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A Note on Two-dimensional Probabilistic Turing Machines

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Summary

This paper introduces *two-dimensional probabilistic Turing machines* (2-ptm's), and investigates several properties of them. We first investigate a relationship between two-dimensional alternating finite automata (2-afa's) and 2-ptm's with error probability less than $\frac{1}{2}$ and with sublogarithmic space, and show that there is a set of square tapes accepted by 2-afa, but not recognized by any $o(\log n)$ space-bounded 2-ptm with error probability less than $\frac{1}{2}$. This partially solves an open problem in [17]. We next investigate a space hierarchy of 2-ptm's with error probability less than $\frac{1}{2}$ and with sublogarithmic space, and show that if $L(n)$ is space-constructible by a two-dimensional Turing machine, $\log\log n < L(n) \leq \log n$ and $L'(n) = o(L(n))$, then, there is a set of square tapes accepted by a strongly $L(n)$ space-bounded two-dimensional deterministic Turing machine, but not recognized by any $L'(n)$ space-bounded 2-ptm with error probability less than $\frac{1}{2}$.

1. Introduction

The classes of sets recognized by (one-dimensional) probabilistic finite automata and probabilistic Turing machines have been studied extensively [3-6,12-14,18,23]. As far as we know, however, there is only one literature concerned with probabilistic automata on a two-dimensional tape [17]. In [17], we introduced two-dimensional probabilistic finite automata (2-pfa's), and showed that

(i) the class of sets recognized by 2-pfa's with error probability less than $\frac{1}{2}$, 2-PFA, is incomparable with the class of sets accepted by two-dimensional alternating finite automata (2-afa's) [9], and

(ii) 2-PFA is not closed under row catenation, column catenation, row + and column + operations in [21].

We believe that it is quite promising to investigate probabilistic machines on a two-dimensional tape.

The classes of sets accepted by two-dimensional (deterministic, nondeterministic, and alternating) finite automata and Turing machines have been studied extensively [1,8-11,15,16,19,22]. In this paper, we introduce a two-dimensional probabilistic Turing machine (2-ptm), and investigate several properties of the class of sets of square tapes recognized by 2-ptm's with error probability less than $\frac{1}{2}$ and with sublogarithmic space.

Section 2 gives some definitions and notations necessary for this paper.

Let 2-PTM$^s(L(n))$ be the class of sets of square tapes recognized by $L(n)$ space-bounded 2-ptm's with error probability less than $\frac{1}{2}$. (See Section 2 for the definition of $L(n)$ space-bounded 2-ptm's.)

In Section 3, we investigate a relationship between 2-afa's and 2-ptm's with sublogarithmic space, and show that there is a set in 2-AFA^s, but not in 2-PTM$^s(L(n))$ with $L(n) = o(\log n)$, where 2-AFA^s denotes the class of sets of square tapes accepted by 2-afa's. As a corollary of this result, it follows that there is a set in 2-AFA^s, but not recognized by any 2-pfa with error probability less than $\frac{1}{2}$. This partially solves an open problem in [17]. Unfortunately, it is still unknown whether there is a set of square tapes recognized by a 2-pfa with error probability less than $\frac{1}{2}$, but not in 2-AFA^s.

In Section 4, we investigate a space hierarchy of 2-ptm's with error probability less than $\frac{1}{2}$ and with sublogarithmic space. It is well known [10,11,15,16] that there is an infinite space hierarchy among classes of sets of square tapes accepted by two-dimensional (deterministic, nondeterministic and alternating) Turing machines with sublogarithmic space. Section 4 shows that if $L(n)$ is space-constructible by a two-dimensional Turing machine, $\log\log n < L(n) \leq \log n$ and $L'(n) = o(L(n))$, then there is a set of square tapes accepted by a strongly $L(n)$ space-bounded two-dimensional deterministic Turing machine, but not in 2-PTM$^s(L'(n))$. As a corollary of this result, it follows that 2-PTM$^s((\log\log n)^k) \subsetneq$ 2-PTM$^s((\log\log n)^{k+1})$ for any positive integer $k \geq 1$.

2. Preliminaries

Let Σ be a finite set of symbols. A *two-dimensional tape* over Σ is a two-dimensional rectangular array of elements of Σ. The set of all the two-dimensional tapes over Σ is denoted by $\Sigma^{(2)}$. Given a tape $x \in \Sigma^{(2)}$, we let $l_1(x)$ be the number of rows and $l_2(x)$ be the number of columns. For each $m, n \geq 1$, let $\Sigma^{m\times n} = \{x \in \Sigma^{(2)} \mid l_1(x) = m \ \&\ l_2(x) = n\}$. If $1 \leq i_k \leq l_k(x)$ for $k = 1, 2$, we let $x(i_1, i_2)$ denote the symbol in x with coordinates (i_1, i_2). Furthermore, we define $x[(i_1, i_2), (i_1', i_2')]$, only when $1 \leq i_1 \leq i_1' \leq l_1(x)$ and $1 \leq i_2 \leq i_2' \leq l_2(x)$, as the two-dimensional tape z satisfying the following (i) and (ii):

(i) $l_1(z) = i_1' - i_1 + 1$ and $l_2(z) = i_2' - i_2 + 1$;

(ii) for each i, j $(1 \leq i \leq l_1(z),\ 1 \leq j \leq l_2(z))$, $z(i,j) = x(i_1 + i - 1, i_2 + j - 1)$.

