数理解析研究所講究録
1042 巻 1998 年 1-10

REMARKS ON GIBBS MEASURES FOR FIBRED SYSTEMS

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1. Gibbs measures for fibred systems

In [4] we investigated the existence and uniqueness of a system of conditional measures for fibred systems which plays a similar role as conformal measures for non-fibred one-dimensional dynamical systems. In this note we add some general properties and make some additional remarks about this concept.

Recall that a fibred system $\mathcal{Y}=(Y,T,X,S,\pi)$ consists of two dynamical systems $T:Y\to Y$ and $S:X\to X$ and a projection $\pi:Y\to X$ which commutes with T and S. In particular, $\mathcal{Y}$ is a skew product if $Y=Y_0\times X$ and if T has the form $T((y_0,x))=(T_x(y_0),S(x))$. We make the further assumption that both spaces are Polish spaces and are equipped with the Borel σ-algebras $\mathcal{B}_Y$ and $\mathcal{B}_X$ and that all maps T, S and π are continuous, although for some concepts below this is not a prerequisite. T preserves the fibres $Y_x=\pi^{-1}(x)$; the restriction of T^n $(n\geq 1)$ to the fibre Y_x will be denoted by T_x^n, so $T_x^n:Y_x\to Y_{S^n(x)}$. If we need to specify a metric on Y, it will be denoted by $d(y,y')$.

The main result in [4] is for fibred systems which have the expanding and exactness property fibrewise. The fibred system $\mathcal{Y}$ is called fibre expanding, if there exists a constant $a>0$ such T_x is a local homeomorphisms on $B(y,a)\cap Y_x$ for every $y\in Y_x, x\in X$ and expands the metric uniformly. $\mathcal{Y}$ is topologically exact along fibres if $T^N(B(x,\epsilon)\cap Y_x)\supset Y_{S^N(x)}$ for any $\epsilon>0$ and any $x\in X$ where N depends on ϵ alone.

Under these assumptions we proved (cf. [1] for a similar result when S is invertible)

Theorem 1.1: Let Y be compact, T be bounded-to-one, $\mathrm{card}Y_x\geq 2$ and $T(Y_x)=Y_{S(x)}$ for $x\in X$. Then for every Hölder continuous function $\phi:Y\to\mathbb{R}$ there exists a unique family of conditional probability measures μ_x $(x\in X)$ and a unique measurable function $A:X\to\mathbb{R}$ satisfying

$$(1)\qquad A(x)\int g(y)\mu_x(dy)=\int V_x^{(1)}g(y)\mu_{S(x)}(dy)$$

for every bounded measurable function $g:Y\to\mathbb{R}$, where

$$(2)\quad V_x^{(k)}g(y)=\sum_{y'\in T^{-k}(y)\cap\pi^{-1}(x)} g(y')\exp\left[\phi(y')+\ldots+\phi(T^{k-1}(\phi(y')))\right]\qquad (k\geq 1).$$

This research was supported by the Deutsche Forschungsgemeinschaft, JSPS, and by INTAS, grant 93-0570

A family of conditional measure μ_x is called Gibbs for the potential ϕ if it satisfies (1) for some measurable function $A : X \to \mathbb{R}$. For short we also say that μ_x is (ϕ, A)-Gibbs.

Under some additional asumptions one can prove continuity of the function A and the map $x \mapsto \mu_x$. These conditions are openess of S and π, and the property that the mapping $y \mapsto (\pi(y), T(y))$ is a local homeomorphism onto its image.

Example 1.2: Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be an entire mapping. Following [6], it is called (d_1, d_2)-regular ($d_1 \in \mathbb{Q}_+, d_2 \in \mathbb{N}$) if there are constants $k_1, k_2 > 0$ and $r \geq 0$ such that for every $z \in \mathbb{C}^n$, $\|z\| \geq r$

$$k_1 \ \|z\|^{d_1} \leq \|T(z)\| \leq k_2 \ \|z\|^{d_2}.$$

Let $n = 2$. For $d_1 = 1/2$ and $d_2 = 2$ these transformations include all Hénon maps (these are automorphisms, but there are also endomorphisms in this class) and for $d := d_1 = d_2 = 2$ the polynomial map

$$T(x, y) = (p(x, y), q(x, y)) \qquad (x, y \in \mathbb{C})$$

is called strict (where p and q are polynomials). A special case are skew products when p does not depend on y. Then $\pi \circ T = p \circ \pi$ where $\pi : \mathbb{C}^2 \to \mathbb{C}$ denotes the projection map onto the first factor.

