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Dynamical Systems for the Frobenius-Perron Operator

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1 Introduction

The Frobenius-Perron and Koopman operators are useful for various mathematical fields. We consider the following transformation.

$$S(x,y)=(ax+by+\alpha, cx+dy+\beta) \pmod 1, \tag{1.1}$$

where $0\leq x,y<1$, $a,b,c,d\in\mathbf{R}$ and $0\leq\alpha,\beta<1$. This transformation may display three levels of irregular behavior (ergodicity, mixing and exactness) depending on the coefficients a,b,c,d,α and β. We investigate the relation between the coefficients and the behavior using these operators. We first give a necessary and sufficient condition for S to be measure preserving [Theorem 4], because measure preserving is supposed in the definition of mixing and exactness. In the case of $a,b,c,d\in\mathbf{Z}$ and $\alpha=\beta=0$ in (1.1), we show a necessary and sufficient condition for S to be mixing [Theorem 9]. In Theorem 10, we show S displays the following behaviors depending on $a,b,c,d\in\mathbf{Z}$ and $0\leq\alpha,\beta<1$ in (1.1):

(i) S is mixing;

(ii) S is ergodic, but not mixing;

(iii) S is not ergodic.

2 The Frobenius-Perron and Koopman Operators

Definition (Markov operator). Let $(X,\mathcal{A},\mu)$ be a measure space. Any linear operator $P:L^1\to L^1$ satisfying

(a) $Pf\geq 0$ for $f\geq 0, f\in L^1$;

(b) $\|Pf\|=\|f\|$, for $f\geq 0, f\in L^1$

is called a **Markov operator.**

Definition (nonsingular). A measurable transformation $S:X\to X$ on a measure space $(X,\mathcal{A},\mu)$ is **nonsingular** if $\mu(S^{-1}(A))=0$ for all $A\in\mathcal{A}$ such that $\mu(A)=0$.

Definition . Let $(X,\mathcal{A},\mu)$ be a measure space. If $S:X\to X$ is a nonsingular transformation, the unique operator $P:L^1\to L^1$ defined by

$$\int_A Pf(x)\mu(dx)=\int_{S^{-1}(A)} f(x)\mu(dx) \quad \text{for } A\in\mathcal{A} \tag{2.1}$$

is called the **Frobenius-Perron operator** corresponding to S.

Definition . Let $(X, \mathcal{A}, \mu)$ be a measure space, $S : X \to X$ a nonsingular transformation, and $f \in L^\infty$. The operator $U : L^\infty \to L^\infty$ defined by

$$Uf(x) = f(S(x))$$

is called the **Koopman operator** with respect to S.

Definition (measure-preserving). Let $(X, \mathcal{A}, \mu)$ be a measure space and $S : X \to X$ a measurable transformation. Then S is said to be **measure preserving** if

$$\mu(S^{-1}(A)) = \mu(A) \qquad \text{for all } A \in \mathcal{A}.$$

Definition (ergodic). Let $(X, \mathcal{A}, \mu)$ be a measure space and let a nonsingular transformation $S : X \to X$ be given. The S is called **ergodic** if every invariant set $A \in \mathcal{A}$ is such that either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

Definition (mixing). Let $(X, \mathcal{A}, \mu)$ be a normalized measure space, and $S : X \to X$ a measure-preserving transformation. S is called **mixing** if

$$\lim_{n \to \infty} \mu(A \cap S^{-n}(B)) = \mu(A)\mu(B) \qquad \text{for all } A, B \in \mathcal{A}.$$

Definition (exact). Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $S : X \to X$ a measure-preserving transformation such that $S(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$. If

$$\lim_{n \to \infty} \mu(S^n(A)) = 1 \qquad \text{for every } A \in \mathcal{A}, \mu(A) > 0,$$

then S is called **exact**.

Remark 1. If S is exact, then S is mixing. If S is mixing, then S is ergodic.

The proof of ergodicity, mixing, or exactness using these definitions is difficult. So we will use the following theorem and proposition.

