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A transversality condition for quadratic family at Collet-Eckmann parameter

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We consider real quadratic maps $Q_t : \mathbf{R} \to \mathbf{R}$, $x \mapsto t - x^2$, where $t \in \mathbf{R}$ is a parameter. We say that Q_t satisfies Collet-Eckmann condition if

$$\liminf_{n\to\infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > 1.$$

This condition implies that the dynamics of Q_t is 'chaotic'(existence of absolutely continuous invariant measure, decay of correlation, *etc.*). We give

Theorem 1 *If Q_t satisfies Collet-Eckmann condition, then*

$$\lim_{n\to\infty} \frac{\frac{\partial}{\partial s}\{Q_s^n(0)\}|_{s=t}}{DQ_t^{n-1}(Q_t(0))} > 0. \tag{1}$$

In a sense, the condition(1) implies that the quadaratic family is transversal to the "manifold" of the maps which is topologically conjugate to Q_t.

Combining theorem 1 with Jacobson's theorem [2], we get

Proposition 2 *Let A be the set of parameters t for which Q_t satisfies Collet-Eckmann condition and*

$$\liminf_{n\to\infty} n^{-1} \log |DQ_t(Q_t^n(0))| = 0. \tag{2}$$

Then every point in A is a density point of A itself in the interval $[0, 2]$.

Remark that A contains $t = 2$. The coondition (2) holds if the crtical point 0 is not recurrent.

We prove theorem 1 as follows. Take $r > 1$ such that

$$\liminf_{n\to\infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > r > 1.$$

We consider Q_t as a map from the complex plain to itself. Let A be a Ruelle operator A on the quadratic differentials:

$$A(\varphi)(x) = \sum_{Q_t(y)=x} \frac{\varphi(y)}{[DQ_t(y)]^2},$$

acting on the space

$$S = \left\{ x = \sum_{i=1}^{\infty} x_i \psi_i \;\middle|\; \sum_i |x_i DQ^i(Q(0))| r^{-i} < \infty \right\}$$

where $\psi_i(z) = (z - Q^i(0))^{-1}$. We endowe S with a norm

$$|x| = \sum_i |x_i DQ^i(Q(0))| r^{-i}.$$

Then we have, formally,

$$\lim_{n\to\infty} \frac{\frac{\partial}{\partial s} Q_s^n(0)|_{s=t}}{DQ_t^{n-1}(Q_t(0))} = \det(\mathrm{Id} - A). \tag{3}$$

Comparing A with the Perron-Frobenius operator, we see that the spectral radius of A is smaller than 1. Hence, if A were a finite-dimensional operator, these would imply (1). Actually, we can't give any appropriate definition for the determinant in (3) since A is an infinite dimensional operator. Instead, we approximate A by a sequence of finite-dimensional operators. For detail, see [3].

References

[1] M.Dunford, J.T.Schwartz, Linear operators 1, Interscience, New York,(1958)

[2] M. Tsujii, Positive Lyapunov exponent in families of one dimensional dynamical systems, *Invent. math.* vol.111 (1993), 113–137

[3] M. Tsujii, A simple proof for monotonicity of entropy in the quadratic family, preprint, Hokkaido University