According to [6], a point $z \in \mathbb{C}^2$ is called weakly normal if there exists an open neighborhood V of z and a family $\{\mathcal{K}_x : x \in V\}$ of at least one-dimensional complex analytic sets $\mathcal{K}_x$ such that $x \in \mathcal{K}_x$ and the family $\{T^n_{|\mathcal{K}_x \cap V} : n \geq 0\}$ is normal in x. The complement of the set of weakly normal points plays the same role as the Julia set in one dimension (and in fact reduces to this set for $n = 1$) and is denoted by $\mathcal{H}(T)$. It is shown in [6] that for strict polynomials $\mathcal{H}(T)$ is compact, fully invariant and contained in (but not equal in general) $\partial K(T)$ where

$$K(T) = \{z \in \mathbb{C}^2 : \sup_k \|T^k(z)\| < \infty\}.$$

It follows that a strict skew product restricted to $\mathcal{H}(T)$ is a fibred system, but not a skew product in general. It is worth noting that $\mathcal{H}_x = \pi^{-1}(x) \cap \mathcal{H}(T)$ is the fibre over x, and is the Julia set of T^n_x in case x is periodic with period n.

This example is the basic motivation for us. The notion of normality introduced by Heinemann seems to be the appropriate notion to study repellers for higher dimensional polynomial mappings and their dynamical properties (despite of other attempts by considering these maps on projective spaces; however, this causes unnecessary difficulties). In some cases, it is known (see [6], [7], [3]) that $\mathcal{H}(T)$ is equal to each of the following sets: (a) the closure of the set of repelling periodic points, (b) the Shilov boundary of $K(T)$, (c) the support of the measure of maximal entropy (this measure can be obtained using the Green current). These cases include Cantor skew products, noodle type maps (which can be studies through generalized Mandelbrot sets) and Torus like maps. Two of these classes belong to the class of fibred systems as considered here, and we expect that our result has further application to the ergodic theory of these dynamical systems as those mentioned in [4].

2. Equivalent conditions for the Gibbs property

In this and the following sections we make the general assumption that T is bounded-to-one on fibres. This means there exists a constant $M > 0$ such that for all $y \in Y$ $\mathrm{card} T^{-1}(y) \cap \pi^{-1}(\pi(y)) \leq M$.

In what follows measurability is always meant with respect to the Borel σ-fields of the Polish spaces under consideration. There is no necessity to consider any extension of Borel σ-fields. General results on countable-to-one Borel maps contained in [8] (and cited there) can be applied to verify the Borel measurability of all sets and functions considered here ('countable' means 'countably infinite or finite'). The function in y under the integral in the right-hand side of (1) may serve as an example.

For a Polish space Z, let $\mathcal{B}_Z$ and B_Z denote the Borel σ-field of Z and the space of bounded measurable functions $f : Z \to \mathbb{R}$, respectively.

The first result gives equivalent conditions for the Gibbs property. Its proof is standard and omitted.

Proposition 2.1: Let $\{\mu_x : x \in X\}$ be a system of conditional probabilities for $\mathcal{Y}$, and let $\varphi : Y \to \mathbb{R}$ and $A : X \to \mathbb{R}$ be measurable functions. Then the following conditions are equivalent and each of the conditions expresses the Gibbs property:

(a) For every $x \in X$ and every $E \in \mathcal{B}_Y$ on which T is invertible and for all $f \in B_Y$ vanishing outside of $T(E)$,

$$\int f(y)\,\mu_{S(x)}(dy) = A(x) \int f(T(y)) \exp[-\varphi(y)]\,\mu_x(dy). \tag{3}$$

(b) For all $h \in B_Y$ vanishing outside a set E as above, for all $x \in X$,

$$A(x) \int h(y)\,\mu_x(dy) = \int h(T^*(y)) \exp[\varphi(T^*(y))]\,\mu_{S(x)}(dy), \tag{4}$$

where T^* denotes the inverse of $T_{|E}$.