Theorem 1 ([1]). *Let $(X, \mathcal{A}, \mu)$ be a normalized measure space, $S : X \to X$ a measure-preserving transformation, and P the Frobenius-Perron operator corresponding to S. Then*

(a) *S is ergodic if and only if*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle P^k f, g \rangle = \langle f, 1 \rangle \langle 1, g \rangle \qquad \textit{for } f \in L^1, g \in L^\infty;$$

(b) *S is mixing if and only if*

$$\lim_{n \to \infty} \langle P^n f, g \rangle = \langle f, 1 \rangle \langle 1, g \rangle \qquad \textit{for } f \in L^1, g \in L^\infty;$$

(c) *S is exact if and only if*

$$\lim_{n \to \infty} \| P^n f - \langle f, 1 \rangle \| = 0 \qquad \textit{for } f \in L^1.$$

Proposition 2 ([1]). *Let $(X, \mathcal{A}, \mu)$ be a normalized measure space, $S : X \to X$ a measure-preserving transformation, and U the Koopman operator corresponding to S. Then*

(a) *S is ergodic if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\langle f, U^k g\rangle = \langle f, 1\rangle\langle 1, g\rangle \qquad \text{for } f \in L^1, g \in L^\infty;$$

(b) *S is mixing if and only if*

$$\lim_{n\to\infty}\langle f, U^n g\rangle = \langle f, 1\rangle\langle 1, g\rangle \qquad \text{for } f \in L^1, g \in L^\infty.$$

3 The dynamics of $S^n(x, y)$

Consider first $\alpha = \beta = 0$ in (1.1), i.e.

$$S(x, y) = (ax + by, cx + dy) \qquad (\text{mod } 1),$$

where $a, b, c, d \in \mathbf{R}$. Let $X = [0, 1) \times [0, 1)$ and $X^\circ = (0, 1) \times (0, 1)$ and O,P,Q and R be the points $(0, 0), (a, c), (a + b, c + d)$ and $(b, d) \in \mathbf{R}^2$, respectively.

Proposition 3. *Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \to X$ is defined by*

$$S(x, y) = (ax + by, cx + dy) \qquad (\text{mod } 1),$$

where $a, b, c, d \in \mathbf{R}$ and the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$detA = ad - bc = 1$$

and $|a + d| < 2$. If there exist $(x_0, y_0) \in X^\circ$ such that $S(x_0, y_0) = (x_0, y_0)$, then S is not ergodic.

Proof. We will show that there exists a nontrivial invariant set.

Let eigenvalues of A be $\mu \pm i\nu$. There exist $\theta \in [0, 2\pi]$ and $r, t \in \mathbf{R}$ satisfying

$$A = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{t} \end{pmatrix}\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}.$$

$S(x_0, y_0) = (x_0, y_0)$ means that there exists $m, n \in \mathbf{Z}$ such that $A(x_0, y_0) + (m, n) = (x_0, y_0)$. By putting $T(x_0, y_0) = A(x_0, y_0) + (m, n)$, we see that the set $\Gamma(x, y) = \{T^n(x, y)|\ n = 0, 1, \cdots\}$ is on the ellipse with center (x_0, y_0), since $ad - bc = 1$ and $|a + d| < 2$. If (x, y) is very near to (x_0, y_0), $\Gamma(x_0, y_0) \subset X^\circ$ and $T^n(x, y) = S^n(x, y)$. So, if we take a sufficiently small set B such that $\mu(B) > 0$ and $(x_0, y_0) \in B$, then $\Gamma(B)$ is an invariant set under S, which implies S is not ergodic. □

Example 1. Suppose $A = \begin{pmatrix} -\frac{13}{10} & -\frac{7}{10} \\ \frac{113}{70} & \frac{1}{10} \end{pmatrix}$ $(detA = 1)$.
Put $(x_0, y_0) = (\frac{9}{32}, \frac{113}{224})$ or $(x_0, y_0) = (\frac{25}{32}, \frac{65}{224})$. Then $S(x_0, y_0) = (x_0, y_0)$. Thus, S is not ergodic by Proposition 3.

