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Analytical Solitary Wave Solutions for the Nonlinear Schrödinger Equation Coupled to the Korteweg-de Vries Equation [1)]

ZHIBIN LI

Department of Mathematics, Lanzhou University, Lanzhou, 730000, P.R. China

Abstract. In order to examine solitary wave solutions for coupled system of the nonlinear Schrödinger equation and the Korteweg-de Vries equation describing the nonlinear interaction between long and short waves, a reduced set of ordinary differential equations are considered by a simple traveling wave transformation. It is then shown that analytical solutions can be obtained systematically by means of a direct algebra method. Seven types of exact solutions are obtained. They are useful for better understanding the interaction of long waves with short waves and the method used here might also be applied in a much wider context. All computation has been completed on the compute algebra systems MATHEMATICA

1. Introduction

Various coupled systems have been proposed to describe the interaction of long waves with short wave packets in nonlinear dispersive media. These systems of equations have been derived for phenomena including fluid dynamics, plasmas and solid-state physics. One of the important system of equations is given in the following coupled form of the nonlinear Schrödinger equation and the Korteweg-de Vries (S-KdV) equation[1)]

$$iS_t + S_{xx} = SL, \quad L_t + \alpha L L_x + \beta L_{xxx} = |S|^2_x, \tag{1}$$

where subscripts t and x denote time and space derivatives, and L and S are the real long wave amplitude and the complex short wave amplitude, respectively, while α and β are control parameters. For $\alpha = \beta = 0$, Eqs.(1) are proved to be integrable or to have an n-soliton solution by means of the inverse scattering transform method,[2)] whereas for $\beta = 1$ the equations are shown to be non-integrable by means of the same method.[3)]

In this paper, we are mainly concerned with the solitary wave solutions which have not been examined sufficiently in relation to the parameters α and β. For this end, introducing a simple traveling wave transformation, we consider a reduced set of ordinary differential equations (ODEs) obtained from Eqs.(1). Then we construct analytically solitary wave solutions to the ODEs thus derived. We obtain seven types of exact solitary wave solutions for a particular choice of α and β, two types of them are first to be found.

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2. Travelling-wave Transformation and a Direct Method

Let us first introduce the following simple travelling wave transformation

$$S = \phi(x - ct)\exp[i\frac{c}{2}(x - \frac{c}{2}t - vt)], \quad L = u(x - ct), \tag{2}$$

where c and v are real constands, while ϕ and u are real function of $\xi = x - ct$ alone. We then obtain the following set of ODEs from Eqs.(1)

$$\phi'' + \frac{cv}{2}\phi = \phi u, \quad \beta u'' + \frac{\alpha}{2}u^2 - cu = \phi^2 - C^2, \tag{3}$$

where the prime denotes differentiation with respect to ξ, and C is integration constant. For solitary wave solutions, we impose on ϕ and u such boundary conditions as $\phi \to C$ and all of ϕ', ϕ'', u, u' and u'' tend to 0 as $|\xi| \to \infty$. It is to be noted that $v = 0$ for $C \neq 0$ in Eqs.(3).

In order to obtain exact solitary wave solutions to Eqs.(3), we represent ϕ and u as polynomials in two elementary solitary waves f and g defined by

$$f(\xi) = \frac{1}{\cosh\xi + r}, \quad g(\xi) = \frac{\sinh\xi}{\cosh\xi + r} \tag{4}$$

with r ($\neq \pm 1$) constant. Functions f and g satisfy the coupled system of projective Riccati equations[4)]

$$f'(\xi) = -f(\xi)\,g(\xi), \qquad g'(\xi) = 1 - g^2(\xi) - r\,f(\xi) \tag{5}$$

which admits the first integral

$$g^2(\xi) = 1 - 2rf(\xi) + (r^2 - 1)f^2(\xi). \tag{6}$$

Balancing the highest order derivative terms with the nonlinear terms in Eqs.(3), we find the polynomial degree of the solutions u in f and g must be 2, and that of ϕ can be 1 or 2 from Eqs.(5). Considering the boundary conditions on ϕ and u as well as Eqs.(6), it is convenient to assume the following forms of ϕ and u in terms of f and g

$$\begin{aligned} \phi &= a_0 + a_1\,f(k\xi) + a_2\,f^2(k\xi) + b_1\,g(k\xi) + b_2\,f(k\xi)g(k\xi), \\ u &= a\,f^2(k\xi), \end{aligned} \tag{7}$$

where $a, a_0, a_1, a_2, b_1, b_2$ and k are some real constants.

