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Lyapunov 方程式と作用素平均
(Lyapunov equation and the operator means)

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1. Introduction.

As an generalization for the Kullback-Leibler information, Umegaki [17] defined the relative entropy in operator algebras, which is now called *the Umegaki entropy.* Also, Nakamura and Umegaki defined the operator entropy in [11]. So Kamei and the author [7] defined the relative operator entropy by generalizing operator means by Kubo and Ando [9] (see also [5,6]). A two variable function, $(A, B) \mapsto A\mathbf{s}B$, from positive invertible operators to selfadjoint operators is called a *solidarity* if $\mathbf{s}$ satisfies the following axioms:

(S1) $\qquad B \leq C$ implies $A\mathbf{s}B \leq A\mathbf{s}C$,

($\mathrm{S}2_r$) $\qquad B_n \downarrow B$ implies $A\mathbf{s}B_n \downarrow A\mathbf{s}B$.

($\mathrm{S}2_\ell$) $\qquad \operatorname*{s-lim}_{n\to\infty} A_n = A$ implies $\operatorname*{s-lim}_{n\to\infty} A_n\mathbf{s}1 = A\mathbf{s}1$.

(S3) $\qquad T^*(A\mathbf{s}B)T \leq T^*AT\mathbf{s}T^*BT$ for all operators T.

Then, for a positive number x, we may consider $f_s(x) = 1 \ \mathbf{s} \ x$ as a scalar by the *transformer inequality* (S3) and the function f_s is operator monotone by (S1). Moreover a map $\mathbf{s} \mapsto f_s$ is a bijection from the solidarities onto the operator monotone functions on $(0, \infty)$ where the inverse map is constructed by

$$A\mathbf{s}B = A^{1/2} f_s(A^{-1/2}BA^{-1/2})A^{1/2}.$$

Here f_s is called the *representing function* for $\mathbf{s}$. For example, the solidarity for the logarithm is the *relative operator entropy* [7]:

$$S(A|B) = A^{1/2}\log(A^{-1/2}BA^{-1/2})A^{1/2}.$$

If f_s is nonnegative, then $\mathbf{s}$ is called *connection* and besides if $f_s(1) = 1$ (or equivalently $A\mathbf{s}A = A$), then $\mathbf{s}$ is called an *operator mean* in the Kubo-Ando theory.

Recently Petz [12] showed that the monotone metrics on the Hilbert space of the matrices correspond exactly to the operator means or connections of the left multiplication operator and the right one. Motivated by this, we consider the operator means or solidarities of multiplication operators and give the integral representations, which is obtained through the Lyapunov equation of operators.

2. Solidarities for multiplication operators.

For a fixed positive linear form φ on a C*-algebra $\mathcal{A}$ on a Hilbert space H, we define multiplication operators in the usual way: Let $\mathcal{N} = \{X|\varphi(X^*X) = 0\}$, the left kernel of φ. For the Hilbert space $\mathcal{H}$ obtained as the completion of the quotient space $\mathcal{A}/\mathcal{N}$, the quotient map from $\mathcal{A} \to \mathcal{H}$ is denoted by $X \mapsto [X]$. Then $L(A)$ and $R(A)$ for $A \in \mathcal{A}$ are defined as

$$\langle L(A)[X],[Y]\rangle_\varphi = \varphi(Y^*AX) \text{ and } \langle R(A)[X],[Y]\rangle_\varphi = \varphi(Y^*XA).$$

The left multiplication $L(A)$ is a bounded operator on $\mathcal{H}$ by

$$\|L(A)[X]\|^2 = \varphi(X^*A^*AX) \leq \|A\|^2_{\mathcal{A}}\varphi(X^*X).$$

Moreover the map $\mathcal{A} \to B(\mathcal{H}),\ A \mapsto L(A)$ is algebraically homomorphic and preserves the positivity as operators, so we have

$$L(A)\mathbf{s}L(B) = L(A\mathbf{s}B).$$

Thus there is no problems in solidarities of left multiplication operators.