Suppose $A = \begin{pmatrix} 1 & -\frac{1}{1000} \\ 1 & \frac{999}{1000} \end{pmatrix}$ $(detA = 1)$. There doesn't exist $(x_0, y_0) \in X^\circ$ such that

$$A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad (m, n \in \mathbf{Z}).$$

Now let's be back to the definitions of mixing and exact. Since measure preserving is supposed in the definition of mixing and exactness (i.e. $\mu(S^{-1}(A)) = \mu(A)$ for $\forall A \in \mathcal{A}$), we first give a necessary and sufficient condition for S to be measure preserving.

Let $A(X) = \bigcup_{l=1}^{M} B_l$, where $B_l \subset [m_l, m_l + 1) \times [n_l, n_l + 1), m_l, n_l \in \mathbf{Z}$. We define ϕ_l as

$$\phi_l(B_l) = \{(x - m_l, y - n_l) | (x, y) \in B_l\} \subset X.$$

Lemma 1. *Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \to X$ is defined by*

$$S(x, y) = (ax + by, cx + dy) \qquad (\text{mod } 1)$$

and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbf{R}$. The following statements are equivalent:

(1) *S is measure preserving;*

(2) *The following statements hold:*

  (i) $|detA| = n \in \mathbf{N}$;

  (i) There exist the sets K_l $(l = 1, 2, \cdots, M)$ and a partition of X $\{D_j\}_{j=1}^{k}$ such that $B_l = \bigcup_{j_l \in K_l} \phi_l^{-1}(D_{j_l})$;

  (iii) The number of elements of the set $\{l \mid D_j^\circ \bigcap \phi_l(B_l) \neq \emptyset\}$ is equal to n.

(3) $|detA| = n \in \mathbf{N}$ and either (a) or (b) holds:

  (a) $a, c \in \mathbf{Z}$ and there exist $(x_0, y_0) \in \mathbf{Z}^2$ on the line $\overline{RQ}$;

  (b) $b, d \in \mathbf{Z}$ and there exist $(x_0, y_0) \in \mathbf{Z}^2$ on the line $\overline{PQ}$.

Proof. We show (1) implies (2). There exists a partition of X $\{D_j\}_{j=1}^{k}$ such that $D_j = \bigcap_{j_l=1}^{t_j} \phi_{j_l}(B_{j_l})$ $(1 \leq \forall j \leq k, \exists t_j \geq 1)$, $\phi_l(B_l) = \bigcup_{l_i=1}^{h_l} D_{l_i}$ $(1 \leq \forall j \leq k)$ and $\mu(D_j) > 0$ $(1 \leq j \leq k)$, where μ is Legesgue measure.

Then for any $j \in \{1, 2, \cdots, k_0\}$ there exists $l \in \mathbf{N}$ such that $\mu(A^{-1}\phi_l^{-1} D_j) > 0$. Put $K_j = \{l \mid D_j^\circ \bigcap \phi_l(B_l) \neq \emptyset\}$ and k_j be the number of elements of K_j. Since $S^{-1}(D_j) = \bigcup_{l \in K_j} A^{-1}\phi_l^{-1}(D_j)$,

$$\begin{aligned} \mu(S^{-1}(D_j)) &= k_j \mu(A^{-1}\phi_l^{-1} D_j) \\ &= k_j |detA|^{-1} \mu(D_j). \end{aligned}$$

We have $k_j = |detA|$ for $1 \le \forall j \le k$ by $\mu(S^{-1}(D_j)) = \mu(D_j)$. Since

$$\begin{aligned}\sum_{j=1}^{k} |detA|\ \mu(D_j) &= \sum_{l=1}^{M} \mu(\phi_l^{-1}(B_l)) \\ &= \mu(A(X)) = |detA|,\end{aligned}$$

we have $\sum_{j=1}^{k} \mu(D_j) = 1$.