(c) For all $x \in X$ the Jacobian with respect to the map T is given by

$$\frac{d\mu_{S(x)} \circ T_x}{d\mu_x} = A(x) \exp[-\varphi] \qquad \mu_x \text{ a.e.} \tag{5}$$

Remark 2.2:

a) The Jacobian in the left-hand side of (5) is, by definition, any Borel measurable nonnegative function J on the support of μ_x satisfiyng $\mu_{S(x)}(T(E)) = \int_E J(y)\,\mu_x(dy)$ for every Borel set E with the property that $T|_E$ is injective (see (3)). Let μ_i be a probability measure on a standard Borel space X_i ($i = 1, 2$). Assume that $R : X_1 \to X_2$ is a Borel map which is nonsingular with respect to μ_1 and μ_2, countable-to-one on the support of μ_1 and which maps μ_1-null-sets to μ_2-null-sets. Then the Jacobian $J_R = J_R^{\mu_1,\mu_2}$ exists, takes values in $(0, \infty)$ and is uniquely defined up to μ_1-null-sets. It can be characterized by the property that

$$\int_{X_1} f(x)\ \mu_1(dx) = \int_{X_2} \sum_{R(x')=x} f(x')\ J_R(x')^{-1}\ \mu_2(dx)$$

for every $f \in L_1(\mu_1)$. Also the following holds for the Jacobian and the Radon-Nikodym derivative: $\frac{d\mu_1 \circ R^{-1}}{d\mu_2}(x) = \sum_{x' \in R^{-1}(x)} J_R(x')^{-1} \quad (x \in X_2)$.

b) Again, consider two standard Borel spaces X_1 and X_2 with σ-finite measures μ_1 and μ_2, respectively, and a countable-to-one Borel map $R : X_1 \to X_2$. If φ is a real-valued Borel function on X_1 and A is a constant then R is called Gibbs with respect to (φ, A) if for every function $f \in B_{X_1}$ we have (cf. (1))

$$A \int f(y)\, \mu_1(dy) = \int \sum_{y' \in R^{-1}(y)} f(y') \exp[\varphi(y')]\, \mu_2(dy).$$

In case of a self-map of a probability space one obtains the definition of a ϕ-conformal measure in [2]. Moreover, this definition applied to any triple $((Y, \mu_x), (Y, \mu_{S(x)}), T_x)$ for $x \in X$ reduces to (1). From a general measure-theoretical viewpoint the Gibbs property in the above general form (for some (φ, A)) is the combination of two properties of R:

(1) R is nonsingular ($\mu_2(A) = 0$ implies $\mu_1(T^{-1}(A) = 0)$;

(2) if $\mu_1(A) = 0$, then $\mu_2(T(A)) = 0$.

More precisely, in this case R is Gibbs with respect to $(-\log J_R, 1)$.

c) The definition of a conditional Gibbs measure does not imply that preimages of sets of positive measure have positive measure. But this is the case if the image of the whole space has full measure.

d) Consider the situation in b) again. It follows from Rokhlin's theory [9] of conditional probabilities for measurable partitions that a measure preserving countable-to-one map has the property that images of null-sets are null-sets and, in view of b), that the measure is Gibbs. Conversely, a map R which is Gibbs with respect to $(\varphi, 1)$, is measure preserving ($\mu_1 \circ R = \mu_2$) if and only if $A^{-1} \sum_{R(x')=x} \exp[\varphi(x')] = 1 \;\; \mu_2 -$ a.e.

3. Uniqueness of Gibbs measures for fibred systems

In this section we state conditions for the uniqueness of Gibbs measures which, in fact, are weaker than those of theorem 1.1. (An interesting peoblem is to weaken the conditions in theorem 1.1 for the existence of Gibbs measures). A system $\{\mu_x : x \in X\}$ of probability measures μ_x on Y is called (weakly) continuous, if for every continuous bounded function f on Y the function $x \to \int f(y)\, \mu_x(dy)$ is continuous. Also, two systems $\{\mu_x : x \in X\}$ and $\{\nu_x : x \in X\}$ are called equivalent if for every $x \in X$ the measures μ_x and ν_x are equivalent (have the same null-sets).

Definition 3.1: The fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called expanding along fibres (with respect to the metric d) if there exist constants $\Lambda > 1$, $\epsilon_0 > 0$, and an integer $N_0 > 0$ such that

$$d(T^{N_0}(y), T^{N_0}(y')) \geq \Lambda^{N_0} d(y, y')$$

for every $y, y' \in Y$ such that $\pi(y) = \pi(y')$, $d(T^{N_0}(y), T^{N_0}(y')) < \epsilon_0$.

