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On the Locus of Crossed Renormalization

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Contents

Abstract

We describe the subsets of the Mandelbrot set for which the dynamics is crossed renormalizable. There are countably many connected components of the locus of crossed renormalization of any given period, and each of them is canonically homeomorphic to a limb of the Mandelbrot set. We discuss similarities and differences to the well-known theory of simple renormalization.

1 Introduction

Renormalization has played an important role in various branches of science including dynamics and physics. Renormalization ideas help to understand universality of certain dynamical features which are observed in a host of different contexts, and why certain dynamical properties can be observed at many different scales.

Holomorphic dynamics offers a substantial collection of powerful tools and is thus a context in which renormalization is understood particularly well. A high iterate of a holomorphic map, investigated at a small scale, can display very similar dynamical properties as a polynomial on a global scale. When a parameter of such a dynamical system is varied, there is a tendency for Mandelbrot sets to appear in parameter space: the Mandelbrot set is a "universal object" in parameter spaces of iterated holomorphic maps. This phenomenon manifests itself in the well-known appearance of countably many little embedded Mandelbrot sets within the entire Mandelbrot set. Every little Mandelbrot set is the locus of parameters for which the dynamics is renormalizable of a certain kind.

The renormalization theory for holomorphic maps was pioneered by Douady and Hubbard [DH]. Since the mid-1980s, two kinds of renormalization have been known (now called "disjoint" and "β-type" renormalization; together, they are known as "simple renormalizations") which correspond to slightly different kinds of embedded Mandelbrot sets. When describing renormalization systematically, McMullen [McM] discovered that there was a third kind of renormalization (now called "α-type" or "crossed renormalization"). From the work of Douady and Hubbard [DH], it is well known that simple n-renormalization is organized in the form of finitely many little embedded Mandelbrot sets in an interesting combinatorial way. The attempt to describe crossed renormalization in a similar way has motivated the present paper.

Our main result is the following. Terminology and background will be explained in Section 2.

Theorem 1 (The Locus of Crossed Renormalizations)
For any period n, the locus of crossed n-renormalization within the Mandelbrot set consists of countably infinitely many connected subsets of $\mathbf{M}$*: there is a finite number of connected sets corresponding to every period of the crossing point of the little Julia sets. Every connected component is canonically homeomorphic to a limb of the Mandelbrot set.*

We will make more precise statements in Sections 3 and 4 where we describe how to find these embedded limbs: we will describe a special case in detail in Section 3, and we will show in Section 4 that the discussion can easily be extended to the general case. A first description of the renormalization locus will be given along with the construction in Section 3.1; in Section 3.2, we will give a second description by chopping off subsets of the Mandelbrot set until only a component of the renormalization locus remains, and a third, combinatorial, description in terms of internal addresses will be presented in Section 3.5.

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2 Background on Quadratic Polynomials

2.1 The Mandelbrot Set and Julia Sets

In this paper, we will discuss exclusively quadratic polynomials; they can all be conjugated to the form $P_c\colon z \mapsto z^2+c$ for a unique complex parameter c. The *filled-in Julia set* K_c for the polynomial P_c is the set of points z in the dynamic plane such that the iterates of z do not escape to infinity, i.e. the sequence $z, P_c(z), P_c^2(z), \ldots$ is bounded. The *Julia set* J_c is the boundary of the filled-in Julia set. Both sets are compact and K_c is full (which means that the complement is connected). The dynamics of a rational map is determined to a large extent by the critical points of the map (those points where the derivative vanishes) and their forward orbits. One example of this observation is that the filled-in Julia set of a polynomial is connected if and only if it contains all the critical points in $\mathbb{C}$ of the polynomial, i.e. if all the critical orbits in $\mathbb{C}$ are bounded.

For our polynomials $P_c(z) = z^2 + c$, the only critical point in $\mathbb{C}$ is 0. Therefore, a filled-in Julia set K_c is connected iff 0 does not escape to ∞ under iteration.

The *Mandelbrot set* **M** is the set of all parameters c such that the filled-in Julia set K_c (or equivalently the Julia set J_c) is connected, so it is often referred to as the *locus of connected quadratic Julia sets* or simply as the *quadratic connectedness locus.* By fundamental work of Douady and Hubbard, it is known to be compact, connected and full.