We next introduce a two-dimensional probabilistic Turing machine which is a natural extension of a two-way probabilistic Turing machine [3, 4] to two dimension. Let S be a finite set. A *coin-tossing distribution on S* is a mapping ψ from S to $\{0, \frac{1}{2}, 1\}$ such that $\Sigma_{a\in S}\psi(a) = 1$. The mapping means "choose a with probability $\psi(a)$".

A *two-dimensional probabilistic Turing machine* (denoted by 2-ptm) is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$, where Q is a finite set of *states*, Σ is a finite *input alphabet* ($\# \notin \Sigma$ is the *boundary symbol*), Γ is a finite *storage tape alphabet* ($B \in \Gamma$ is the *blank symbol*), δ is a *transition function*, $q_0 \in Q$ is the *initial state*, $q_a \in Q$ is the *accepting state*, and $q_r \in Q$ is the *rejecting state*. As shown in Fig.1, the machine M has a read-only rectangular input tape over Σ surrounded by the boundary symbols # and has one semi-infinite storage tape, initially blank. The transition function δ is defined on $(Q - \{q_a, q_r\}) \times (\Sigma \cup \{\#\}) \times \Gamma$ such that for each $q \in Q - \{q_a, q_r\}$, each $\sigma \in \Sigma \cup \{\#\}$ and each $\gamma \in \Gamma$, $\delta[q, \sigma, \gamma]$ is a coin-tossing distribution on $Q \times (\Gamma - \{B\}) \times$ {Left, Right, Up, Down, Stay} $\times$ {Left, Right, Stay}, where Left means "moving left", Right "moving right", Up "moving up", Down "moving down" and Stay "staying there". The meaning of δ is that if M is in state q with the input head scanning the symbol σ and the storage tape head scanning the symbol γ, then with probability $\delta[q, \sigma, \gamma](q', \gamma', d_1, d_2)$ the machine enters state q', rewrites the symbol γ by the symbol γ', either moves the input head one symbol in direction d_1 if $d_1 \in$ {Left, Right, Up, Down} or dose not move the input head if d_1 =Stay, and either moves the storage tape head one symbol in direction d_2 if $d_2 \in$ {Left, Right} or dose not move the storage tape head if d_2 =Stay.

Given an input tape $x \in \Sigma^{(2)}$, M starts in the initial state q_0 with the input head on the upper left-hand corner of x, with all the cells of the storage tape blank and with the storage tape head on the left end of the storage tape. The computation of M on x is then governed (probabilistically) by the transition function δ until M either accepts by entering the accepting state q_a or rejects by entering the rejecting state q_r. We assume that δ is defined so that the input head never falls off an input tape out of the boundary symbols #, the storage tape head cannot write the blank symbol, and fall off the storage tape by moving left. M halts when it enters state q_a or q_r.

Let $L \subseteq \Sigma^{(2)}$ and $0 \leq \epsilon < \frac{1}{2}$. A 2-ptm M recognizes L with error probability ϵ if for all $x \in L$, M accepts x with probability at least $1 - \epsilon$, and for all $x \notin L$, M rejects x with probability at least $1 - \epsilon$.

In this paper, we are concerned with 2-ptm's whose input tapes are restricted to square ones. Let $L : N \to N \cup \{0\}$ be a function, where N denotes the set of all the positive integers. We say that a 2-ptm M is *L(n) space-bounded* if for each $n \geq 1$, and for each input tape x with $l_1(x) = l_2(x) = n$, M uses at most $L(n)$ cells of the storage tape. By 2-PTM$^s(L(n))$, we denote the class of sets of square tapes recognized by $L(n)$ space-bounded 2-ptm's with error probability less than $\frac{1}{2}$ (whose input tapes are restricted to square ones). Especially, by 2-PFA^s, we denote 2-PTM$^s(0)$, i.e, the class of sets of square tapes recognized by two-dimensional probabilistic finite automata [17] with error probability less than $\frac{1}{2}$.

A *two-dimensional alternating finite automaton* (2-afa) is a two-dimensional analogue of the alternating finite automaton [2] with the exception that the input tape head moves left, right, up or down on the two-dimensional tape. See [9] for the formal definition of 2-afa's. By 2-AFA^s, we denote the class of sets accepted by 2-afa's whose input tapes are restricted to square ones. Throughout this paper, we assume that logarithms are base 2.

3. 2-AFA^s versus 2-PTM$^s(L(n))$ with $L(n) = o(\log n)$

This section investigates a relationship between 2-AFA^s and 2-PTM$^s(L(n))$ with $L(n) = o(\log n)$. We first give some preliminaries necessary for getting our desired result.