We show (2) implies (1). Let $G \in \mathcal{A}$. There exist $k_0 \in \mathbf{N}$, $\{G_i\}_{i=1}^{k_0}$ and $\{j_i\}_{i=1}^{k_0}$ ($j_i \in \{1, 2, \cdots, k\}$) such that $G \cap D_{j_i} = G_i$ and $G = \bigcup_{i=1}^{k_0} G_i$ ($G_i^\circ \cap G_j^\circ = \emptyset \ \ i \neq j$). There exist $\{i_m\}_{m=1}^{n}$ such that $G_i \subset \phi_{i_m}(B_{i_m})$. We have

$$\begin{aligned}\mu(S^{-1}(G)) &= \mu(S^{-1}(\cup_{i=1}^{k_0} G_i)) = \sum_{m=1}^{n}\sum_{i=1}^{k_0} \mu(A^{-1}\phi_{i_m}^{-1}(G_i)) \\ &= n\sum_{i=1}^{k_0} \mu(A^{-1}\phi_{i_1}^{-1}(G_i)) = \sum_{i=1}^{k_0} \mu(G_i) = \mu(G).\end{aligned}$$

We show (3) implies (2). Put $B'_l = B_l$ mod 1. Since there exist $(t_i, s_i) \in \mathbf{Z}^2$ $(i = 1, 2)$ such that the line $\{(x - t_i, y - s_i) | (x, y) \in A(X)\} \bigcap A(X)$ is parallel to either $y = \frac{c}{a}x$ or $y = \frac{d}{b}x$, there exists $l' \in \{1, \cdots, M\}$ for any $l \in \{1, \cdots, M\}$ such that the line $B'_l \bigcap B'_{l'}$ is parallel to either $y = \frac{c}{a}x$ or $y = \frac{d}{b}x$. Then there exists a partition $\{D_j\}$ which satisfies the condition (2).

We show (2) implies (3). Consider the case of $a, b, c, d > 0$, $detA > 0$ and $d > c$. There exists $j \in \{1, \cdots, M\}$ such that $(0, 0) \in B_j$. Then there exist l_1, l_2 and $l_3 \in \{1, \cdots, k\}$ such that $(0, 0) \in D_{l_1} \bigcap D_{l_2} \bigcap D_{l_3}$, $D_{l_1}^\circ \bigcap D_{l_2}^\circ \bigcap D_{l_3}^\circ = \emptyset$, the line $D_{l_1} \bigcap D_{l_2}$ is parallel to $y = \frac{c}{a}x$ and the line $D_{l_1} \bigcap D_{l_3}$ is parallel to $y = \frac{d}{b}x$. The following statements hold:

(I) There exists $j \in \{1, \cdots, M\}$ and $(m_1, n_1) \in \mathbf{Z}^2$ such that $(m_1, n_1) \in \phi_j(D_{l_2})$;

(II) There exists $i \in \{1, \cdots, M\}$ and $(m_2, n_2) \in \mathbf{Z}^2$ such that $(m_2, n_2) \in \phi_i(D_{l_3})$.

Suppose $(m_1, n_1) \neq (b, d)$ and $(m_2, n_2) \neq (a, c)$. There exists $l_4 \in \{1, \cdots, k\}$ such that $(1, 1) \in D_{l_4}$, $\partial D_{l_4} \bigcap X^\circ$ is parallel to $y = \frac{c}{a}x$ and there is $\phi_{j_0}^{-1}(\partial D_{l_4} \bigcap X^\circ)$ on $y = \frac{c}{a}x$ for $\exists j_0 \in \{1, 2, \cdots, M\}$. There exists $(m_3, n_3) \in \mathbf{Z}^2$ such that $(m_3, n_3) \in \phi_{j_0}^{-1}(D_{l_4})$ and $(m_3, n_3) \neq (a, c)$. The parallelogram which has the vertices $(0, 0), (m_3, n_3), (b + m_3, d + n_3)$ and (b, d) satisfies the condition of (3). Put the parallelogram be B'. Suppose $B'' = \{(x - m_3, y - n_3) | (x, y) \in A(X) \setminus B'\}$. If we repeat a similar procedure for B'', there are no lattice point on the line $\overline{OP}$, which contradicts the assumption. Either $(a, c) \in \mathbf{Z}^2$ or $(b, d) \in \mathbf{Z}^2$ holds. So (3) follows from (I) and (II). In the other cases, we may prove in a similar way. □

By Lemma 1, we shall show the following theorem.

