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Three dimensional hypersurface purely elliptic singularities

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Abstract

Three dimensional hypersurface purely elliptic singularities are classified into three classes according to the shape of their Newton boundaries.

A lot of examples of their defining equations are obtained from the defining equations of hypersurface simple $K3$ singularities. There are at least 95 types for the defining equations of hypersurface purely elliptic singularities of the type $(0,1)$ or $(0,0)$.

1 Introduction

In the theory of normal two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the most reasonable class of singularities after rational double points. They are characterized as two-dimensional purely elliptic singularities of (0,1)-type and of (0,0)-type, respectively. What are natural generalizations in three-dimensional case of simple elliptic singularities. The notion of a simple $K3$ singularity was defined in [4] as a three-dimensional isolated Gorenstein purely elliptic singularity of (0,2)-type. A simple $K3$ singularity is characterized as a normal three-dimensional isolated singularity such that the exceptional set of any $\mathbf{Q}$-factorial terminal modification is a three-dimensional $K3$ surface (see [4]). Here we are interested in three-dimensional hypersurface purely elliptic singularities of $(0,i)$-type for $i=0$ or $i=1$. Let $f \in \mathbf{C}[z_0, z_1, z_2, z_3]$ be a polynomial which

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is non-degenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [5], and whose zero locus $X = \{f = 0\}$ in $\mathbf{C}^4$ has an isolated singularity at the origin $0 \in \mathbf{C}^4$. Then the condition for the singularity (X, x) to be a purely elliptic singularity of $(0, 0)$-type is given by a property of the Newton boundary of $\Gamma(f)$ of f.

In this paper, we give the method to obtain the principal parts of defining equations, which define three-dimensional hypersurface purely elliptic singularities of $(0, i)$-type for $i = 0$ to $i = 1$.

2 Preliminaries

In this section, we recall some definitions and results from [1], [4] and [6].

First we define the plurigenera $\delta_m, m \in \mathbf{N}$, for normal isolated singularities and define purely elliptic singularities. Let (X, x) be a normal isolated singularity in an n-dimensional analytic space X, and $\pi : (M, E) \to (X, x)$ a good resolution. In the following, we assume that X is a sufficiently small Stein neighbourhood of x.

DEFINITION. ([6]) Let (X, x) be a normal isolated singularity. For any positive integer m,

$$\delta_m(X, x) := dim_{\mathbf{C}}\Gamma(X - \{x\}, \mathcal{O}(mK))/L^{2/m}(X - \{x\}),$$

where K is the canonical line bundle on $X - \{x\}$.

Then δ_m is finite and does not depend on the choice of a Stein neighborhood.

DEFINITION. ([6]) A singularity (X, x) is said to be purely elliptic if $\delta_m = 1$ for every $m \in \mathbf{N}$.

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there is a nowhere vanishing holomorphic 2-form on $X - \{x\}$ (see [2]). But in higher dimension, purely elliptic singularities are not always quasi-Gorenstein (see [3]).

In the following, we assume that (X, x) is quasi-Gorenstein. Let $E = \bigcup E_i$ be the decomposition of the exceptional set E into its irreducible components, and write

$$K_M = \pi^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_J$$

with $m_i \geq 0, m_j \geq 0$. Ishii [1] defined the essential part of the exceptional set E as $E_j = \sum_{j \in J} m_j E_J$, and showed that if (X, x) is purely elliptic, then $m_j = 1$ for all $j \in J$.

DEFINITION. ([1],[6]) A quasi-Gorenstein purely elliptic singularity (X, x) is of (0,i)-type if $H^{n-1}(E_J, \mathcal{O}_E)$ consists of the (0,i)-Hodge component $H^{0,i}(E_J)$, where

$$\mathbf{C} \simeq H^{n-1}(E_J, \mathcal{O}_E) = Gr_F^0 H^{n-1}(E_J) = \bigoplus_{i=1}^{n-1} H^{0,i}(E_J)$$

n-dimensional quasi-Gorenstein purely elliptic singularities are classified into $2n$ classes, including the condition that the singularity is Cohen-Macaulay or not.