Let M be a 2-ptm and Σ be the input alphabet of M. For each $m \geq 2$ and each $1 \leq n \leq m-1$, an (m,n)*-chunk* over Σ is a pattern as shown in Fig. 2, where $v_1 \in \Sigma^{(m-1)\times n}$ and $v_2 \in \Sigma^{m \times (m-n)}$. By $ch_{(m,n)}(v_1, v_2)$, we denote the (m,n)-chunk as shown in Fig. 2. For any (m,n)-chunk v, we denote by $v(\#)$ the pattern obtained from v by attaching the boundary symbols # to v as shown in Fig. 3. Below, we assume without loss of generality that M enters or exits the pattern $v(\#)$ only at the face designated by the bold line in Fig. 3. Thus, the the number of the entrance points to $v(\#)$ (or the exit points from $v(\#)$) for M is $n+3$. We suppose that these entrance points (or exit points) are named $\overline{(2,0)}$, $\overline{(2,1)}, \ldots, \overline{(2,n)}$, $\overline{(1,n+1)}$, $\overline{(0,n+1)}$ as shown in Fig. 4. Let $PT(v(\#))$ be the set of these entrance points (or exit points). To each cell of $v(\#)$, we assign a position as shown in Fig. 4. Let $PS(v(\#))$ be the set of all the positions of $v(\#)$. For each $n \geq 1$, an *n-chunk* over Σ is a pattern in $\Sigma^{1\times n}$. For any n-chunk u, we denote by $u(\#)$ the pattern obtained from u by attaching the boundary symbols # to u as shown in Fig. 5. We again assume without loss of generality that M enters or exits the pattern $u(\#)$ only at the face designated by the bold line in Fig. 5. The number of the entrance points to $u(\#)$ (or the exit points from $u(\#)$) for M is again $n+3$, and these entrance points (or exit points) are named $\overline{(2,0)}'$, $\overline{(2,1)}', \ldots, \overline{(2,n)}'$, $\overline{(1,n+1)}'$, $\overline{(0,n+1)}'$ as shown in Fig. 5. Let $PT(u(\#))$ be the set of these entrance points (or exit points). For any (m,n)-chunk v over Σ and any n-chunk u over Σ, let $v[u]$ be the tape in $\Sigma^{m\times m}$ consisting of v and u as shown in Fig. 6.

Let M be a 2-ptm. A *storage state* of M is a combination of the state of the finite control, the non-blank contents of the storage tape, and the storage tape head position. Let q_a and q_r be the accepting and rejecting states of M, respectively and x be an (m,n)-chunk (or an n-chunk) over the input alphabet of M ($m \geq 2, n \geq 1$). We define the *chunk probabilities* of M on x as follows. A *starting condition* for the chunk probability is a pair (s,l), where s is a storage state of M and $l \in PT(x(\#))$; its intuitive meaning is "M has just entered $x(\#)$ in storage state s from entrance point l of $x(\#)$". A *stopping condition* for the chunk probability is either:

(i) a pair (s,l) as above, meaning that M exits from $x(\#)$ in storage state s at exit point l,
(ii) "Loop" meaning that the computation of M loops forever within $x(\#)$,
(iii) "Accept" meaning that M halts in the accepting state q_a before exiting from $x(\#)$ at exit points of $x(\#)$, or
(iv) "Reject" meaning that M halts in the rejecting state q_r before exiting from $x(\#)$ at exit points of $x(\#)$.

For each starting condition σ and each stopping condition τ, let $p(x, \sigma, \tau)$ be the probability that stopping condition τ occurs given that M is started in starting condition σ on an (m,n)-chunk (or n-chunk) x.

Computations of a 2-ptm are modeled by Markov chains [20] with finite state space, say $\{1, 2, \ldots, s\}$ for some s. A particular Markov chain is completely specified by its matrix $R = \{r_{ij}\}_{1 \leq i,j \leq s}$ of transition probabilities. If the Markov chain is in state i, then it next moves to state j with probability r_{ij}. The chains we consider have the designated starting state, say, state 1, and some set T_r of trapping states, so $r_{tt} = 1$ for all $t \in T_r$. For $t \in T_r$, let $p^*[t, R]$ denote the probability that Markov chain R is trapped in state t when started in state 1. The following lemma which bounds the effect of small changes in the transition probabilities of a Markov chain is used below.

Let $\beta \geq 1$. Say that two numbers r and r' are *β-close* if either (i) $r = r' = 0$ or (ii) $r > 0, r' > 0$ and $\beta^{-1} \leq \frac{r}{r'} \leq \beta$. Two Markov chains $R = \{r_{ij}\}_{i,j=1}^{s}$ and $R' = \{r'_{ij}\}_{i,j=1}^{s}$ are *β-close* if r_{ij} and r'_{ij} are β-close for all pairs i, j.

Lemma 3.1 [3]. Let R and R' be two s-state Markov chains which are β-close, and let t be a trapping state of both R and R'. Then $p^*[t, R]$ and $p^*[t, R']$ are β^{2s}-close.

Theorem 3.1 There exists a set in 2-AFA^s, but not in 2-PTM$^s(L(n))$ for any $L(n) = o(\log n)$.

Proof. Let $T_1 = \{x \in \{0,1\}^{(2)} | \exists n \geq 2[l_1(x) = l_2(x) = n \ \& \ \exists i(2 \leq i \leq n)[x[(1,1),(1,n)] = x[(i,1),(i,n)]$ (i.e., the top row of x is identical with some another row of x)]]$\}$.

T_1 is accepted by the 2-afa M_1 which acts as follows. Given an input tape x with $l_1(x) = l_2(x) \geq 2$, M_1 existentially chooses some row other than the top row, say the i-th row, of x. Then M_1 universally tries to check that, for each $j(1 \leq j \leq l_2(x)), x(i,j) = x(1,j)$. That is, on the i-th row and j-th column of $x(1 \leq j \leq l_2(x))$, M_1 enters a universal state to choose one of two further actions. One action is to pick up the symbol $x(i,j)$, move up with the symbol stored in the finite control, compare the stored symbol with the symbol $x(1,j)$, and enter an accepting state if both the symbols are identical. The other action is to continue to move right one tape cell (in order to pick up the symbol $x(i,j+1)$ and compare it with the symbol $x(1,j+1)$). It will be obvious that M_1 accepts T.

