{||\bar{\partial}\frac{\partial g}{\partial t}||^2_{D(t)} + \frac{1}{2}||\sqrt{c}\frac{\partial g}{\partial t}||^2_{D(t)}\}. \quad (1')$$

In particular, if $\mathcal{D}$ is pseudoconvex in $B\times M$, then $-\lambda(t)$ is subharmonic on B.

Remark 1. Formula (1′) is valid under the weaker assumption that the *complex torsion* of the metric $g_{a\bar{b}}$ vanishes. We do not discuss this notion here. Note that (1′) reduces to (1) if $g_{a\bar{b}} = \delta_{ab}$ and $c \equiv 0$.

We consider the same situation as in the corollary. From the variation formula (1′) and continuity of $g(t,z)$ up to $\partial D(t)$, we get the following result.

Lemma 1.2 (rigidity). *Assume $\mathcal{D}$ is pseudoconvex in $B \times M$, ds^2 is a Kähler metric on M and that there exists $t_0 \in B$ such that $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t_0) = 0$. If $c(z) \not\equiv 0$ on $D(t_0)$, then*

$$\frac{\partial g}{\partial t}(t_0, z) \equiv 0 \text{ on } \overline{D(t_0)}.$$

Remark 2. The same conclusion is valid if we assume that $\partial D(t_0)$ has one strictly pseudoconvex boundary point (instead of (or in addition to) assuming $c(z) \not\equiv 0$ on $D(t_0)$). However, the importance of the above formulation of the rigidity lemma is that, as we will see below, the function c gives us extra flexibility in order to *deduce* strict pseudonvexity in certain cases.

2. Complex Lie groups. We apply the rigidity lemma to the study of complex Lie groups. Let M be a complex Lie group of complex dimension n with identity e equipped with a Kähler metric ds^2 and let $c = c(z)$ be a nonnegative C^∞ function on M. Let $D \subset M$ be a domain in M with smooth boundary. For $z \in D$, let

$$D(z) := \{wz^{-1} \in D : w \in D\} = D \cdot z^{-1}$$

be right-translation (multiplication) of D by z^{-1}. Note that $D(z)$ is a smoothly bounded domain in M which contains e if $z \in D$; if D and hence $D(z)$ is unbounded, the c–Green function for $(D(z), e)$ can be defined as a limit of c–Green functions for $(D_k(z), e)$ where $\{D_k(z)\}$ are bounded domains with $D_k(z) \subset\subset D_{k+1}(z)$ and $\cup D_k(z) = D(z)$. Let $\Lambda(z)$ denote the c–Robin constant for $(D(z), e)$ (we assume, apriori, that a fundamental solution Q_0 for Δ and c at e is fixed). Our first main result is the following.

Theorem 2.1. *Suppose $D \subset\subset M$ is pseudoconvex. Then*

1. $-\Lambda(z)$ is a plurisubharmonic exhaustion function for D;

2. if $c > 0$, then $-\Lambda(z)$ is a strictly plurisubharmonic exhaustion function for D if and only if D is Stein; indeed, if the complex Hessian matrix $[\frac{\partial^2(-\Lambda)}{\partial z_j \partial \bar{z}_k}(\zeta)]$ has a zero eigenvalue with (geometric) multiplicity $k \geq 1$ at some *point $\zeta \in D$, than the complex Hessian matrix of* any *plurisubharmonic exhaustion function $s(z)$ for D has a zero eigenvalue with (geometric) multiplicity at least k* at each *point $z \in D$.*

We will sketch the proof of Theorem 2.1. First we remark that there exist n linearly independent left-invariant holomorphic vector fields $X_1, ..., X_n$ such that $\text{Exp}tX_j$, $j = 1, ..., n$ form local coordinates in a neighborhood V of the identity $e \in M$; then $\zeta\text{Exp}tX_j$, $j = 1, ..., n$ form local coordinates in a neighborhood ζV of $\zeta \in M$. If we fix a direction vector α and consider the complex disk $t \to \zeta + \alpha t$ for small $|t|$, we can assume that $\zeta \text{Exp}tX_1 = \zeta + \alpha t$; for simplicity, we write $X := X_1$. This suggests, as in the variation of domains case described in the introduction, how to set up a variation of domains in the setting of the complex Lie group M. We note, for future use, that $t \to z\text{Exp}tX$ is the unique integral curve to X taking the value $z \in M$ for $t = 0$.