But the right multiplication $R(B)$ is not always bounded even if B is a matrix. In fact, consider

$$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we have

$$\|[X]\|^2 = \varphi(X^*X) = \varphi\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

while

$$\|R(A)[X]\|^2 = \varphi(A^*X^*XA) = \varphi(A) = 1.$$

So, from now on, we must assume that φ is *tracial*, i.e., φ satisfies

$$\varphi(AB) = \varphi(BA)$$

for all operators $A, B \in \mathcal{A}$. In this case, the map $\mathcal{A} \to B(\mathcal{H})$, $A \to R(A)$ is anti-homomorphic and preserves positivity, so we may identify $R(A)\mathbf{s}R(B)$ with $A\mathbf{s}B$. However, the structure of $L(A)\mathbf{s}R(B)$ (or $R(A)\mathbf{s}L(B)$) is not clear although $L(A)$ and $R(B)$ always commute. Petz discussed these types of means in [12].

Our main problem in this note is to consider the structure of the above solidarity. So we examine $\big(L(A)\mathbf{s}R(B)\big)[X]$ for operators $X \in \mathcal{A}$. There are several elementary examples:

$$\begin{array}{cl} \text{the arithmetic mean} & \big(L(A)\mathbf{a}R(B)\big)[X] = \left[\dfrac{AX+XB}{2}\right] \\ \text{the geometric mean} & \big(L(A)\mathbf{g}R(B)\big)[X] = \left[A^{1/2}XB^{1/2}\right] \\ \begin{array}{c}\text{the relative}\\ \text{operator entropy}\end{array} & S\big(L(A)|R(B)\big)[X] = [-A\log AX + AX\log B]\,. \end{array}$$

Since $L(A)$ and $R(B)$ commute, the relative operator entropy is

$$S\big(L(A)|R(B)\big) = -L(A)\log L(A) + L(A)\log R(B),$$

which assures the above last formula.

Now we see the difference for relative entropies: If $\mathcal{A}$ is a semifinite von Neumann algebra and φ is its trace, then the *Umegaki entropy* is

$$S_U(A|B) = -\langle S\big(L(A)|R(B)\big)[1], [1]\rangle_\varphi.$$

On the other hand, the *Belavkin-Staszewski entropy* in [3], which is $-\varphi\big(S(A|B)\big)$ in this case, is

$$S_{BS}(A|B) = -\langle S\big(L(A)|L(B)\big)[1], [1]\rangle_\varphi.$$

3. Lyapunov equation.

The operator $\big(L(A) + R(B)\big)[X] = [AX + XB]$ reminds us of the Lyapunov equation $AX + XB = C$. So we review it in this section.

First we see a solution which is known even for operators (e.g., [4]). The spectrum $\sigma(A)$ never grows for the map $A \mapsto L(A)$ or $R(A)$, that is, $\sigma(A) \supset \sigma(L(A)), \sigma(R(A))$ which are the spectra in $B(\mathcal{H})$. According to the system theory, an operator A is called *stable* (resp. *anti-stable*) if $\sigma(A) \subset \mathbb{C}^-$ (resp. $\mathbb{C}^+$), where $C^\pm$ is the open half plane whose real part is $\pm$ respectively. So the stability preserves under the map $A \mapsto L(A)$ or $R(A)$. Hereafter we deal with only anti-stable operators because positive invertible operators belong to this class. Now we confirm for completeness that the Lyapunov equation for two anti-stable operators is solvable by consulting Ando's lecture note [2], which was also shown by Rosenblum [13]:

Proposition. *Let A and B be anti-stable operators. Then*

$$\|e^{tA}\| \to 0 \qquad \text{as} \qquad t \downarrow -\infty$$

and the Lyapunov equation $AX + XB = C$ has a solution

$$(*) \qquad X = \int_{-\infty}^{0} e^{tA} C e^{tB} dt.$$

Proof. For a closed Jordan curve Γ in $\mathbb{C}^+$ containing $\sigma(A)$ in its interior, we can use Cauchy integral representation (see also [16]):

$$e^{tA} = \frac{1}{2\pi i} \int_\Gamma e^{tz} (z - A)^{-1} dz.$$

Let ℓ be the length of L and ε the distance between L and the imaginary axis. Then we have

$$\begin{aligned} \|e^{tA}\| &\leq \frac{1}{2\pi} \int_\Gamma |e^{tz}| \|(z - A)^{-1}\| |dz| \\ &\leq e^{t\varepsilon} \frac{\ell}{2\pi} \sup_{z \in \Gamma} \|(z - A)^{-1}\|. \end{aligned}$$