We next show that $T_1 \notin$ 2-PTM$^s(L(n))$ with $L(n) = o(\log n)$. Suppose to the contrary that there exists a 2-ptm M recognizing T_1 with error probability $\epsilon < \frac{1}{2}$. For large n, let

- $U(n) =$ the set of all the n-chunks over $\{0,1\}$,
- $W(n) = \{0,1\}^{(m_n-1)\times n}$, where $m_n = 2^n + 1$, and
- $V(n) = \{ch_{(m_n,n)}(w_1, w_2) | w_1 \in W(n) \ \& \ w_2 \in \{0\}^{m_n \times (m_n - n)}\}$.

We shall below consider the computations of M on the input tapes of side-length m_n. For large n, let $C(n)$ be the set of all the storage states of M using at most $L(m_n)$ storage tape cells, and let $c(n) = |C(n)|$. Then $c(n) = b^{L(m_n)}$ for some constant b. Consider the chunk probabilities $p(v, \sigma, \tau)$ defined above. For each (m_n, n)-chunk v in $V(n)$, there are a total of

$$d(n) = c(n) \times |PT(v(\#))| \times (c(n) \times |PT(v(\#))| + 3) = O(n^2 t^{L(m_n)})$$

chunk probabilities for some constant t. Fix some ordering of the pairs (σ, τ) of starting and stopping conditions and let $\mathbf{p}(v)$ be the vector of these $d(n)$ probabilities according to this ordering.

We first show that if $v \in V(n)$ and if p is a nonzero element of $\mathbf{p}(v)$, then $p \geq 2^{-c(n)a(n)}$, where $a(n) = |PS(v(\#))| = O(m_n^2) = O(e^n)$ for some constant e.

Form a Markov chain $K(v)$ with states of the form (s, l), where s is a storage state of M and $l \in PS(v(\#)) \cup PT(v(\#))$. The chain state (s, l) with $l \in PS(v(\#))$ corresponds to M being in storage state s scanning the symbol at position l of $v(\#)$. Transition probabilities from such states are obtained from the transition probabilities of M in the obvious way. For example, if the symbol at position (i,j) of $v(\#)$ is 0, and if M in storage state s reading a 0 can move its input head left and enter storage state s' with probability 1/2, then the transition probability from state $(s,(i,j))$ to state $(s',(i,j-1))$ is 1/2. Chain states of the form $(s, \overline{(i,j)})$ with $\overline{(i,j)} \in PT(v(\#))$ are trap states of $K(v)$ and correspond to M just having exited from $v(\#)$ in storage state s at exit point $\overline{(i,j)}$ of $v(\#)$. Now consider, for example, $p = p(v, \sigma, \tau)$, where $\sigma = (s, \overline{(i,j)})$ and $\tau = (s', \overline{(k,l)})$ with $\overline{(i,j)}, \overline{(k,l)} \in PT(v(\#))$. If $p > 0$, then there must be some path of nonzero probability in $K(v)$ from $(s, \overline{(i,j)})$ to $(s', \overline{(k,l)})$, and since $K(v)$ has at most $c(n)a(n)$ nontrapping states, there is such a path of length at most $c(n)a(n)$. Since 1/2 is the smallest nonzero transition probability of M, it follow that $p \geq 2^{-c(n)a(n)}$. If $\sigma = (s, \overline{(i,j)})$ with $\overline{(i,j)} \in PT(v(\#))$ and $\tau =$ Loop, there must be a path of nonzero probability in $K(v)$ from state $(s,(i,j))$ to some state $(s',(i',j'))$ such that there is no path of nonzero probability from $(s',(i',j'))$ to any trap state of the form $(s'', \overline{(k,l)})$ with $\overline{(k,l)} \in PT(v(\#))$. Again, if there is such a path, there is one of length at most $c(n)a(n)$. The remaining cases are similar.

For each $v = ch_{(m_n,n)}(w_1, w_2) \in V(n)$, let

$$contens(v) = \{u \in U(n) | u = w_1[(i,1),(i,n)] \text{ for some } i(1 \leq i \leq 2^n)\}.$$

Divide $V(n)$ into contents-equivalence classes by making v and v' *contents-equivalent* if $contents(v) = contents(v')$. There are

$$contents(n) = \binom{2^n}{1} + \binom{2^n}{2} + \ldots + \binom{2^n}{2^n} = 2^{2^n} - 1$$

contents-equivalence classes of (m_n, n)-chunks in $V(n)$. (Note that $contents(n)$ corresponds to the number of all the nonempty subsets of $U(n)$.) We denote by $CONTENTS(n)$ the set of all the representatives of these $contents(n)$ contents-equivalence classes. Of course, $|CONTENTS(n)| = contents(n)$. Divide $CONTENTS(n)$ into M-equivalence classes by making v and v' *M-equivalent* if $\mathbf{p}(v)$ and $\mathbf{p}(v')$ are zero in exactly the same coordinates. Let $E(n)$ be a largest M-equivalence class. Then we have

$$|E(n)| \geq contents(n)/2^{d(n)}.$$

Let $d'(n)$ be the number of nonzero coordinates of $\mathbf{p}(v)$ for $v \in E(n)$. Let $\hat{\mathbf{p}}(v)$ be the $d'(n)$-dimensional vector of nonzero coordinates of $\mathbf{p}(v)$. Note that $\hat{\mathbf{p}}(v) \in [2^{-c(n)a(n)}, 1]^{d'(n)}$ for all $v \in E(n)$. Let $log\ \hat{\mathbf{p}}(v)$ be the componentwise log of $\hat{\mathbf{p}}(v)$.