We now let ζ be a fixed point in D and choose $B = \{t \in \mathbf{C} : |t| < \rho\}$ with ρ sufficiently small so that

$$\eta := \zeta\text{Exp}tX = \zeta + \alpha t \in D \text{ for all } t \in B. \tag{2}$$

Let $T : B \times M \to B \times M$ via $T(t, z) = (t, F(t, z)) := (t, w)$ where $w = F(t, z) := z(\zeta\text{Exp}tX)^{-1}$. Then $\mathcal{D} := T(B \times D)$ defines a variation of domains $D(t) := F(t, D) = \{z(\zeta\text{Exp}tX)^{-1} \in M : z \in D\} = D \cdot (\zeta\text{Exp}tX)^{-1}$.

Let $g(t, w)$ be the c–Green function for $(D(t), e)$ and let $\lambda(t) := \Lambda(\zeta\text{Exp}tX)$ for $t \in B$; this is the c–Robin constant for $(D(t), e)$ (note $e \in D(t)$ if $t \in B$ by (2)). Then

$$\sum_{j,k=1}^{n} \frac{\partial^2(-\Lambda)}{\partial \eta_j \partial \overline{\eta}_k}(\zeta)\alpha_j \overline{\alpha}_k = \frac{\partial^2(-\Lambda)}{\partial t \partial \bar{t}}(\zeta\text{Exp}tX)|_{t=0} = \frac{\partial^2(-\lambda)}{\partial t \partial \bar{t}}(0). \tag{3}$$

The plurisubharmonicity of $-\Lambda(z)$ now follows from Corollary 1.1 and the fact that $\mathcal{D} := T(B \times D)$ is the biholomorphic image of the pseudoconvex set $B \times D$; indeed, for each $t \in B$, the function $z = \phi(t, w) = (\phi_1(t, w), ..., \phi_n(t, w)) := w\zeta\text{Exp}tX = F^{-1}(t, w)$ is the well-defined holomorphic inverse map of $z \to w = F(t, z)$ for all $w \in M$. Standard arguments show that $\Lambda(z) \to -\infty$ as $z \to z' \in \partial D$ which proves 1. of the theorem.

We will prove 2. in the case where $k=1$; here, we use the assumption that $c>0$ and apply the rigidity lemma. The key observation is the following.

Claim: Suppose that $\frac{\partial^2\lambda}{\partial t\partial\bar{t}}(0)=0$.

a. *$z\in D$ (resp. ∂D, $\overline{D}^c$) if and only if $z\mathrm{Exp}tX\in D$ (resp. ∂D, $\overline{D}^c$) for* **all** *$t\in\mathbf{C}$;*

b. *$D\cdot z^{-1}=D\cdot(z\mathrm{Exp}tX)^{-1}$ (resp. ∂D, $\overline{D}^c$) for* **all** *$t\in\mathbf{C}$ and for each $z\in M$.*

To prove the claim, we apply the rigidity lemma to show that the left-invariant holomorphic vector field X is a non-vanishing holomorphic vector field on M satisfying the property that ***any integral curve*** $z(t)$ ***of*** *X with initial value $X(z_0)$ for $z_0=z(0)\in\partial D$ remains in ∂D for all $t\in\mathbf{C}$*. This is one implication in part a. of the claim for ∂D.

Recall that $z=\phi(t,w)=(\phi_1(t,w),...,\phi_n(t,w)):=w\zeta\mathrm{Exp}tX=F^{-1}(t,w)$ for all $w\in M$. Let $t\to\phi(t,e)$ be the (moving) image under ϕ of the identity element. Note that if $ds_t^2(z)$ denotes the pull-back of the metric $ds^2(w)$ under $F(t,z)$, then the Green function $G(t,z)$ for D with pole at $\phi(t,e)$ (with respect to $ds_t^2(z)$) equals $g(t,w)$. The assumption that $\frac{\partial^2\lambda}{\partial t\partial\bar{t}}(0)=0$ yields, by the rigidity lemma, $\frac{\partial g}{\partial t}(0,w)\equiv 0$ for $w\in\overline{D(0)}$; this becomes