Theorem 4. *Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \to X$ is defined by*

$$S(x, y) = (ax + by, cx + dy) \qquad (\text{mod } 1)$$

and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *where* $a, b, c, d \in \mathbf{R}$. *The following* (1) *and* (2) *are equivalent:*

(1) S *is measure preserving;*

(2) $|detA| = n \in \mathbf{N}$ *and* $(a, c)|n \quad (a, c \in \mathbf{Z})$
or
$|detA| = n \in \mathbf{N}$ *and* $(b, d)|n \quad (b, d \in \mathbf{Z})$,
where (a, c) *indicates a greatest common divisor of* a *and* c.

Proof. We shall show that Lemma 1 (3) and Theorem 4 (2) are equivalent.
(Lemma 1 (3) $\Longrightarrow$ Theorem 4 (2))
Let $a, c \in \mathbf{Z}, ad - bc = n$ and let l be the line $\overline{RQ}$. Then $l : y = \frac{c}{a}(x - b) + d$. Let $(x_0, y_0) \in \mathbf{Z}^2$,

$$\begin{aligned} y_0 &= \tfrac{c}{a}(x_0 - b) + d \\ &= \tfrac{c}{a}x_0 + \tfrac{n}{a} \end{aligned}$$

Suppose $(a, c) = p$, and $(n, p) = m < p$. Put $p = mp', a = pa', c = pc', n = mn'$ then $(n', p') = 1$
So $y_0 = \frac{c'}{a'}x_0 + \frac{n'}{a'p'}$ and $a'y_0 - c'x_0 = \frac{n'}{p'}$ holds. $a'y_0 - c'x_0 \in \mathbf{Z}$ contradicts $\frac{n'}{p'} \notin \mathbf{Z}$. So $(n, p) = p$ holds.
(Theorem 4 (2) $\Longrightarrow$ Lemma 1 (3))
Let $|detA| = n, a, c \in \mathbf{Z}, l : y = \frac{c}{a}(x - b) + d = \frac{c}{a}x + \frac{n}{a}$. Put $(a, c) = p \in \mathbf{Z}$ then $a = pa', c = pc', (a', c') = 1$ (i.e.$\exists s, t \in \mathbf{Z}$ s.t. $a's + c't = 1$) holds. By $p|n$, put $n = pn'$ $(n' \in \mathbf{Z})$. $a'n's + c'n't = n'$ holds. If $x_1 = -n't \in \mathbf{Z}$ then

$$y_1 = \frac{c'}{a'}(-n't) + \frac{n'}{a'} = \frac{-c'tn' + n'}{a'} = \frac{a'n's}{a'} = n's \in \mathbf{Z}.$$

Hence $(x_1, y_1) \in \mathbf{Z}^2$. □

By the above theorem, we can consider the case of $detA \in \mathbf{Z}$ hereafter.

Lemma 2. *Let* $(X, \mathcal{A}, \mu)$ *be a normalized measure space. Suppose* $S : X \to X$ *is defined by*

$$S(x, y) = (ax + by, cx + dy) \pmod 1$$

and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *where* $a, b, c, d \in \mathbf{R}$. *Put* $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ *and* $detA = \pm m (m \geq 1)$. *Then*

$$\begin{cases} a_{n+1} &= az_n \mp mz_{n-1} \\ b_{n+1} &= bz_n \\ c_{n+1} &= cz_n \\ d_{n+1} &= dz_n \mp mz_{n-1}, \end{cases}$$

where

$$\begin{cases} z_{-1} = 0 \\ z_0 = 1 \\ z_{n+1} = (a + d)z_n \mp mz_{n-1}. \end{cases}$$

Put $D = \{f(x,y) = \exp[2\pi i(px+qy)] | p,q \in \mathbf{Z}\}$. Since the linear span of D is dense in $L^1(X)$, we have the following.