Then, $log\ \hat{\mathbf{p}}(v) \in [-c(n)a(n), 0]^{d'(n)}$. By dividing each coordinate interval $[-c(n)a(n), 0]$ into subintervals of length μ, we divide the space $[-c(n)a(n), 0]^{d'(n)}$ into at most $(c(n)a(n)/\mu)^{d(n)}$ cells, each of size $\mu \times \mu \times \ldots \times \mu$. We want to choose μ so large enough that the number of cells is smaller than the size of $E(n)$, that is,

$$(\frac{c(n)a(n)}{\mu})^{d(n)} < \frac{contents(n)}{2^{d(n)}}(\leq |E(n)|) \tag{1}$$

Concretely, we choose $\mu = 2^{-n}$. (From the assumption that $L(n) = o(\log n)$, we have $L(m_n) = o(\log m_n)$. Thus, $L(m_n) = o(n)$. From this, by a simple calculation, we can easily see that for large n, (1) holds for $\mu = 2^{-n}$). Assuming (1), there must be two different (m_n, n)-chunks $v, v' \in E(n)$ such that $log\ \hat{\mathbf{p}}(v)$ and $log\ \hat{\mathbf{p}}(v')$ belong to the same cell. Therefore, if p and p' are two nonzero probabilities in the same coordinate of $\mathbf{p}(v)$ and $\mathbf{p}(v')$, respectively, then

$$|log\ p - log\ p'| \leq \mu.$$

It follows that p and p' are 2^{μ}-close. Therefore, $\mathbf{p}(v)$ and $\mathbf{p}(v')$ are componentwize 2^{μ}-close.

For this v and v', we consider an n-chunk $u \in contents(v) - contents(v')$. We describe two Markov chains, R and R', which model the computations of M on $v[u]$ and $v'[u]$, respectively. The state space of R is

$$C(n) \times (PT(v(\#)) \cup PT(u(\#))) \cup \{\text{Accept,Reject,Loop}\}.$$

Thus the number of states of R is

$$z = c(n)(n+3+n+3)+3 = 2c(n)(n+3)+3.$$

The state $(s, \overline{(i,j)}) \in c(n) \times PT(v(\#))$ of R corresponds to M just having entered $v(\#)$ in storage state s from entrance point $\overline{(i,j)}$ of $v(\#)$, and the state $(s', \overline{(k,l)'}) \in c(n) \times PT(u(\#))$ of R corresponds to M just having entered $u(\#)$ in storage state s' from entrance point $\overline{(k,l)'}$ of $u(\#)$. For convenience sake, we assume that M begins to read any input tape x in the initial storage state $s_0 = (q_0, \lambda, 1)$, where q_0 is the initial state of M, by entering $x(1,1)$ from the lower edge of the cell on which $x(1,1)$ is written. Thus, the starting state of R is Initial$\stackrel{\triangle}{=} (s_0, \overline{(2,1)'})$. The states Accept and Reject correspond to the computations halting in the accepting state and the rejecting state, respectively, and Loop means that M has entered an infinite loop. The transition probabilities of R are obtained from the chunk probabilities of M on $u(\#)$ and $v(\#)$. For example, the transition probability from $(s, \overline{(i,j)})$ to $(s', \overline{(k,l)'})$ with $\overline{(i,j)} \in PT(v(\#))$ and $\overline{(k,l)'} \in PT(u(\#))$ is just $p(v, (s, \overline{(i,j)}), (s', \overline{(k,l)}))$, the transition probability from $(s', \overline{(k,l)'})$ to $(s, \overline{(i,j)})$ with $\overline{(i,j)} \in PT(v(\#))$ and $\overline{(k,l)'} \in PT(u(\#))$ is $p(u, (s', \overline{(k,l)'}), (q, \overline{(i,j)'}))$, the transition probability from $(s, \overline{(i,j)})$ to Accept is $p(v, (s, \overline{(i,j)}),\text{Accept})$, and the transition probability from $(s', \overline{(k,l)'})$ to Accept is $p(u, (s', \overline{(k,l)'}),\text{Accept})$. The states Accept, Reject, and Loop are trap states. The chain R' is defined similarly, but using $v'[u]$ in place of $v[u]$.

Let $acc(v[u])$ (resp., $acc(v'[u])$) be the probability that M accepts input $v[u]$ (resp., $acc(v'[u])$). Then, $acc(v[u])$ (resp., $acc(v'[u])$) is exactly the probability that the Markov chain R (resp., R') is trapped in state Accept when started in state Initial. From the fact that $v[u]$ is in T_1, it follows that $acc(v[u]) \geq 1 - \epsilon$. Since R and R' are 2^{μ}-close, Lemma 3.1 implies that

$$\frac{acc(v'[u])}{acc(v[u])} \geq 2^{-2\mu z}.$$

$2^{-2\mu z}$ approaches 1 as n increases. Therefore, for large n, we have

$$acc(v'[u]) \geq 2^{-2\mu z}(1-\epsilon) > \frac{1}{2},$$

because $\epsilon < \frac{1}{2}$. This is a contradiction, because $v'[u] \notin T_1$. ∎

We conjecture that there is a set in 2-PFA^s, but not in 2-AFA^s. The candidate set is $T_2 = \{x \in \{0,1\}^{n \times n} | n \geq 2$ & (the numbers of 0's and 1's in x are the same)$\}$. By using the idea in [4], we can show that T_2 is in 2-PFA^s. But, we have no proof of "$T_2 \notin$2-AFA^s".