The next property of fibred systems (see (6) below) needs some terminology related to covering theory with applications to differentiation of set functions. Except for the definition of the universal Vitali relation we follow section 2.8 of [5]. A covering relation $\mathcal{C}$ on a metric space Z is a subset of $\{(z, S) : z \in S \subset Z\}$ and it is fine at z if $\inf\{\mathrm{diam}(S) : (z, S) \in \mathcal{C}\} = 0$. For $A \subset Z$ let $\mathcal{C}(A) = \{S : (z, S) \in \mathcal{C}$ for some $z \in A\}$. A Vitali relation for a Borel measure m (finite on bounded sets) is a covering relation $\mathcal{V}$ such that $\mathcal{V}(Z)$ is a family of Borel sets, $\mathcal{V}$ is fine at each point of Z and the following condition holds: if $\mathcal{C} \subset \mathcal{V}$ is a covering relation, $A \subset Z$ and if $\mathcal{C}$ is fine at each point of A, then $\mathcal{C}(A)$ has a countable disjoint subfamily which covers a subset of A of full measure in A. A covering relation is said to be a universal Vitali relation if it is a Vitali relation for any measure m as above.

We consider the following property for expanding fibred systems $\mathcal{Y} = (Y, T, X, S, \pi)$ where Y is a compact metric space :
There exist $r_0 > 0$ and a covering relation $\mathcal{C} \subset \{(y, B) : y \in B \in \mathcal{B}\}$ on Y such that for any $x \in X$ the following two conditions are satisfied:

(6a) $\mathcal{C}_x = \{(y, B \cap Y_x) : y \in Y_x, (y, B) \in \mathcal{C}\}$ is a universal Vitali relation on Y_x.

(6b) For every $(y, B) \in \mathcal{C}_x$ there exists $n \geq 0$ such that

(i) $T^n|_B : B \to T^n(B)$ is invertible, $\mathrm{diam}(T^n(B)) \leq \epsilon_0$ (where ϵ_0 is the same as in definition 3.1) and

(ii) $T^n(B) \supset B_2(T^n y, r_0)$, where $B_2(z, r)$ denotes the ball of radius r in $Y_{\pi(z)}$ centered at z.

There are two basic examples satisfying (6) and which are expanding along fibres.

Example 3.2: Let Y be a compact space and $T : Y \to Y$ be continuous with a Markov partition for T, respecting fibres and for which the fibre maps are uniformly expanding (if $\mathcal{R}$ is a Markov partition then for every $x \in X$, $R \in \mathcal{R} \cap Y_x$, TR is a union of sets in $\mathcal{R} \cap Y_{S(x)}$; also note that the sets from $\mathcal{R} \cap Y_x$ can be assumeded to have arbitrary small diameters). Let us suppose also that each element of $\mathcal{R}$ is the closure of its interior. Then property (6) can be verified (see theorem 2.8.19 of [5]) for the relation
$\mathcal{C} = \{(y, B) : \ y \in B = \bigcap_{k=0}^{m} T^{-k} R_{i_k},\ R_{i_k} \in \mathcal{R}; 0 \leq k \leq m;\ m \geq 0\}$.