Substituting the expressions (7) into Eqs.(3), eliminating any derivative of (f, g) and any power of g higher than one with Eqs.(5) and Eqs.(6), and setting to zero the coefficients of the different powers of fg, we obtain a system of algebraic equations for parameters α and β consisting 18 equations of fourth degree in ten unknowns k, c, v, r and $a, a_0, a_1, a_2, b_1, b_2$:

$$\begin{gathered} cva_0 = 0,\ cvb_1 = 0,\ a_0b_1 = 0,\ a_2b_2 = 0,\ a_1b_1 + = a_0b_2 = 0,\ a_2b_1 + a_1b_2 = 0, \\ (cv + 2k^2)a_1 = 0,\ (a + 6k^2 + 6k^2r^2)a_2 = 0,\ (a + 6k^2 - 6k^2r^2)b_2 = 0, \\ a_0a_1 + b_1b_2 - b_1^2r = 0,\ C + a_0^2 + b_1^2 = 0, \\ 2k^2b_1r - cvb_2 - 2k^2b_2 = 0,\ 2k^2b_1 + ab_1 - 2k^2b_1r^2 + 6k^2b_2r = 0, \\ 2aa_0 + 6k^2a_1r - cva_2 - 8k^2a_2 = 0,\ 2k^2a_1 + aa_1 - 2k^2a_1r^2 + 10k^2a_2r = 0, \\ 5\beta ak^2r + a_1a_2 - b_1b_2 + b_1b_2r^2 - b_2^2r = 0, \\ ac - 4\beta ak^2 + a_1^2 + 2a_0a_2 - b_1^2 + b_1^2r^2 - 4b_1b_2r + b_2^2 = 0, \\ 12\beta ak^2 - \alpha a^2 - 12\beta ak^2r^2 + 2a_2^2 - 2b_2^2 + 2b_2^2r^2 = 0. \end{gathered} \tag{8}$$

3. Solitary Wave Solutions of S-KdV Equations

In order to solve system (8), one may use the Ritt-Wu Elimination[5] or the technique of Gröbner bases. However, we have used instead a much more effective algorithm which exploits the special structure of the system (8). Its main idea is to consider several alternative cases, such as $v = 0, v \neq 0$, and several subcases inside each case, etc. Applying this method and carrying out all computations in the interactive mode of the computer algebra system MATHEMATICA, we have found all the non-trivial solutions of system (8). For example, for the case $v = 0$, the system (8) leads to three possibilities: 1) $a_2 \neq 0, r = 0$, 2) $a_2 = 0, r = 0$ and 3) $a_2 = 0, r \neq 0$. In subcase 1), the system (8) have non-trivial solutions

$$a = -6k^2,\ a_0 = C,\ a_1 = 0,\ a_2 = -\frac{3}{2}C,\ b_1 = b_2 = 0;\ c = -4k^2(\alpha + \beta) \tag{9}$$

where $C^2 = 8k^4(\alpha + 2\beta)$ and k is an arbitrary constant. =20 In this way, we find that seven types of coupled solitary waves exist. The results are summed up as follows:

1) If $\alpha + 2\beta > 0$ and $\alpha + \beta \neq 0$, then Eqs.(1) admit exact solitary wave solutions

$$\begin{aligned} S_1(x,t) &= C\{1 - \tfrac{3}{2}\mathrm{sech}^2[k(x-ct)]\} \exp[i\tfrac{c}{2}(x - \tfrac{c}{2}t)], \\ L_1(x,t) &= -6k^2\,\mathrm{sech}^2[k(x-ct)] \end{aligned} \tag{10}$$

where $c = -4k^2(\alpha + \beta)$, $C^2 = 8k^2(\alpha + 2\beta)$, and k is a constant;

2) If $\alpha + 6\beta = 0$, then Eqs.(1) admit exact solitary wave solutions

$$\begin{aligned} S_2(x,t) &= C \tanh[k(x-ct)] \exp[i\tfrac{c}{2}(x - \tfrac{c}{2}t)], \\ L_2(x,t) &= -2k^2\,\mathrm{sech}^2[k(x-ct)] \end{aligned} \tag{11}$$

where $C^2 = 2k^2(4\beta k^2 - c)$ with k, c constants satisfying $c < 4\beta k^2$.

3) If $3\alpha + 4\beta = 0$ and $\beta < 0$, then Eqs.(1) admit exact solitary wave solutions

$$\begin{aligned} S_3(x,t) &= C\sqrt{5}\frac{(\pm 2\sqrt{2} + \sqrt{5}\cosh[k(x-ct)])\sinh[k(x-ct)]}{(\sqrt{2} \pm \sqrt{5}\cosh[k(x-ct)])^2} \exp[i\tfrac{c}{2}(x - \tfrac{c}{2}t)], \\ L_3(x,t) &= -18k^2 \frac{1}{(\sqrt{2} \pm \sqrt{5}\cosh[k(x-ct)])^2} \end{aligned} \tag{12}$$

where $c = 13\beta k^2$, $C^2 = -18\beta k^4$ with k constant.

4) If $\alpha + 2\beta > 0$ and $\beta \neq 0$, then Eqs.(1) have exact solitary wave solutions

$$\begin{aligned} S_4(x,t) &= A\,\mathrm{sech}^2[k(x-ct)] \exp[i\tfrac{c}{2}(x - \tfrac{c}{2}t - vt)], \\ L_4(x,t) &= -6k^2\,\mathrm{sech}^2[k(x-ct)] \end{aligned} \tag{13}$$

where $c = 4\beta k^2$, $v = -2/\beta$, $A^2 = 18k^4(\alpha + 2\beta)$ and k is a constant.

5) If $\alpha + 2\beta < 0$ and $3\alpha + 2\beta \neq 0$, then Eqs.(1) have exact solitary wave solutions

$$\begin{aligned} S_5(x,t) &= A\,\mathrm{sech}[k(x-ct)] \tanh[k(x-ct)] \exp[i\tfrac{c}{2}(x - \tfrac{c}{2}t - vt)], \\ L_5(x,t) &= -6k^2\,\mathrm{sech}^2[k(x-ct)] \end{aligned} \tag{14}$$

where $c = -k^2(3\alpha + 2\beta)$, $v = 2/(3\alpha + 2\beta)$, $A^2 = -18k^4(\alpha + 2\beta)$ and k is a constant;

6) If $\alpha + 6\beta = 0$, then Eqs.(1) have exact solitary wave solutions

$$\begin{aligned} S_6(x,t) &= A\,\mathrm{sech}[k(x-ct)]\exp[i\tfrac{c}{2}(x - \tfrac{c}{2}t - vt)], \\ L_6(x,t) &= -2k^2\,\mathrm{sech}^2[k(x-ct)] \end{aligned} \tag{15}$$

where $v = -2k^2/c$, $A^2 = 2k^2(c - 4\beta k^2)$ with k, c constants satisfying $c > 4\beta k^2$;

7) If $3\alpha + 4\beta = 0$ and $\beta > 0$, then Eqs.(1) have exact solitary wave solutions

$$\begin{aligned} S_7(x,t) &= A\frac{\pm 2\sqrt{2} + \sqrt{7}\cosh[k(x-ct)]}{(\sqrt{2} \pm \sqrt{7}\cosh[k(x-ct)])^2}\exp[i\tfrac{c}{2}(x - \tfrac{c}{2}t - vt)], \\ L_7(x,t) &= -30k^2\frac{1}{(\sqrt{2} \pm \sqrt{7}\cosh[k(x-ct)])^2}. \end{aligned} \tag{16}$$

where $c = 9\beta k^2$, $v = -2/(9\beta)$, $A^2 = 150\beta k^4$, and k is a constant.