Therefore $\|e^{tA}\| \to 0$ as $t \downarrow -\infty$. Substituting (*) into the equation, we have

$$\begin{aligned} AX + XB &= \int_{-\infty}^{0} A e^{tA} C e^{tB} + e^{tA} C e^{tB} B dt \\ &= \int_{-\infty}^{0} \frac{d\, e^{tA} C e^{tB}}{dt} dt = \left[e^{tA} C e^{tB} \right]_{-\infty}^{0} = C - 0 = C. \end{aligned}$$

Next we refer the equivalent condition that the Lyapunov equation has a solution. Let $\mathcal{A}$ be a C*-algebra with a faithful state φ on a Hilbert space H with a fixed basis $\{e_i\}$. Here we define the *transpose* operator A^T with respect to $\{e_j\}$ by

$$\langle A^T e_i, e_j \rangle = \overline{\langle A^* e_i, e_j \rangle}.$$

So we have $\sigma(A^T) = \sigma(A)$ immediately.

Then $\mathcal{A}$ is a Hilbert space with the inner product $\langle A, B\rangle = \varphi(B^*A)$ and equal to the quotient space $\mathcal{H}$ itself since φ is faithful. Moreover we can identify $\mathcal{A}$ with the subspace $\tilde{\mathcal{A}}$ of $H \otimes H$ as Hilbert spaces by the unitary operator U defined by

$$U : e_i \otimes \overline{e_j} \mapsto e_i \otimes e_j,$$

where $x \otimes \overline{y}$ is a dyad (or von Neumann-Schatten operator) with $(x \otimes \overline{y})\, z = \langle z, y\rangle x$ for all $z \in H$. Then U induces the isomorphism from $B(\mathcal{A})$ onto $B(\tilde{\mathcal{A}})$ by $Z \mapsto UZU^*$ and hence we have

$$UL(A)U^* = A \otimes 1 \text{ and } UR(B)U^* = 1 \otimes B^T$$

and the image of the Lyapunov equation$\big(L(A) + R(B)\big)X = AX + XB = C$ is

$$(A \otimes 1 + 1 \otimes B^T)UXU^* = UCU^*.$$

Applying Schechter's theorem [15];

$$\sigma\big(p(X \otimes 1, 1 \otimes Y)\big) = p\big(\sigma(X), \sigma(Y)\big)$$

for all polynomials p of two variables, we have

$$\sigma(A \otimes 1 + 1 \otimes B^T) = \sigma(A) + \sigma(B^T) = \sigma(A) + \sigma(B) = \sigma(A \otimes 1 + 1 \otimes B),$$

which shows the required condition:

Theorem 1. *The Lyapunov equation $AX + XB = C$ in a C^*-algebra $\mathcal{A}$ with a faithful state φ is solvable if and only if zero never belongs to the sum of the spectra of A and B:*

$$0 \notin \sigma(A \otimes 1 + 1 \otimes B) = \sigma(A) + \sigma(B).$$

Remark. The assumption that $\mathcal{A}$ is a C*-algebra with a faithful state in Theorem 1 is not essential. In fact, Kleinecke's theorem in [10] says that the condition $0 \notin \sigma(A) + \sigma(B)$ is equivalent to the solvable one for a Banach algebra of all bounded linear operators on a Banach space. These facts are well-known for matrices as in [8]. Lyapunov operator equations were studied in 50's by M.A.Rutman [14], M.Rosenblum [13] et. al..

4. Parallel sum and means.

Let A and B be invertible operators in $\mathcal{A}$. Then $A+B$ is invertible if and only if $A^{-1}+B^{-1}$ is. In fact, if $A+B$ is invertible, then one can easily verify that $A^{-1}+B^{-1}$ has the following inverse:

$$A(A+B)^{-1}B = B(A+B)^{-1}A.$$

So, for invertible operators A and B such that $A+B$ is also invertible, one can define the *parallel sum* $A{:}B$ by

$$A{:}B = (A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B = B(A+B)^{-1}A,$$

see [1] (Note that the positivity for A and B is not assumed here).