4. Space hierarchy between $\log\log n$ and $\log n$

This section shows that there is an infinite space hierarchy for 2-ptm's with error probability less than $\frac{1}{2}$ whose spaces are between $\log\log n$ and $\log n$.

A *two-dimensional deterministic Turing machine* (2-dtm) is a two-dimensional analogue of the two-way deterministic Turing machine [7], which has one read-only input tape and one semi-infinite read-write storage tape, with the exception that the input head moves left, right, up or down on the two-dimensional tape. The 2-dtm *accepts* an input tape x if it starts in the initial state with the input head on the upper left-hand corner of x, and eventually enters an accepting state. See [9,16] for the formal definition of 2-dtm's.

Let $L(n) : N \rightarrow N \cup \{0\}$ be a function. A 2-dtm M is *strongly L(n) space-bounded* if it uses at most $L(n)$ cells of the storage tape for each $n \geq 1$ and each input tape x with $l_1(x) = l_2(x) = n$. Let strong 2-DTM$^s(L(n))$ be the class of sets of square tapes accepted by strongly $L(n)$ space-bounded 2-dtm's.

A function $L(n) : N \rightarrow N \cup \{0\}$ is *space-constructible* by a two-dimensional Turing machine (2-tm) if there is a strongly $L(n)$ space-bounded 2-dtm M such that for each $n \geq 1$, there exists some input tape x with $l_1(x) = l_2(x) = n$ on which M halts after its storage tape head has marked off exactly $L(n)$ cells of the storage tape. In this case, we say that M *constructs* the function $L(n)$.

Let Σ_1, Σ_2 be finite sets of symbols. A *projection* is a mapping $\bar{\tau} : \Sigma_1^{(2)} \to \Sigma_2^{(2)}$ which is obtained by extending the mapping $\tau : \Sigma_1 \to \Sigma_2$ as follows:

$$\bar{\tau}(x) = x' \Leftrightarrow \left(\begin{array}{ll} \text{(i) } l_k(x) = l_k(x') & \text{for each } k = 1, 2, \text{ and} \\ \text{(ii) } \tau(x(i,j)) = x'(i,j) & \text{for each } (i,j)(1 \le i \le l_1(x) \\ & \quad \text{and } 1 \le j \le l_2(x)). \end{array} \right.$$

Theorem 4.1 If $L(n)$ is space-constructible by a 2-tm, $\log\log n < L(n) \le \log n$, and $L'(n) = o(L(n))$, then, there exists a set in strong 2-DTM$^s(L(n))$, but not in 2-PTM$^s(L'(n))$.

Proof. Let $L : N \to N$ be a function space-constructible by a two-dimensional Turing machine such that $\log\log n < L(n) \le \log n$ $(n \ge 1)$, and M be a strongly $L(n)$ space-bounded 2-dtm which constructs the function L, and $T[L, M]$ be the following set, which depends on L and M:

$T[L,M] = \{x \in (\Sigma \times \{0,1\})^{(2)} | \exists n \ge 2\, [l_1(x) = l_2(x) = n \ \&\ \exists r (r \le L(n))$ [(when the tape $\bar{h}_1(x)$ is presented to M, it uses r cells of the storage tape and halts) & $\exists i(2 \le i \le n)[\bar{h}_2(x[(1,1),(1,r)]) = \bar{h}_2(x[(i,1),(i,r)])]]]\}$, where Σ is the input alphabet of M, and $\bar{h}_1$ ($\bar{h}_2$) is the projection obtained by extending the mapping $h_1 : \Sigma \times \{0,1\} \to \Sigma$ ($h_2 : \Sigma \times \{0,1\} \to \{0,1\}$) such that for any $c = (a,b) \in \Sigma \times \{0,1\}, h_1(c) = a$ ($h_2(c) = b$).

We first show that $T[L,M] \in$ strong 2-DTM$^s(L(n))$. The set $T[L,M]$ is accepted by a strongly $L(n)$ space-bounded 2-dtm M_1 which acts as follows. When an input tape $x \in (\Sigma \times \{0,1\})^{(2)}$ with $l_1(x) = l_2(x) = n, n \ge 2$, is presented to M_1, M_1 directly simulates the action of M on $\bar{h}_1(x)$. If M does not halt, then M_1 also does not halt, and will not accept x. If M_1 finds out that M halts (in this case, note that M_1 has used at most $L(n)$ cells of the storage tape, because M is a strongly $L(n)$ space-bounded), then M_1 checks by using the non-blank part of the storage tape that $\bar{h}_2(x)$ is a desired form. M_1 enters an accepting state only if this check is successful.