Theorem 5. *Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \to X$ is defined by*

$$S(x,y) = (ax+by, cx+dy) \qquad (\text{mod } 1),$$

where $a,b,c,d \in \mathbf{Z}$. Put $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Then the following statements are equivalent:

(1) *S is not mixing;*

(2) *There exist $\{n_j\}_{j=1}^{\infty}$ and $(p,q,k,l) \neq (0,0,0,0)$ $(p,q,k,l \in \mathbf{Z})$ such that*

$$ka_{n_j} + lc_{n_j} - p = kb_{n_j} + ld_{n_j} - q = 0;$$

(3) *There exists $\{z_{n_j}\}_{j=1}^{\infty}$ which satisfies either* (i) *or* (ii).

(i) *$z_{n_j} = z_{n_1}$ and $z_{n_j - 1} = z_{n_1 - 1}$ for any j.*

(ii) *There exists an eigenvalue λ of matrix A such that*

$$\lambda \in \mathbf{Q} \quad \text{and} \quad \frac{z_{n_j} - z_{n_l}}{z_{n_j-1} - z_{n_l-1}} = \frac{\det A}{\lambda} \qquad \text{for any } j, l (j \neq l).$$

Proof. ((1) $\Longrightarrow$ (2))
If S is not mixing, then $\lim_{n\to\infty}\langle f, U^n g\rangle \neq \langle f,1\rangle\langle 1,g\rangle = \begin{cases} 1 & k=l=p=q=0 \\ 0 & \text{otherwise} \end{cases}$, i.e. for any n_0, there exists $n_1 \geq n_0$ such that $\langle f, U^{n_1} g\rangle = 1$ with $(k,l,p,q) \neq (0,\cdots,0)$. Repeating the relation, we can show that there exists $n_2 \geq n_1$ such that $\langle f, U^{n_2} g\rangle = 1$ with $(k,l,p,q) \neq (0,\cdots,0)$. Taking this sequence $\{n_j\}_{j=1}^{\infty}$, the next holds : $\langle f, U^{n_j} g\rangle = 1$ with $(k,l,p,q) \neq (0,\cdots,0)$, i.e. $ka_{n_j} + lc_{n_j} - p = 0$ and $kb_{n_j} + ld_{n_j} - q = 0$. This means (2) holds.
((2) $\Longrightarrow$ (1))
We shall show that S is not mixing by Proposition 2(b). (S is mixing $\Leftrightarrow$ $\lim\langle f, U^n g\rangle = \langle f,1\rangle\langle 1,g\rangle$ with g in a linearly dense set in $L^\infty(X)$. We define the Koopman operator as $U^n g(x,y) = g(S^n(x,y))$. If we take $g(x,y) = \exp[2\pi i(kx+ly)]$ and $f(x,y) = \exp[-2\pi i(px+qy)]$ with $k,l,p,q \in Z$ then we have $U^n g(x,y) = g(a_n x + b_n y, c_n x + d_n y)$ and

$$\langle f, U^n g\rangle = \int_0^1 \int_0^1 \exp[2\pi i \{(ka_n + lc_n - p)x + (kb_n + ld_n - q)y\}] dx dy$$

$$= \begin{cases} 1 & \text{if } ka_n + lc_n - p = kb_n + ld_n - q = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \cdots (A)$$

On the other hand,

$$\langle f,1\rangle\langle 1,g\rangle = \begin{cases} 1 & k=l=p=q=0 \\ 0 & \text{otherwise} \end{cases}$$

By (2), for any $n_0 \in N$, there exists $t \geq n_0$ $(t \in \{n_j\})$ and $p,q,k,l \in Z$ such that $(p,q,k,l) \neq (0,0,0,0)$ and $ka_t + lc_t - p = kb_t + ld_t - q = 0 \cdots (B)$. By (A) and (B), $\langle f, U^n g\rangle$ does not converge to $\langle f,1\rangle\langle 1,g\rangle$. So S is not mixing.