Among the above solutions, (10), (11), (13), (14) and (15) with $\alpha \neq 0, \beta \neq 0$ have already obtained by applying a modified Hirota's method[6]. In addition to this, (11) and (15) with $\alpha = \beta = 0$, and (13) and (14) with $\alpha \neq 0, \beta \neq 0$ have also been obtained by means of the direct integration of Eqs.(3).[7] Nevertheless, we would like to emphasize that all of these solutions can be obtained systematically by determining only a finite number of coefficients. This method is much simpler and obtains more solutions than (modified) Hirota's method or Hereman's method[8] which consists of summing a perturbation series build from exponential solutions of the linearized equations.

4. Solitary Wave Solutions of S-B Equations

There is a similar situation for another long and short wave interaction system

$$iS_t = S_{xx} + LS, \qquad L_{xx} + \alpha L_{xxxx} + \beta(L)^2_{xx} - \gamma L_{tt} = (|S|^2)_{xx} \tag{17}$$

which are the coupled system of the nonlinear Schrödinger and Boussinesq equation. The solutions of Eqs.(17) describing coupled solitary waves can be obtained by assuming

$$S = \phi(x-ct)\exp[i\frac{c}{2}(x - \frac{c}{2}t - vt)], \qquad L = u(x-ct), \tag{18}$$

where $\xi = x - ct$, and c, v are real constants. In fact, transformation (18) reduces Eqs.(17) to an ODEs which are similar with Eqs.(3)

$$\phi'' + \frac{cv}{2}\phi = \phi u, \qquad \alpha u'' + \beta u^2 + (1 - c^2\gamma)u = \phi^2 + A\xi - B^2, \tag{19}$$

where A and B are integration constants.

For Eqs.(19), we only consider the boundary conditions $\phi, \phi', \phi'', u, u', u'' \to 0$ as $\xi \to \pm\infty$ for simplicity. Then $A = B = 0$. Applying the same technique as before, we find that four types of coupled solitary waves exist. Except for the well known solutions,[9] Eqs.(17) admit also the following exact solitary wave solutions

$$\begin{aligned} S(x,t) &= \pm 5\sqrt{-6\alpha}k^2\frac{\pm 2\sqrt{2} + \sqrt{7}\cosh[k(x-ct)]}{(\sqrt{2} \pm \sqrt{7}\cosh[k(x-ct)])^2}\exp[i\tfrac{c}{2}(x - \tfrac{c}{2}t + \tfrac{2k^2}{c}t)], \\ L(x,t) &= 30k^2\frac{1}{(\sqrt{2} \pm \sqrt{7}\cosh[k(x-ct)])^2} \end{aligned} \tag{20}$$

where $2\alpha - 3\beta = 0$, $\alpha < 0$ and $c^2 = (1 + 9\alpha k^2)/\gamma$, and k is a constant satisfying $(1 + 9\alpha k^2)/\gamma > 0$.

As we have studied for S-KdV equations before, S-B equations have three other types of solutions if imposed boundary conditions as $\phi \to B \neq 0$ and all of ϕ', ϕ'', u, u' and u'' tend to 0 as $|\xi| \to \infty$.

One can see from above that if we make some constraints on the parameters of the system to the type of coupled nonlinear equations such as in this paper, we can get some exact solutions as polynomial in two elementary bell-shaped and kink-shaped functions by the determination of a finite number of coefficients. The present method can evidently be applied to the higher dimensional nonlinear equations. It is an exercise to check our results in MATHEMATICA.

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