Example 3.3: The second example is from conformal dynamics. Let $Y \subset \mathbb{C}^2$ and let each Y_x be contained in the complex plane $\mathbb{C}$, denoted by $\widehat{Y}_x$, and assume that each map T_x extends to a holomorphic map $\widehat{T}_x : \widehat{Y}_x \to \widehat{Y}_{S(x)}$. If T is expanding along fibres (we conjecture that it is true for strict polynomials if the forward orbit of the set of critical points of T does not have any accumulation point in Y), then T can be shown to satisfy (6) for the relation $\mathcal{C}$ defined by all pairs (y, B) where B is any ball (with respect to the maximum metric of $\mathbb{C}^2$) of sufficiently small diameter and centered at $y \in Y$ (note that in this case $\mathcal{C}_x$ consist of pairs of the form $(y, B_2(y, r)), y \in Y_x$). This can be seen as follows. Property (6a) is implied by theorem 2.8.18 and section 2.8.9 of [5]. Furthermore, there exist constants $K > 0$ and $r_1 > 0$ such that for each $n \geq 1$ and every $y \in Y$ all inverse branches $\widehat{T}_\nu^{-n}$ of $\widehat{T}^n$ are well defined on $\widehat{B}_2(y, r_1) = \{y' \in \widehat{Y}_{\pi(y)} : d_{\pi(y)}(y, y') < r_1\}$ and, by Koebe's theorem, $|D_2 \widehat{T}_\nu^{-n}(y)| \leq K |D_2 \widehat{T}_\nu^{-n}(y')|\ y' \in \widehat{B}_2(y, r_1)$, where D_2 denotes the partial derivative in the fibre $\widehat{Y}_{\pi(y)}$. This estimate bounds the distortion of fibres under the inverse branches of $\widehat{T}^n$ and under $\widehat{T}^n$ itself (restricted to the range of an inverse branch) uniformly in n and over all fibres. We may (and shall) suppose that r_1 is sufficiently small (in particular, $r_1 \leq \epsilon_0/2$). It is not difficult to show that (6b) holds for the family $\mathcal{C}$.

Proposition 3.4: Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system with compact metric space Y. Assume that T is expanding along fibres and satisfies (6). Let $\{\mu_x : x \in X\}$ be a conformal system of conditional probabilities for some continuous function φ which is also uniformly Hölder continuous in each fibre for some exponent $s > 0$. Assume that

$$\text{(7)} \qquad \inf\{\mu_x(B) : x \in X;\ B \text{ is a ball of radius } r_0 \text{ with center in } Y_x\} > 0.$$

Then $\{\mu_x : x \in X\}$ is uniquely determined by the above properties up to equivalence.

Proof. We may assume that the system is $(\varphi, 1)$-Gibbs. Note that by (6) $r_0 \leq \epsilon_0$. Fix $x \in X$ and $(y,\ B) \in \mathcal{C}_x$. Set $D = T^n(B)$ and denote by T^{n*} the inverse of T^n on B sending $T^n(y)$ to y. Then by the expanding property in definition 3.1 for $y', y'' \in D$ we have that $d(y', y'') = d(T^{n-k}(T^k(T^{n*}(y'))), T^{n-k}(T^k(T^{n*}(y''))))$ and hence

$$d(T^k(T^{n*}(y')), T^k(T^{n*}(y''))) \leq \Lambda^{-n+k} \operatorname{diam}(D).$$

Then, using Hölder continuity, we obtain

$$\left| \sum_{k=0}^{n} \varphi(T^k(T^{n*}(y'))) - \varphi(T^k(T^{n*}(y''))) \right| \leq ||\varphi||_s \sum_{k=0}^{n} \Lambda^{-sk} (\operatorname{diam}(D))^s = \log K_1,$$

where $||\varphi||_s$ denotes the upper bound of all Hölder seminorms of exponent s taken over the fibres. Therefore, $\mu_x(B) \exp[-\varphi(y) - \ldots - \varphi(T^n(y))] = K_1 \mu_{S^n(x)}(T^n(B)) \leq K_1$. Similarly, $\mu_x(B) \exp[-\varphi(y) - \ldots - \varphi(T^n(y))] \geq K_1^{-1} \inf_{B'} \{\mu_{S^n(x)}(B')\} > 0$, where B' denotes a ball in $Y_{S^n(\pi(y))}$ of radius r_0 (as in (6)).

Let $\{\nu_x : x \in X\}$ denote another Gibbs system of conditional measures for the function φ satisfying (7). We derive for some constant $K_2 > 0$

$$\text{(8)} \qquad K_2^{-1} \nu_x(B) \leq \mu_x(B) \leq K_2 \nu_x(B)$$

for all sets B such that $(y, B) \in \mathcal{C}$ for some $y \in Y$. In order to extend (8) to arbitrary Borel sets, consider a relatively open $G \subset Y_x \subset Y$. There exists a covering relation $\mathcal{C}_x^G \subset \mathcal{C}_x$ which is fine at any $y \in G$ and such that $B \subset G$ for every $(y, B) \in \mathcal{C}_x^G$. Then $\mathcal{C}_x^G$ contains a countable disjoint subfamily $\{B_i : i \in I\}$ with $\mu_x(\bigcup_{i \in I} B_i) = \mu_x(G)$, hence