(2)⇔(3)

Put $\mid \det A \mid = N$.

(2) $\Leftrightarrow \exists \{n_j\}$ and $\exists (p,q,k,l) \neq (0,0,0,0)$ s.t. $ka_{n_j} + lc_{n_j} - p = kc_{n_j} + ld_{n_j} - q = 0$

$$\Leftrightarrow \begin{cases} ka_{n_j} + lc_{n_j} - p = (ka+lc)zn_j - 1 \mp kNz_{n_j-2} - p = 0 \\ ka_{n_i} + lc_{n_i} - p = (ka+lc)zn_i - 1 \mp kNz_{n_i-2} - p = 0 \end{cases}$$

$$\Leftrightarrow k(a_{n_j} - a_{n_i}) + l(c_{n_j} - c_{n_i}) = (ka+lc)(z_{n_j-1} - z_{n_i-1}) \mp kN(z_{n_j-2} - z_{n_i-2}) = 0$$

$$\Leftrightarrow \begin{cases} k = c = p = 0, l \neq 0, d \neq 0, \frac{z_{n_j-1} - z_{n_i-1}}{z_{n_j-2} - z_{n_i-2}} = \pm\frac{N}{d} \\ \text{or} \\ k = d = 0, l \neq 0, z_{n_j-1} = z_{n_1-1}, z_{n_j-2} = z_{n_1-2} \\ \text{or} \\ l = b = q = 0, k \neq 0, a \neq 0, \frac{z_{n_j-1} - z_{n_i-1}}{z_{n_j-2} - z_{n_i-2}} = \pm\frac{N}{a} \\ \text{or} \\ l = a = 0, k \neq 0, z_{n_j-1} = z_{n_1-1}, z_{n_j-2} = z_{n_j-2} \\ \text{or} \\ ka + lc \neq 0, kb + ld \neq 0, l \neq 0, \frac{z_{n_j-1} - z_{n_i-1}}{z_{n_j-2} - z_{n_i-2}} = \pm\frac{kN}{ka+lc} = \pm\frac{lN}{kb+ld} \end{cases}$$

$\Leftrightarrow$ (3). □

Theorem 6. *Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \to X$ is defined by*

$$S(x,y) = (ax + by, cx + dy) \qquad (\text{mod } 1),$$

where $a,b,c,d \in \mathbf{Z}$. Put $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. If there exist $\{n_j\}_{j=1}^{\infty}$ and $(p,q,k,l) \neq (0,0,0,0)$ $(p,q,k,l \in \mathbf{Z})$ such that $n_{j+1} - n_j = n_2 - n_1$ for any j and

$$ka_{n_j} + lc_{n_j} - p = kb_{n_j} + ld_{n_j} - q = 0,$$

then S is not ergodic.

In order to obtain a criterion for demonstrating either mixing, exactness or ergodicity, we first show the following propositions using Theorem 5.

Proposition 7. *Suppose $\det A > 0$. Then the following statements holds:*

(1) *If $a + d = \det A + 1$, then $\{z_n\}$ satisfies the condition of Theorem 5(3)(ii);*

(2) *If $a + d = -(\det A + 1)$, then $\{z_{2n}\}$ satisfies the condition of Theorem 5(3)(ii);*

(3) *If $|a + d| \neq \det A + 1$ $(\det A \neq 1)$, then there doesn't exist $\{z_{n_j}\}$ which satisfies the condition of Theorem 5(3);*

(4) *Let $\det A = 1$.*

(i) *If* $|a+d| = 0$, *then* $\{z_{4n}\}$ *satisfies the condition of Theorem 5*(3)(i).

(ii) *If* $|a+d| = 1$, *then* $\{z_{6n}\}$ *satisfies the condition of Theorem 5*(3)(i).