For every period $n \geq 1$, there are open sets in parameter space for which the polynomial P_c has an attracting periodic orbit of period n. These sets are contained in the Mandelbrot set, and their connected components are called *hyperbolic components* of the Mandelbrot set. For every fixed period, their number is finite. Every hyperbolic component is conformally parametrized by the multiplier of the attracting orbit: the multiplier map supplies a biholomorphic map between the component and the open unit disk, and it extends as a homeomorphism to the closures. Every hyperbolic component has a unique center and a unique root: these are the points where the multiplier map takes values 0 and +1, respectively.

2.2 Dynamic Rays, Parameter Rays and Equipotentials

Let us consider some parameter $c \in \mathbf{M}$. The dynamics outside the filled-in Julia set can conveniently be described by *dynamic rays* (also known as *external rays*) and *equipotentials*, which are a dynamic variant of polar coordinates. Since K_c is full, there is a conformal isomorphism $\varphi_c\colon \overline{\mathbb{C}} - K_c \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$ fixing ∞; it is unique up to rotation and can be fixed so that $\lim_{z\to\infty} \varphi_c(z)/z$ is real positive; in fact, since our polynomials are normalized so that their leading coefficient is 1, the limit will be equal to 1. The map φ_c conjugates the dynamics in $\overline{\mathbb{C}} - K_c$ to the dynamics of $z \mapsto z^2$ in $\overline{\mathbb{C}} - \overline{\mathbb{D}}$: we have $(\varphi_c(z))^2 = \varphi(z^2 + c)$.

For every $\vartheta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, the set $R_\vartheta := \varphi_c^{-1}(\{e^r \cdot e^{2\pi i\vartheta} : 0 < r < \infty\})$ is called the *dynamic ray of P_c at angle ϑ*. External angles are counted in full turns. By the conjugation property we have

$$P_c(R_\vartheta) = R_{2\vartheta} .$$

In our normalization, a dynamic ray is periodic if its angle is rational with odd denominator when written in lowest terms; it is strictly preperiodic if the angle is rational with even denominator; and it has an infinite forward orbit if the angle is irrational.

For any $r \in (0, \infty)$, the set $E_r := \varphi_c^{-1}(\{e^r \cdot e^{2\pi i\vartheta} : \vartheta \in \mathbb{S}^1\})$ is called the *equipotential of P_c at potential r*. The conjugation property yields

$$P_c(E_r) = E_{2r} .$$

The dynamic rays together with the equipotentials form a coordinate system in $\mathbb{C} - K_c$ in which the dynamics is simply doubling of external angles and potentials. It is often useful, but not always possible, to extend this coordinate system to the Julia set. A dynamic ray at angle ϑ is said to *land* at a point z of the Julia set if

$$\lim_{r\to 0} \varphi_c^{-1}(e^r \cdot e^{2\pi i\vartheta}) = z.$$

In general, not every dynamic ray needs to land; its limit set will be a connected subset of the Julia set. It is well known that every dynamic ray at a rational angle lands at a periodic or preperiodic point of the Julia set; conversely, every repelling periodic or preperiodic point is the landing point of some dynamic rays with rational angles, and all the rays landing at the same point have the same periods and preperiods.

Since the Mandelbrot set is compact, connected and full, we can define external rays and equipotentials of the Mandelbrot set as well. In order to distinguish these rays from the rays in dynamical planes, we call them *parameter rays.* Again, it is known that all parameter rays at rational angles land. Parameter rays at periodic angles land in pairs at roots of hyperbolic components such that the periods of both angles and of the component are equal, and every root is the landing point of exactly two such parameter rays. Parameter rays at preperiodic angles land at parameters for which the critical orbit is strictly preperiodic; such parameters are known as Misiurewicz points. The number of parameter rays landing at any given Misiurewicz point is positive and finite.