$$\mu_x(G) = \mu_x(\bigcup_{i \in I} B_i) = \sum_{i \in I} \mu_x(B_i) \leq K_2 \sum_{i \in I} \nu_x(B_i) = \nu_x(\bigcup_{i \in I} B_i) \leq \nu_x(G),$$

and, by symmetry arguments, $\nu_x(G) \leq K_2 \mu_x(G)$. Approximating $B \in \mathcal{B}_{Y_x}$ by such G simultaneously with respect to μ_x and ν_x we obtain (8) for any measurable set B. This proves the proposition.

Corollary 3.5: Let $\{\mu_x : x \in X\}$ be in addition continuous and assume that the set of periodic points of $S : X \to X$ is dense in X. Then $\{\mu_x : x \in X\}$ is unique as a continuous Gibbs measure for ϕ.

Proof. Let $x \in X$ be periodic for S with period n. Then $T_0 := T^n_{|Y_x} : Y_x \to Y_x$ and hence μ_x is Gibbs for T_0 and the function $\Phi_x(y) = \varphi(y) + ... + \varphi(T^{n-1}(y))$. Such μ_x is unique because T_0 is expanding and Φ_x is Hölder continuous. This follows from the previous proposition together with the ergodic decomposition of Gibbs measures (this can be shown as in [10]). Now let $\{\nu_x : x \in X\}$ be another continuous Gibbs system for φ. Then $\mu_x = \nu_x$ for every periodic point x of S, and since the systems are continuous, they must coincide because periodic points are dense.

4. Absolutely continuous invariant measures

In this section we construct invariant and other measures of special interest from a given system of conditional probabilities for (Y, T). It is clear that for a T-invariant probability measure ν on Y the measure $\mu = \nu \circ \pi^{-1}$ is S-invariant. Conversely, every S-invariant probability measure μ on X can be lifted to a T-invariant probability measure ν on Y. However, we are interested in such liftings when the conditional measures given π are prescribed. A system $\{\mu_x : x \in X\}$ of conditional probabilities for $\mathcal{Y}$ is called invariant if $\mu_x \circ T^{-1} = \mu_{S(x)}$ for all $x \in X$. Integrating such a system over x with respect to any S-invariant probability μ one obtains a T-invariant measure ν. Moreover, assuming the hypotheses of corollary 3.5 it can be easily seen that a system $\{\mu_x : x \in X\}$ is invariant if such an integral is a T-invariant measure μ for any S-invariant probability ν. For invertible S any T-invariant probability measure is obtained by integration of a (measurable) invariant family of conditional probabilities. But this is not the case for non-invertible S. In this case invariant systems only form a subclass within the class of systems of conditional measures on fibres arising from T-invariant probabilities on Y. The following proposition gives some necessary and sufficient conditions that a Gibbs measure admits an equivalent invariant system.

Assume that $\varphi \in B_Y$. We define the conditional transfer operators $V_x^{(k)} : B_x \to B_{S^k(x)}$, $(k \geq 0, x \in X)$, by (2) where $B_x := B_{Y_x}$.

Proposition 4.1: Let $\{\mu_x : x \in X\}$ be (φ, A)-Gibbs.
(A) Assume that there exists a family $\{h_x : Y_x \to \mathbb{R},\ x \in X\} \cup \{\lambda : X \to \mathbb{R}\}$ of measurable nonnegative functions satisfying

$$V_x^{(1)} h_x = \lambda(x) h_{S(x)} \quad \text{and} \quad \int h_x(y)\, \mu_x(dy) > 0, \quad (x \in X).$$

Then

$$A(x) = \lambda(x) \frac{\int h_{S(x)}(y) \mu_{S(x)}(dy)}{\int h_x(y) \mu_x(dy)} \quad \text{and} \quad \tilde{\mu}_x = \frac{h_x}{\int h_x(y)\, \mu_x(dy)} \mu_x, \quad (x \in X)$$

defines an invariant system of conditional probabilities absolutely continuous with respect to $\{\mu_x : x \in X\}$.
(B) If there exists a T-invariant system of conditional probabilities $\{\tilde{\mu}_x : x \in X\}$ absolutely continuous with respect to $\{\mu_x : x \in X\}$ so that $\tilde{\mu}_x = h_x \mu_x$ for some family of nonnegative measurable functions $h_x : Y_x \to \mathbb{R}$, then

$$V_x^{(1)} h_x = A(x)\; h_{S(x)} \quad \mu_{S(x)}\text{-a.e.}, \quad (x \in X).$$