Next we consider the case where (X, x) is a hypersurface singularity defined by a non-degenerate polynomial $f = \sum a_\nu z^\nu \in \mathbf{C}[z_0, z_1, ..., z_n]$, and $x = 0 \in \mathbf{C}^{n+1}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_\nu \neq 0}(\nu + \mathbf{R}_0^{n+1})$ in $\mathbf{R}^{n+1}$. For any face Δ of $\Gamma_+(f)$, set $f_\Delta := \sum_{\nu \in \Gamma} a_\nu z^\nu$. We say f to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial z_0} = \frac{\partial f_\Delta}{\partial z_1} = \cdots = \frac{\partial f_\Delta}{\partial z_n} = 0$$

has no solution in $(\mathbf{C}^*)^{n+1}$ for any face Δ. Where f is nondegenerate, the condition for (X, x) to be a purely elliptic singularity of (0,i)-type is given as follows:

THEOREM 2. 1 *Let f be a nondegenerate polynomial and suppose $X = \{f = 0\}$ has an isolated singularity at $x = 0 \in \mathbf{C}^{n+1}$.*

(1) *(X, x) is purely elliptic if and only if $(1, 1, ..., 1) \in \Gamma(f)$.*

(2) *Let $n = 3$ and let Δ_0 be the face of $\Gamma(f)$ consisting the point $(1, 1, 1, 1)$ in the relative interior of Δ_0. Then we have*

(i) *(X, x) is a singularity of $(0, 2)$-type if and only if $dim_{\mathbf{R}} \Delta_0 = 3$.*

(ii) *(X, x) is a singularity of $(0, 1)$-type if and only if $dim_{\mathbf{R}} \Delta_0 = 2$.*

(iii) *(X, x) is a singularity of $(0, 0)$-type if and only if $dim_{\mathbf{R}} \Delta_0 = 1$ or $dim_{\mathbf{R}} \Delta_0 = 0$.*

3 Principal parts

In this section, we give examples of the principal parts of hypersurface purely elliptic singularities of $(0, i)$-type defined by a nondegenerate polynomial for $i = 0$ to $i = 1$.

EXAMPLE. Let (X, x) be the hypersurface purely elliptic singularity

$$xyzw + x^{5+p} + y^{5+q} + z^{5+r} + w^{5+s} = 0$$

in $\mathbf{C}^4$. Blow up the point $O = (0,0,0,0)$, let F be the exceptional set, and let Y be the strict transform of X. In this case the morphism $\pi : Y \to X$ is the canonical resolution of X. The exceptional set E consists of four 2-dimensional projective spaces in F, forming a tetrahedron.

EXAMPLE. Let (X, x) be the hypersurface purely elliptic singularity

$$x^2 + y^3 + z^7 + w^{43+s} + xyzw = 0$$

in $\mathbf{C}^4$. Blow up the point $O = (0,0,0,0)$ with weight (21,14,6,1), let F be the exceptional set, and let Y be the strict trans form of X. In this case the morphism $\pi : Y \to X$ is the canonical resolution of X. The exceptional set E is a rational surface with a singularity $T_{2,3.7}$ in a weighted projective space F, i.e., $\mathbf{P}(21, 14, 6, 1)$.

EXAMPLE. Let (X, x) be the hypersurface singularity defined by the equation

$$x^2 + y^3 + z^7 + z^6w^6 + w^{43+s} + xyzw = 0$$

in $\mathbf{C}^4$. Then the singularity (X, x) is a purely elliptic singularity of $(0, 1)$-type.

EXAMPLE. Let (X, x) be the hypersurface singularity defined by the equation

$$x^2 + y^3 + z^7 + \lambda z^6w^6 + \mu w^{42} + w^{43+s} + xyzw = 0$$

in $\mathbf{C}^4$. Then the we obtain:

(1) $\mu \neq 0 \Leftrightarrow (0, 2)$-type.

(2) $\mu = 0, \lambda \neq 0 \Leftrightarrow (0, 1)$-type.

(3) $\mu = 0, \lambda = 0 \Leftrightarrow (0, 0)$-type.

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