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ON THE RUMIN COMPLEX

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As is well known, Rumin introduced his complex for contact manifolds. Obviously, his method is applicable to the strongly pseudo convex boundary case(see [Ru], and also [A-M1]). Namely, let $(M, {}^0T'')$ be a strongly pseudo convex CR manifold in a complex manifold N, then by Rumin we have a differential complex on M,

$$\wedge^{n-1}({}^0T')^* \xrightarrow{D} \theta\wedge\wedge^{n-2}({}^0T')^*\wedge^1({}^0T'')^* \xrightarrow{\overline{\partial}_b^{(1)}} \theta\wedge\wedge^{n-2}({}^0T')^*\wedge^2({}^0T'')^*$$

$$\xrightarrow{\overline{\partial}_b^{(p-1)}} \theta\wedge\wedge^{n-2}({}^0T')^*\wedge^p({}^0T'')^* \xrightarrow{\overline{\partial}_b^{(p)}} \theta\wedge\wedge^{n-2}({}^0T')^*\wedge^{p+1}({}^0T'')^*,$$

which recovers the Kohn-Rossi cohomology.

$$\begin{aligned} &for\ p=1,\ Ker\ \partial_b^{(1)}/Im\ D \simeq H_b^{(1)}(M,\wedge^{(n-1)}(T')^*),\\ &for\ p>1,\ Ker\ \overline{\partial}_b^{(p)}/Im\ \overline{\partial}_b^{(p-1)} \simeq H_b^{(p)}(M,\wedge^{(n-1)}(T')^*),\end{aligned}$$

where D is a second order differential operator which is introduced by Rumin, and ${}^0T' = \overline{{}^0T''}$ and θ means a contact form(this is a part of Rumin's result but a typical one. More precisely,see Sect.1 in this paper).
However this differential complex has an essential weak point. Namely, $\overline{\partial}_b^{(p)}$ has a natural extension to an ambient space N, but it is not clear if the above second order differential operator D has a natural extension to an operator on N or not. By this reason, we propose a new complex, based on Rumin complex, which is applicable to several complex variables(see Sect.2).

Sect.1. CR structures and Rumin complex

Let $(M, {}^0T'')$ be a CR structure. This means that: M is a real 2n-1 dimensional C^∞ manifold and ${}^0T''$ is a complex subbundle of the complexfied tangent bundle satisfying:

$$1)\ {}^0T'' \cap {}^0T' = 0,\ dim_C(C \otimes TM/({}^0T'' + {}^0T')) = 1$$
$$2)\ [\Gamma(M, {}^0T''), \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''),$$

where ${}^0T' = \overline{{}^0T''}$. In this paper we assume more. Namely, we assume that there is a real global C^∞ vector bundle ξ satisfying:

$$3)\ \xi_p \notin {}^0T''_p + {}^0T'_p\ for\ every\ point\ p\ of\ M,$$

For brevity, we use the following notation.

$$4)\ T' = C \otimes \xi + {}^0T'.$$

Now we set a real one form θ by

$$\theta(\xi) = 1,$$
$$\theta\,|_{{}^0T'' + {}^0T'} = 0.$$

Let

$$\Omega = d\theta.$$

If this 2-form is positive or negative definite, then our CR-structure $(M, {}^0T'')$ is called strongly pseudo convex. From now on, we assume that our CR is strongly pseudo convex. By using these notations, we can introduce a C^∞ vector bundle decomposition:

$$5)\ \wedge^k (C \otimes TM)^* = \sum\nolimits_{p+q=k-1, p,q \geq 0} \theta \wedge \wedge^p({}^0T')^* \wedge \wedge^q({}^0T'')^*$$
$$\sum\nolimits_{r+s=k, r,s \geq 0} \wedge^r ({}^0T')^* \wedge \wedge^s({}^0T'')^*$$

We fix this decompostion. And we would like to consider a double complex (for the precise definition, see [A4]). For $u \in \Gamma(M, \theta \wedge \wedge^{p-1}({}^0T')^* \wedge \wedge^{q-1}({}^0T'')^*)$, we set an element of $\Gamma(M, \wedge^p({}^0T')^* \wedge \wedge^q({}^0T'')^*)$ by

$$u \rightarrow (du)_{\wedge^p({}^0T')^* \wedge \wedge^q({}^0T'')^*}$$

Proposition 1.1. *This map is a bundle map.*

(The proof is a direct computation, and more precisely, see Sect.3.)