Now we see the relation between the parallel sum and the Lyapunov equation:

Theorem 2. *Let A and B be anti-stable operators in a C^*-algebra $\mathcal{A}$ with a fixed state φ. Then, in the above quotient Hilbert space $\mathcal{H}$,*

$$\big(L(A){:}R(B)\big)[X] = \left[\int_{-\infty}^{0} e^{tA}AXBe^{tB}dt\right]$$

for all operators $X \in \mathcal{A}$.

Proof. By the commutativity and invertibility, we have

$$\big(L(A){:}R(B)\big)[X] = \big(L(A)+R(B)\big)^{-1}L(A)R(B)[X] = \big(L(A)+R(B)\big)^{-1}[AXB].$$

Thereby, to find Y with $[Y] = \big(L(A){:}R(B)\big)[X]$, , we have only to solve the following equation:.

$$[AXB] = \big(L(A)+R(B)\big)[Y] = [AY+YB],$$

that is, the Lyapunov equation for the unknown Y

$$AY+YB = AXB.$$

The requaired solution follows from Proposition.

Now we go back to means of positive invertible operators in $\mathcal{A}$. Since positive invertible operators are anti-stable, we can use Theorem 2. In [9], Kubo and Ando showed that every connection $\mathbf{m}$ has the integral representation for the positive Radon measure μ on $[0,\infty]$:

$$A\mathbf{m}B = aA + bB + \int_{(0,\infty)} (tA){:}B \; \frac{1+t}{t} d\mu(t)$$

where $a = f_m(0) = \mu(\{0\})$ and $b = \inf \; tf_m(1/t) = \mu(\{\infty\})$. So we can express connections of multiplication operators:

Theorem 3. *Let $\mathbf{m}$ be a connection in the sense of Kubo and Ando. Then, in the above quotient Hilbert space $\mathcal{H}$,*

$$(L(A)\mathbf{m}R(B))[X] = \left[aAX + bXB + \int_{(0,\infty)} \left\{ \int_{-\infty}^{0} e^{tsA}AXBe^{tB}\frac{1+s}{s}dt \right\} d\mu(s)\right]$$

for positive invertible A and B and arbitrary X in $\mathcal{A}$, where a, b and μ is in the above representation for connections $\mathbf{m}$.

For a solidarity $\mathbf{s}$, define an nonnegative operator monotone function $f_m(x) = f_s(x+\varepsilon) - f_s(\varepsilon)$ for a fixed $\varepsilon > 0$ with the corresponding connection $\mathbf{m}$. By the integral representation of $\mathbf{m}$, we have

$$\begin{aligned} C\mathbf{s}D &= f_s(\varepsilon)C + C\mathbf{m}(D - \varepsilon C) \\ &= (a + f_s(\varepsilon) - b\varepsilon)CA + bD + \int_{(0,\infty)} (tD){:}(D-\varepsilon C) \; \frac{1+t}{t} d\mu(t). \end{aligned}$$

Since

$$\begin{aligned} (tD){:}(D-\varepsilon C) &= (tC)(tC + D - \varepsilon C)^{-1}(D - \varepsilon C) \\ &= \frac{t}{t-\varepsilon}(t-\varepsilon)C(tC + D - \varepsilon C)^{-1}D(1 - \varepsilon D^{-1}C) \\ &= \frac{t}{t-\varepsilon}\Big((t-\varepsilon)C{:}D\Big)(1 - \varepsilon D^{-1}C), \end{aligned}$$

we have the following formula for solidarities:

Theorem 4. *For a solidarity $\mathbf{s}$ and $\varepsilon > 0$, let $\mathbf{m}$ be the corresponding connection for $1\mathbf{s}x = 1\mathbf{s}\varepsilon + 1\mathbf{m}(x - \varepsilon)$ with the integral representation as in Theorem 3. Then*

$$\begin{aligned} \big(L(A)\mathbf{s}R(B)\big)X &= (a + f_s(\varepsilon) - b\varepsilon)AX + bXB \\ &+ \int_{(0,\infty)} \left(\int_{-\infty}^{0} te^{(t-\varepsilon)rA}AXBe^{tB}\frac{1+r}{r}dt \right) d\mu(r) \\ &- \int_{(0,\infty)} \left(\int_{-\infty}^{0} t\varepsilon e^{(t-\varepsilon)rA}A^2Xe^{tB}\frac{1+r}{r}dt \right) d\mu(r). \end{aligned}$$

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