We next show that $T[L,M] \notin$ 2-PTM$^s(L'(n))$, where $L'(n) = o(L(n))$. For each $n \ge 1$, let $t(n) \in \Sigma^{(2)}$ be a fixed tape such that (i) $l_1(t(n)) = l_2(t(n)) = n$ and (ii) when $t(n)$ is presented to M, M marks off exactly $L(n)$ cells of the storage tape and halts. (Note that for each $n \ge 1$, there exists such a tape $t(n)$, because M constructs the function L.) Now, suppose that there exists a 2-ptm M_2 recognizing $T[L,M]$ with error probability $\epsilon < \frac{1}{2}$. We can derive a contradiction by using the same idea as in the proof of Theorem 3.1. The main difference is

(i) to replace

- "$U(n)$ = the set of all the n-chunks over $\{0,1\}$",
- "$W(n) = \{0,1\}^{(m_n - 1)\times n}$, where $m_n = 2^n + 1$",
- "$V(n) = \{ch_{(m_n,n)}(w_1,w_2) | w_1 \in W(n) \ \&\ w_2 \in \{0\}^{m_n \times (m_n - n)}\}$",
- "$c(n) = |C(n)| = b^{L(m_n)}$ for some constant b",
- "$d(n) = c(n) \times |PT(v(\#))| \times (c(n) \times |PT(v(\#))| + 3) = O(n^2 t^{L(m_n)})$",
- "$p \ge 2^{-c(n)a(n)}$, where $a(n) = |PS(v(\#))| = O(m_n^2) = O(e^n)$ for some constant e",
- "for each $v = ch_{(m_n,n)}(w_1,w_2) \in V(n)$, $contens(v) = \{u \in U(n) | u = w_1[(i,1),(i,n)]$ for some $i(1 \le i \le 2^n)\}$",
- "$contents(n) = \binom{2^n}{1} + \binom{2^n}{2} + \ldots + \binom{2^n}{2^n} = 2^{2^n} - 1$ contents-equivalence classes of (m_n, n)-chunks in $V(n)$",
- "$\mu = 2^{-n}$",
- "n-chunk $u \in contents(v) - contents(v')$", and
- "$z = c(n)(n+3+n+3)+3 = 2c(n)(n+3)+3$",

in the proof of Theorem 3.1, with

- "$U(n)$ = the set of all the $L(n)$−chunks u over $\Sigma \times \{0,1\}$ such that $\bar{h}_1(u) = t(n)[(1,1),(1,L(n))]$",
- "$W(n) = \{w \in (\Sigma \times \{0,1\})^{(n-1)\times L(n)} | \bar{h}_1(w) = t(n)[(2,1),(n,L(n))]\}$",
- "$V(n) = \{ch_{(n,L(n))}(w_1,w_2) | w_1 \in W(n) \ \&$ (w_2 is a tape in $(\Sigma \times \{0\})^{n \times (n - L(n))}$ such that $\bar{h}_1(w_2) = t(n)[(1, L(n+1)),(n,n)])\}$",
- "$c(n) = |C(n)| = b^{L'(n)}$ for some constant b",
- "$d(n) = c(n) \times |PT(v(\#))| \times (c(n) \times |PT(v(\#))| + 3) = O(L(n)^2 t^{L'(n)})$ for some constant t",
- "$p \ge 2^{-c(n)a(n)}$, where $a(n) = |PS(v(\#))| = O(n^2)$",
- "for each $v = ch_{(n,L(n))}(w_1,w_2) \in V(n)$, $contents(v) = \{u \in U(n) | u = w_1[(i,1),(i,L(n))]$ for some $i(1 < i \le n-1)\}$",
- "
$$contents(n) = \begin{cases} \binom{2^{L(n)}}{1} + \ldots + \binom{2^{L(n)}}{n-1} & \text{if } 2^{L(n)} \ge n-1 \\ \binom{2^{L(n)}}{1} + \ldots + \binom{2^{L(n)}}{2^{L(n)}} = 2^{2^{L(n)}} - 1 & \text{otherwise} \end{cases}$$
contents-equivalence classes of $(n, L(n))$-chunks in $V(n)$",
- "$\mu = 2^{-L(n)}$",
- "$L(n)$-chunk $u \in contents(v) - contents(v')$", and

- "$z = c(n)(L(n) + 3 + L(n) + 3) + 3 = 2c(n)(L(n) + 3) + 3$",

respectively, and

(ii) to consider the computations on the input tapes of side-length n and on $(n, L(n))$-chunks, instead of considering the computations on the input tapes of side-length m_n and on (m_n, n)-chunks.

The details of the proof is left to the reader as an exercise. We note that by making a simple calculation, we can easily ascertain that

$$\left(\frac{c(n)a(n)}{\mu}\right)^{d(n)} < \frac{contents(n)}{2^{d(n)}} (\leq |E(n)|)$$

for large n and for our new $c(n), a(n), d(n), \mu$, and $contents(n)$, because $\log\log n < L(n) \leq \log n$ and $L'(n) = o(L(n))$. ∎

Since $(\log\log n)^k$, $k \geq 1$, is space-constructible by a 2-tm (in fact, $(\log\log n)^k$ is space-constructible by one-dimensional Turing machine [7]), it follows from Theorem 4.1 that the following corollary holds.

Corollary 4.1 For any integer $k \geq 1$,
$2\text{-PTM}^s((\log\log n)^k) \subsetneqq 2\text{-PTM}^s((\log\log n)^{k+1})$.