Proposition 1.2. *If $p+q \geq n$, then this map is surjective and especially, if $p+q=n$, then by comparing dimensions, this is isomorphic.*

(For the proof, see lemma 3.3 in [A4])

We use the notation κ^p for this isomorphic map from $\theta \wedge \wedge^{p-1}({}^0T'') \wedge \wedge^{q-1}({}^0T'')$ to $\wedge^p({}^0T')^* \wedge \wedge^q({}^0T'')^*$, where $q = n-p$. By using this κ^p, we set an element ψ_u of $\Gamma(M, \theta \wedge \wedge^{p-1}({}^0T'')^* \wedge \wedge^{q-1}({}^0T'')^*)$ by

$$\psi_u = (\kappa^p)^{-1}((\overline{\partial}_b u)_{\wedge^p({}^0T')^* \wedge \wedge^{n-p}({}^0T'')^*})$$

where $(\overline{\partial} u)_{\wedge^p({}^0T')^* \wedge \wedge^{n-p}({}^0T'')^*}$ means the projection of $\overline{\partial} u$ to $\wedge^p({}^0T')^* \wedge \wedge^{n-p}({}^0T'')^*$ according to (5). So by the definition of ψ_u, our ψ_u includes the first derivatiove of u.

Sect.2. New complex

Now we introduce a new complex. For a simplicity, we discuss only in the case $p = n-2$, which is quite related to deformation theory. We set

$H^0 = \{u : u \in \Gamma(M, \wedge^{n-1}(T')^*), (\partial_b u)_{\wedge \wedge^{n-1}({}^0T')^* \wedge \wedge({}^0T')^*} = 0\}$

$H^1 = \{u : u \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^* \wedge \wedge({}^0T'')^*), (\overline{\partial}_b^{(1)} u)_{\wedge \wedge^{n-1}({}^0T')^* \wedge \wedge^2({}^0T')^*} = 0\}$

$H^2 = \{u : u \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^* \wedge \wedge^2({}^0T'')^*), (\overline{\partial}_b^{(2)} u)_{\wedge \wedge^{n-1}({}^0T')^* \wedge \wedge^3({}^0T')^*} = 0\}$

Then by defintion our $(H^i, \overline{\partial}_b)$ is a differential complex

$$H^0 \xrightarrow{\overline{\partial}_b} H^1 \xrightarrow{\overline{\partial}_b^{(1)}} H^2$$

Of course H^1(resp. H^2) is nothing but our $F^{n-2,1}$(resp. $F^{n-2,2}$) (see [A-M1],[A4]). And we note that our H^0 doesn't come from C^∞ sections of any C^∞ vector bundle on M. This is purely a vector space of some $\wedge^{n-1}({}^0T')^*$-valued C^∞ sections. We put an L^2 norm on these spaces and discuss the Kodaira Hodge type decomposition theorem on H^1. For this, we have to show an a priori estimate. And a difficult problem is to compute the adjoint operator of $\overline{\partial}_b$. By the definition of H^0, H^0 is a subspace of

$$\Gamma(M, \wedge^{n-1}(T')^*) = \Gamma(M, \wedge^{n-1}({}^0T')^*) + \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T'')^*).$$

And in this canonical decomposition of $\Gamma(M, \wedge^{n-1}(T')^*)$, our H^0 can be regarded as a graph of the following map.

For $u \in \Gamma(M, \wedge^{n-1}({}^0T')^*)$, we set

$$\psi_u \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^*),$$

where ψ_u is introduec in Sect.1 in this paper. So

$$H^0 = \{v : v = u + \psi_u,\ u \in \Gamma(M, \wedge^{n-1}({}^0T')^*)\} \subset \Gamma(M, \wedge^{n-1}(T')^*).$$

For this correspondence, we call the *graph map i*. On $\Gamma(M, \wedge^{n-1}(T')^*)$, and $\Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^* \wedge \wedge^p({}^0T'')^*), p = 1, 2, ..,$ we put L^2 norm and consider the Kodaira Hodge decomposition theorem on $(H^p, \overline{\partial}_b)$. The problem is to show an a priori estimate. In proving an a priori estimate, we have to compute, explicitly " the adjoint operator " on H^p spaces (namely we have to write down the term of " the adjoint operator "., otherwise, it is impossible to obtain an a apriori estimate. We discuss this in the next section.