$$\frac{\partial G}{\partial t}(0,z)+\sum_{a=1}^{n}[\frac{\partial G}{\partial z_a}(0,z)\frac{\partial\phi_a}{\partial t}(0,F(0,z))+\frac{\partial G}{\partial\bar{z}_a}(0,z)\frac{\partial\bar{\phi}_a}{\partial t}(0,F(0,z))]=0$$

for $z\in\overline{D}$. But $\frac{\partial\bar{\phi}_a}{\partial t}(0,F(0,z))=0$ since $\phi(t,w)=(\phi_1(t,w),...,\phi_n(t,w))$ is holomorphic in t; and $\frac{\partial G}{\partial t}(0,z)=0$ for $z\in\partial D$ since $G(t,z)=0$ for $z\in\partial D$ and $t\in B$. Thus

$$\sum_{a=1}^{n}\frac{\partial G}{\partial z_a}(0,z)\frac{\partial\phi_a}{\partial t}(0,F(0,z))=0 \tag{4}$$

for $z\in\partial D$. Since $\phi(t,w)$ is defined for all $w\in M$, the vector field

$$Y(z):=\sum_{a=1}^{n}\frac{\partial\phi_a}{\partial t}(0,F(0,z))\frac{\partial}{\partial z_a}$$

is a globally defined (on M) non-vanishing holomorphic vector field; using the fact that

$$(\frac{\partial G}{\partial z_1}(0,z),...,\frac{\partial G}{\partial z_n}(0,z))$$

is a (complex) normal vector to ∂D at z, it can be shown that (4) implies that any integral curve $z(t)$ of Y with initial value $Y(z_0)$ for $z_0=z(0)\in\partial D$ remains in ∂D for all $t\in\mathbf{C}$. Thus, to verify the italicised statement, it suffices to show that $Y=X$.

Since X is left-invariant, if $X(z)=\sum_{a=1}^{n}\eta_a\frac{\partial}{\partial z_a}$, then $[\frac{\partial}{\partial t}(z\mathrm{Exp}tX)_a]|_{t=0}=\eta_a(z)$, $a=1,...,n$. But for $w=z\zeta^{-1}$,

$$\frac{\partial\phi_a}{\partial t}(0,F(0,z))=\frac{\partial\phi_a}{\partial t}(0,w)=[\frac{\partial}{\partial t}(w\zeta\mathrm{Exp}tX)_a]|_{t=0}=\eta_a(w\zeta)=\eta_a(z),$$

which gives the result.

The proof of the claim is now immediate. For example, to establish a. for ∂D; i.e., to show $z\in\partial D$ if and only if $z\mathrm{Exp}tX\in\partial D$ for all $t\in\mathbf{C}$, the "only if" direction has already been proved. Suppose now that $z\mathrm{Exp}tX\in\partial D$ for all $t\in\mathbf{C}$. Since

$$z=z(\mathrm{Exp}tX)(\mathrm{Exp}(-tX)):=z'\mathrm{Exp}(-tX)$$

where $z'=z\mathrm{Exp}tX\in\partial D$, the previous argument shows that $z\in\partial D$. Since ∂D is a smooth, closed $(2n-1)$-dimensional real hypersurface in M, the analogous results for D and $\overline{D}^c$ follow from uniqueness of the integral curve $t\to z\mathrm{Exp}tX$. Similarly we prove b. only for ∂D. Let $z_1\in\partial D$ and $z\in M$. Since $z_1\mathrm{Exp}tX\in\partial D$ for all $t\in\mathbf{C}$ from a. of the claim, the equation

$$z_1z^{-1}=z_1(\mathrm{Exp}tX)(\mathrm{Exp}(-tX))z^{-1}=z_1(\mathrm{Exp}tX)(z\mathrm{Exp}tX)^{-1}$$

yields b. of the claim.