Proof. Our assumptions and (1) imply that $A(x) = \lambda(x) \frac{\int h_{S(x)}(y)\mu_{S(x)}(dy)}{\int h_x(y)\mu_x(dy)}$. Replacing $\varphi(y)$ by $\varphi(y) - \log A(\pi(y))$ and h_x by $h_x / \int h_x(y)\,\mu_x(dy)$ we may assume that $\{\mu_x : x \in X\}$ is a $(\varphi, 1)$-Gibbs measure and that h_x satisfies $V_x^{(1)} h_x = h_{S(x)}$ and $\int h_x(y)\,\mu_x(dy) = 1, \quad (x \in X)$. Let $d\tilde{\mu}_x = h_x\, d\mu_x$. Then, by (1) and the definition of the conditional transfer operator, we have for any $f \in B_Y$:

$$\int f(T(y))\tilde{\mu}_x(dy) = \int f(T(y))h_x(y)\mu_x(dy) = \int V_x^{(1)}[h_x\; f \circ T](y)\mu_{S(x)}(dy)$$
$$= \int f(y)V_x^{(1)}h_x(y)\mu_{S(x)}(dy) = \int f(y)h_{S(x)}(y)\mu_{S(x)}(dy) = \int f(y)\tilde{\mu}_{S(x)}(dy).$$

This proves the invariance of $\tilde{\mu}_x$ and (A).
In order to show (B), we obtain from the above

$$\int f(y)V_x^{(1)}h_x(y)\mu_{S(x)}(dy) = \int f \circ T d\tilde{\mu}_x = \int f d\tilde{\mu}_{S(x)} = \int f h_{S(x)} d\mu_{S(x)}$$

for every integrable f. Thus $V_x^{(1)} h_x = h_{S(x)} \;\; \mu_{S(x)}$-a.e.

The following proposition 4.2 is concerned with general conditions under which a measurable system of conditional probabilities on fibres, together with a probability measure on the base, gives rise to a conformal (or invariant conformal) measure on the total space.

Proposition 4.2: Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system with a bounded-to-one map $S : X \to X$ and let φ and A be measurable functions on Y and X.
(A) Let ν be a $(\varphi, 1)$-conformal probability on Y and let $\{\nu_x : x \in X\}$ be a version of the conditional probabilities for ν with respect to π. Then $\mu = \nu \circ \pi^{-1}$ is a $(\log B, 1)$-conformal measure for (X, S) where $B(x) = \int_{Y_{S(x)}} V_x^{(1)} 1(y) \nu_{S(x)}(dy) \quad (x \in X)$. Moreover, for μ-a.e. $x \in X$, $T_x : Y_x \to Y_{S(x)}$ is a (φ, B)-Gibbs map (cf. remark 2.2 b)) for (Y_x, ν_x), i.e. the Jacobian of T_x is $\frac{d\nu_{S(x)} \circ T}{d\nu_x}(y) = B(x) \exp[-\varphi(y)]\;\; \nu_x$ – a.e.
(B) Conversely, if $\{\nu_x : x \in X\}$ is (φ, A)-Gibbs for (Y, T), and if μ is a $(\psi, 1)$-conformal measure for (X, S) then $\nu(dy) = \int_X \nu_x(dy)\; \mu(dx)$ is $(\Lambda, 1)$-conformal for (Y, T) where $\Lambda = \varphi + (\psi - \log A) \circ \pi$.