Sect.3. The adjoint operators on $\mathbf{H^p}$ spaces

We consider the projection of $\Gamma_2(M, \wedge^{n-1}(T')^*)$ to $\tilde{H}^0$, where $\Gamma_2(M, \wedge^{n-1}(T')^*)$ means the L_2 - completion of $\Gamma(M, \wedge^{n-1}(T')^*)$, and $\tilde{H}^0$ means the L_2 closure of H^0 in $\Gamma_2(M, \wedge^{n-1}(T')^*)$. We recall the graph map i.

$$\Gamma(M, \wedge^{n-1}({}^0T')^*) \xrightarrow{graph\ map\ i} H^0$$

We use the notation A for the composition map of this graph map i and the inclusion map of H^1 to $\Gamma(M, \wedge^{n-1}(T')^*)$. So A is a map from $\Gamma(M, \wedge^{n-1}({}^0T')^*)$ to $\Gamma(M, \wedge^{n-1}(T')^*)$. By the way, if we put a L_2 norm on $\Gamma(M, \wedge^{n-1}({}^0T'')^*)$ by : for $v \in \Gamma(M, \wedge^{n-1}({}^0T'')^*)$,

$$\|v\|^2_{\Gamma(M, \wedge^{n-1}({}^0T'')^*)} = \|v\|^2 + \|\psi_v\|^2 \ (\ a\ graph\ norm\ ),$$

our i is a norm preserving map (almost tautology). Therefore

$$\begin{aligned}(i^* iv, w)_{\Gamma(M, \wedge^{n-1}({}^0T'')^*)} &= (iv, iw)\\ &= (v, w) + (\psi_v, \psi_w)\\ &= (v, w)_{\Gamma(M, \wedge^{n-1}({}^0T'')^*)}\end{aligned}$$

Namely

$$i^* i = identity\ on\ \Gamma(M, F)$$

Especially,

$$i^*(v + \psi_v) = v,\ for\ v \in \Gamma(M, \wedge^{n-1}({}^0T'')^*).$$

Here i^* means the adjoint operator of i with respect to this graph norm (on $\Gamma(M, \wedge^{n-1}({}^0T')^*)$, we use the graph norm defined by i, and on $\Gamma(M, \wedge^{n-1}(T')^*)$, the standard L_2 is used). Then, our main theorem is

Main Theoem. *The projection map* $= i \cdot A^*$ *on* $\Gamma(M, T')$.

Proof. For this, it suffices to show that:

(1) *for* $w = w_1 + w_2$, *which is orthogonal to* H^0, *we have* $i \cdot A^* w = 0$,

(2) *for* $u \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^*)$, *we have* $i \cdot A^*(u + \psi_u) = u + \psi_u$.

where

$$w \in \Gamma(M, \wedge^{n-1}(T')^*),$$
$$w_1 \in \Gamma(M, \wedge^{n-1}({}^0T')^*),$$
$$w_2 \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T'')^*).$$

For the proof of (1), by the definition of $w = w_1 + w_2$,

$$(w_1, v) + (w_2, \psi_v) = 0 \ for \ v \in \Gamma(M, \wedge^{n-1}({}^0T')^*),$$

But this means

$$(w, Av) = 0 \ for \ v \in \Gamma(M, \wedge^{n-1}({}^0T')^*).$$

So

$$(A^* w, v) = 0 \ for \ v \in \Gamma(M, \wedge^{n-1}({}^0T')^*).$$

So we have (1).
For the proof of (2),

$$\begin{aligned}(i \cdot A^*(u + \psi_u) - (u + \psi_u), v + \psi_v) &= ((u + \psi_u), Ai^* \cdot (v + \psi_v)) - (u + \psi_u, u + \psi_u), \\ &= (u + \psi_u, Av) - (u + \psi_u, u + \psi_u)\end{aligned}$$

for $u, v \in \Gamma(M, \wedge^{n-1}({}^0T')^*)$. This becomes

$$\begin{aligned}(A^*(u + \psi_u) - u, v) &= (u + \psi_u, v + \psi_v) - (u + \psi_u, v + \psi_v) \\ &= 0.\end{aligned}$$

Sect.4. Kodaira – Hodge type decomposition theorem

In order to establish a Kodaira-Hodge type decomoposition theorem on H^1, we have to show the following two conditions.

1) $H^1 \cap Dom\ S$ is dense in $\tilde{H}^1$.

2) The following type a priori estimate.

$$\|S\phi\| + \|\partial_b^{(1)}\phi\| + \|\phi\| \geq C\|\phi\|_{1/2}$$

for $\phi \in H^1 \cap Dom\ S$, where S means the adjoint operator of ∂_b in (H^p, ∂_b) complex, and C is a positive constant. While, by the result in Sect.2, we have

$$S = the\ composition\ of\ \overline{\partial}_b^*\ and\ "the\ projection\ operator\ of\ \Gamma_2(M, \wedge^{n-1}(T')^*)\ to\ \tilde{H}^0",$$

where ∂_b^* means the adjoint operator ∂_b in *the standard complex*. Because our complex is a subcomplex, this result follows from functional analysis. Namely, S is nothing but the adjoint operator of D, which Rumin finds. So, by his estimate((1) is now obvious), we have a Kodaira-Hodge type decomposition theorem over H^1.

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