(iii) *If* $|a+d| = -1$, *then* $\{z_{3n}\}$ *satisfies the condition of Theorem 5*(3)(i).

Proposition 8. *Suppose* $detA < 0$. *Then the following statements holds:*

(1) *If* $a+d = detA + 1 \neq 0$, *then* $\{z_n\}$ *satisfies the condition of Theorem 5*(3)(ii);

(2) *If* $a+d = -(detA+1) \neq 0$, *then* $\{z_{2n}\}$ *satisfies the condition of Theorem 5*(3)(ii);

(3) *If* $a+d = detA + 1 = 0$, $\{z_{2n}\}$ *satisfies the condition of Theorem 5*(3)(i);

(4) *If* $|a+d| \neq |detA| - 1$, *there doesn't exist* $\{z_{n_j}\}$ *which satisfies the condition of Theorem 5*(3).

Using the next theorem, we can know the behavior of S calculating $detA$ and $|a+d|$.

Theorem 9. *Let* $(X, \mathcal{A}, \mu)$ *be a normalized measure space. Suppose* $S : X \to X$ *is defined by*

$$S(x,y) = (ax+by, cx+dy) \pmod 1,$$

where $a, b, c, d \in \mathbf{Z}$. *The following statements are equivalent:*

(i) S *is mixing;*

(ii) S *is ergodic;*

(iii) *Either* (a), (b) *or* (c) *holds:*

(a) $detA \geq 2$ *and* $|a+d| \neq detA + 1$;

(b) $detA = 1$ *and* $|a+d| \geq 3$;

(c) $detA < 0$ *and* $|a+d| \neq |detA| - 1$.

We consider the following transformation:

$$S(x,y) = (ax+by+\alpha, cx+dy+\beta) \pmod 1,$$

where $a, b, c, d \in \mathbf{Z}$ and $0 \leq \alpha, \beta < 1$.

Theorem 10. *Let* $(X, \mathcal{A}, \mu)$ *be a normalized measure space. Suppose* $S : X \to X$ *is defined by*

$$S(x,y) = (ax+by+\alpha, cx+dy+\beta) \pmod 1,$$

where $a, b, c, d \in \mathbf{Z}$ *and* $0 \leq \alpha, \beta < 1$. *Let* $S_0(x,y) = (ax+by, cx+dy) \pmod 1$. *The following statements hold:*

(1) *If either* $detA = 1$ *and* $|a+d| \geq 3$ *or* $|a+d| \neq \mathrm{sgn}(detA)(detA+1)$, *then* S *is mixing, where* $\mathrm{sgn}(detA)$ *indicates the sign of* $detA$;

(2) *If either* (i) *or* (ii) *holds, then* S *is ergodic, but not mixing;*

(i) $|a+d|$ =sgn$(detA)(detA+1)$, $A=\pm I$ (*I is an 2×2 identity matrix) and* $\alpha,\beta\notin\mathbf{Q}$.

(ii) $a+d=detA+1, A\neq I$ *and either* $\alpha c-(a-1)\beta\notin\mathbf{Q}$ *or* $\alpha(d-1)-\beta b\notin\mathbf{Q}$.

(3) *If either* (i),(ii),(iii) *or* (iv) *holds, S is not ergodic.*

(i) $detA=1$ *and* $|a+d|\leq 1$.

(ii) $|a+d|=\mathrm{sgn}(detA)(detA+1)$, $A\neq\pm I$ *and either* $\alpha\in\mathbf{Q}$ *or* $\beta\in\mathbf{Q}$.

(iii) $|a+d|=detA+1$, $A\neq I$ *and either* $\alpha c-(a-1)\beta\in\mathbf{Q}$ *or* $\alpha(d-1)-\beta b\in\mathbf{Q}$.

(iv) $|a+d|=-detA-1$ *and* $A\neq -I$.

References

[1] A. Lasota and M. C. Mackey, Chaos, Fractals, and Noise, Springer Verlag (1995)