Proof. (A) Let ν be $(\varphi, 1)$-conformal. Then the conformality of μ follows from remark 2.2 b). In fact, we only need to prove that T is a nonsingular map and sends null-sets to null-sets. If $E \in \mathcal{B}_X$ then $S^{-1}(E) = \pi(T^{-1}(\pi^{-1}(E)))$ and $S(E) = \pi(T(\pi^{-1}(E)))$, since π is onto. Also, since $\pi : (Y,\nu) \to (X,\mu)$ is measure preserving, it follows that S is nonsingular and sends null-sets to null-sets as well. Hence S is Gibbs. Let us calculate the Jacobian of S (and prove once more that S is Gibbs). Take a funcion $f \in L_1(X,\mu)$. Then, since π is measure preserving and ν is conformal, we have

$$\begin{aligned}
\int_X f(x)\mu(dx) &= \int_Y f(\pi(y))\ \nu(dy) = \int_Y \sum_{y' \in T^{-1}(y)} f(\pi(y'))\ \exp[\varphi(y')]\ \nu(dy) \\
&= \int_X \Big(\int_Y \sum_{x' \in S^{-1}(x)} \ \sum_{y' \in T^{-1}(y) \cap \pi^{-1}(x')} f(\pi(y')) \exp[\varphi(y')]\ \nu_x(dy) \Big)\ \mu(dx) \\
&= \int_X \sum_{x' \in S^{-1}(x)} f(x') \Big(\int_{Y_x} \sum_{y' \in T^{-1}(y) \cap \pi^{-1}(x')} \exp[\varphi(y')]\ \nu_x(dy) \Big)\ \mu(dx) \\
&= \int_X \sum_{x' \in S^{-1}(x)} J_S(x')^{-1}\ f(x')\ \mu(dx),
\end{aligned}$$

where $J_S(x)^{-1} = \int_{Y_{S(x)}} \sum_{y' \in T^{-1}(y) \cap \pi^{-1}(x)} \exp[\varphi(y')]\ \nu_{S(x)}(dy)$.

Similarly one shows the second part of (A) using (1) repeatedly and conformality of μ. Let $f : Y \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be bounded measurable functions. Then

$$\begin{aligned}
\int_X B(x)g(x) \int_{Y_x} f(y)\nu_x(dy)\mu(dx) &= \int_Y B(\pi(y))g(\pi(y))f(y)\ \nu(dy) \\
= \int_X \sum_{x' \in S^{-1}(x)} g(x')B(x') &\Big(\int_{Y_x} \sum_{y' \in T^{-1}(y) \cap \pi^{-1}(x')} f(y') \exp[\varphi(y')]\ \nu_x(dy) \Big)\ \mu(dx) \\
= \int_X g(x) \int_{Y_{S(x)}} &\sum_{y' \in T^{-1}(y) \cap \pi^{-1}(x)} f(y') \exp[\varphi(y')] \nu_{S(x)}(dy)\mu(dx).
\end{aligned}$$

It follows from this that for μ-a.e. $x \in X$ $B(x) \int_{Y_x} f(y)\nu_x(dy) = \int_{Y_{S(x)}} V_x^{(1)} f(y)\nu_{S(x)}(dy)$. Applying remark 2.2 b) to triples $((Y_x,\nu_x),(Y_{S(x)},\nu_{S(x)}),T_x)$ the claim follows.

(B) We shall show (1) for a ν-integrable function f and for ν using conformality of μ and the Gibbs property of ν_x in the form (1).

$$\begin{aligned}
\int f(y)\nu(dy) &= \int \sum_{x' \in S^{-1}(x)} \int f(y)\nu_{x'}(dy) \exp[\psi(x')]\ \mu(dx) \\
&= \int \sum_{x' \in S^{-1}(x)} A(x')^{-1} \exp[\psi(x')] \int \sum_{y' \in T^{-1}(y) \cap \pi^{-1}(x')} f(y') \exp[\varphi(y')] \nu_{S(x')}(dy)\ \mu(dx) \\
&= \int \sum_{y' \in T^{-1}(y)} f(y') \exp[\Lambda(y')] \nu(dy).
\end{aligned}$$

Corollary 4.3:
(A) In the situation of proposition 4.2 (A) μ is S-invariant iff $\int \sum_{y' \in T^{-1}(y)} \exp[\varphi(y')]\nu_x(dy)$

$= 1$ μ-a.e., and ν is T-invariant iff $\sum_{y' \in T^{-1}(y)} \exp[\varphi(y')] = 1$ ν-a.e.
(B) In the situation of proposition 4.2 (B) let h be a density with respect to ν. Let $\tilde{\nu}(dy) = h(y)\,\nu(dy)$. Then $\tilde{\nu}$ is T-invariant iff h satisfies $h(y) = \sum_{y' \in T^{-1}(y)} \exp[\Lambda(y')]h(y')$, ν-a.e.

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