Remark 4.1 It is well-known [7] that, in the one-dimensional case, there exists no space-constructible function which grows more slowly than the order of $\log\log n$. On the other hand, Morita et al. [15] and Szepietowski [22] showed that the function $\log^{(k)}(n)$ $(k \geq 1), \log^* n$ and $\log^{(1)} \log^* n$ are all space-constructible by a two-dimensional Turing machine, where these functions are defined as follows:

$$\log^{(1)} n = \begin{cases} 0 & (n = 0) \\ \lceil \log_2 n \rceil & (n \geq 1) \end{cases}$$

$$\log^{(k+1)} n = \log^{(1)}(\log^{(k)} n) \text{ for } k \geq 1$$

$$\exp^* 0 = 1, \ \ \exp^*(n+1) = 2^{exp^* n}$$

$$\log^* n = min\{x | \exp^* x \geq n\}$$

It is shown in [10,11,16] that for two-dimensional (deterministic, nondeterministic and alternating) Turing machines whose input tapes are restricted to square ones, $\log^{(k)}$ space-bounded machines are more powerful than $\log^{(k+1)}$ space-bounded machines $(k \geq 1)$. We conjecture that for each $k \geq 2$, $2\text{-PTM}^s(\log^{(k+1)} n) \subsetneqq 2\text{-PTM}^s(\log^{(k)} n))$, but we have no proof of this conjecture.

5. Conclusion

We conclude this paper by giving the following open problems.

(1) For what $L(n)$, is there a set in 2-PFA^s, but not accepted by any $L(n)$ space-bounded two-dimensional alternating Turing machine?

(2) Is there an infinite space hierarchy for 2-ptm's with error probability $\epsilon < \frac{1}{2}$ whose spaces are below $\log\log n$?

It will be also interesting to investigate the relationship among the accepting powers of 2-ptm's with error probability $\epsilon < \frac{1}{2}$, 2-atm's with only universal states, and two-dimensional nondeterministic Turing machines [9]. We will discuss this topics in a forthcoming paper.

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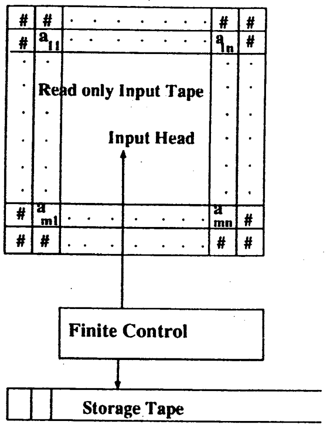

Figure 1: Two-dimensional probabilistic Turing machine.

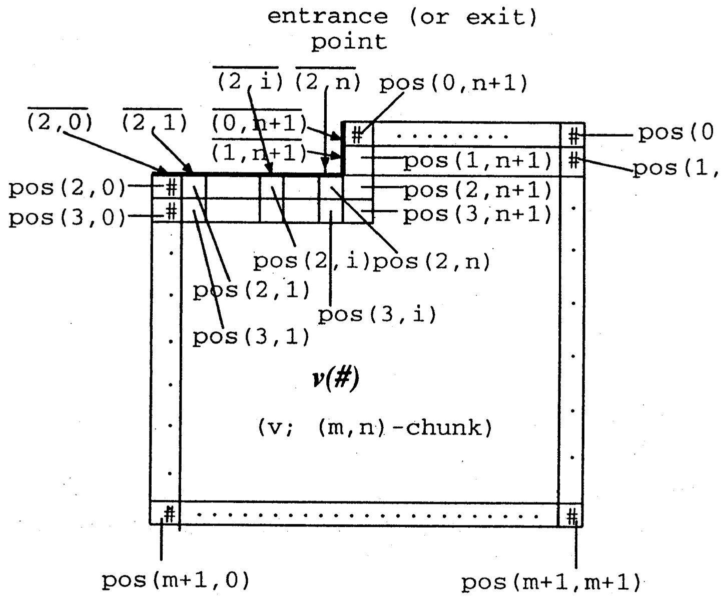

Figure 4: An Illustration for $v(\#)$ (v : (m,n)-chunk).

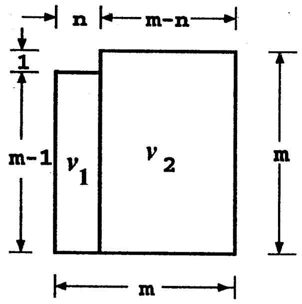

Figure 2: (m,n)-chunk.

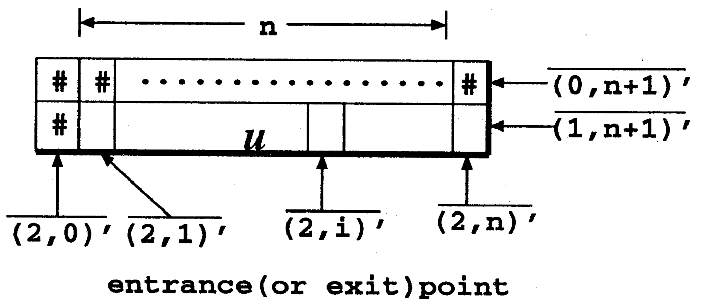

Figure 5: An Illustration for $u(\#)$.

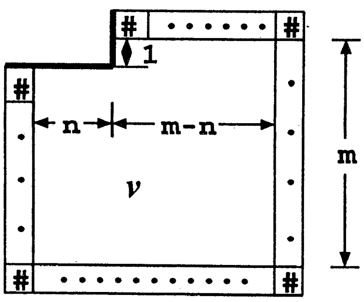

Figure 3: $v(\#)$.

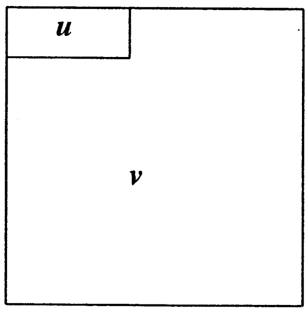

Figure 6: $v[u]$.