We can now finish the proof of 2. of the theorem. For a point $\zeta \in D$, let $a_i(\zeta)$, $i = 1,\dots,n$ denote the eigenvalues of $[\frac{\partial^2(-\Lambda)}{\partial z_j \partial \bar{z}_k}(\zeta)]$ at ζ. To prove 2. in the case $k = 1$, we suppose there exists a point $\zeta \in D$ with $a_1(\zeta) = 0$; without loss of generality, we can assume that $\zeta \mathrm{Exp} tX_1 = \zeta + \alpha t$ gives the direction of the corresponding eigenvector; i.e.,

$$\frac{\partial^2(-\Lambda)}{\partial t \partial \bar{t}}(\zeta + \alpha t)|_{t=0} = 0. \tag{5}$$

Taking $X = X_1$ in the previous claim, $D_1(t) := D \cdot (\zeta \mathrm{Exp} tX_1)^{-1}$ and $\lambda_1(t) := \Lambda(\zeta \mathrm{Exp} tX_1)$, (5) becomes $\frac{\partial^2(-\lambda_1)}{\partial t \partial \bar{t}}(0) = 0$. Then the integral curve $t \to z\mathrm{Exp} tX_1$, $t \in \mathbf{C}$, satisfies the conditions of the claim. In particular, if $z \in D$, then $D \cdot z^{-1} = D \cdot (z\mathrm{Exp} tX_1)^{-1}$ for all $t \in \mathbf{C}$ which implies that

$$\Lambda(z\mathrm{Exp} tX_1) = \Lambda(z) \text{ for all } t \in \mathbf{C}.$$

But $-\Lambda$ is an exhaustion function for D; hence the image C_z of the integral curve $t \to z\mathrm{Exp} tX_1$, $t \in \mathbf{C}$ is compactly contained in D and $-\Lambda$ is constant on C_z. In particular, $-\Lambda$ is *harmonic* on C_z for each $z \in D$; i.e., $[\frac{\partial^2(-\Lambda)}{\partial z_j \partial \bar{z}_k}(z)]$ has a zero eigenvalue $a_1(z)$ for each $z \in D$. But then if s is *any* plurisubharmonic exhaustion function for D, s is also subharmonic and entire on each complex curve C_z and hence constant (and harmonic) on this curve, which implies that $[\frac{\partial^2 s}{\partial z_j \partial \bar{z}_k}(z)]$ has a zero eigenvalue for each $z \in D$. In particular, D is not Stein.

Remark 3. Note that if M is a Stein manifold, then each pseudoconvex $D \subset\subset M$ is Stein; this occurs, for example, if M is a simply connected solvable Lie group or if M is connected and semi-simple (cf. [GR]).

3. Complex homogeneous spaces. In this section, we let M be a complex space with the property that there exists a complex Lie group $G \subset \mathrm{Aut} M$ of complex dimension n which acts transitively on M. As prototypical examples, we can take $M = \mathbf{P}^N$ = complex projective space, or, more generally, we can take $M = G(k, N)$ = complex Grassmann manifold (and $G = \mathrm{Aut} M$). Let $D \subset\subset M$ be a domain with smooth boundary. For $z \in M$, we let

$$D(z) := \{g \in G : g(z) \in D\}$$

be a (possibly unbounded) domain in G. Note that if $z \in D$, then the identity element e of G lies in $D(z)$. Thus if we let ds^2 be a Kähler metric on G and let c be a nonnegative smooth function on G, we can form the c–Robin constant $\lambda(z)$ for $(D(z), e)$ (recall that the c–Green function is defined by the usual exhaustion method for unbounded domains). Using the ideas and techniques from the previous section, we can prove the following result.

Theorem 3.1. *Suppose D is pseudoconvex in M. Then for $z \in D$, $D(z)$ is pseudoconvex in G and $-\lambda(z)$ is a plurisubharmonic exhaustion function for D. Furthermore, if $c > 0$ in G and G is doubly transitive on M, then $-\lambda(z)$ is strictly plurisubharmonic; i.e., D is Stein.*

Recall that G is *doubly transitive on* M if for pairs of points (a, b), $(c, d) \in M$, there exists $g \in G$ with $g(a) = c$ and $g(b) = d$. This is equivalent to the *three point property of* (M, G): for each triple of points $a, b, c \in M$, there exists $g \in G$ with $g(a) = a$ and $g(b) = c$. Details of the proof of Theorem 3.1 will be given elsewhere.

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