We have seen above that the boundary of any hyperbolic component W of period n can canonically be parametrized by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Any boundary point at a rational "internal angle" $p/q \neq 0/1$ is a bifurcation point: at this point, a hyperbolic component of period qn is attached, together with an interesting structure of "decorations". The bifurcation point is the root of the qn-periodic component, and the pair of periodic parameter rays landing at this root separates the component of period qn from the component of period n and the origin. The open region which is separated from the origin by this parameter ray pair is called the *p/q-subwake of W*. The intersection of this wake with $\mathbf{M}$ is the *p/q-sublimb of* $\mathbf{M}$. The landing point of the two bounding parameter rays is the *root* of limb and wake, but we will not consider it part of limb or wake (this convention differs between various authors). Subwake and sublimb at internal angle p/q of the unique hyperbolic component of period 1 are called the *p/q-wake* and *p/q-limb* of the Mandelbrot set; the p/q-limb will be called $\mathbf{M}_{p/q}$.

Now let $c \in \mathbb{C}$ be a parameter (not necessarily in $\mathbf{M}$) which is not contained in the real interval $[1/4, \infty)$. Then P_c has exactly two fixed points, one of which is the landing point of the dynamic ray at angle $\vartheta = 0$. This fixed point is called the β-fixed point of P_c. The other fixed point is the α-fixed point; it may be attracting, indifferent, or repelling, and it may or may not be the landing point of rational dynamic rays. If it is, the period of these dynamic rays must be some finite number $q \geq 2$ (because the only ray of period 1 is already taken). The combinatorial rotation number of the dynamics of these q rays is then p/q for some integer p coprime to q. It turns out that the subset of $\mathbb{C}$ for which the α-fixed point is repelling and the landing point of q rays with combinatorial rotation number p/q is exactly the p/q-wake of $\mathbf{M}$ as defined above. Between these wakes, there is still a Cantor set of external angles for which the number of dynamic rays landing at α is infinite.

For any $c \in \mathbf{M}$, the α-fixed point is either attracting or indifferent (which happens in the interior respectively on the boundary of the hyperbolic component of period 1), or it is repelling and c is in some p/q-limb of $\mathbf{M}$; in the latter case, the α-fixed point must disconnect the filled-in Julia set. The β-fixed point never disconnects K_c.

2.3 Polynomial-like Maps

A *polynomial-like map* $f: U \to V$ is a proper holomorphic map f between two bounded, open, connected and simply connected domains $U, V \subset \mathbb{C}$ such that $\overline{U} \subset V$. Such a map has a mapping degree $d \geq 1$. If this degree is 2, we call it a *quadratic-like map.* Every polynomial p becomes a polynomial-like map when V is a sufficiently large disk and $U = p^{-1}(V)$. But often the dynamics of a high iterate of a polynomial, which itself is a polynomial of large degree, can be understood better by restricting it to an appropriate subset on which the dynamics is polynomial-like of much smaller degree.

The filled-in Julia set $K(f)$ of a polynomial-like mapping $f : U \to V$ is the set of all points $z \in U$ which never leave U under iteration of f. The Julia set $J(f)$ of f is the boundary of $K(f)$. As for actual polynomials, these sets are connected iff all the critical points of f have bounded orbits so they are contained in $K(f)$.

An important statement is the *Straightening Theorem* of A. Douady and J. Hubbard [DH, Theorem 1]. For details, as well as for the definition of quasiconformal maps, we refer to this paper and to the references given there.

Theorem 2 (The Straightening Theorem)
Let $f : U \to V$ be a polynomial-like map of degree $d \geq 2$. Then there exists a polynomial P of degree d such that f and P are quasiconformally conjugate in a neighborhood of the filled-in Julia sets $K(f)$ and $K(P)$, i.e. there exists a quasiconformal map φ which maps a neighborhood of $K(f)$ to a neighborhood of $K(P)$ such that

$$\varphi \circ f \circ \varphi^{-1} = P \, .$$

The conjugation φ can be chosen so that its complex dilatation vanishes on $K(f)$. For such a φ, the polynomial P is unique up to affine conjugation if $K(f)$ is connected. □

Douady and Hubbard call a quasiconformal conjugation with vanishing complex dilatation on the filled-in Julia set a *hybrid equivalence.* For quadratic polynomials in the normalization $z^2 + c$, every parameter c represents its own affine conjugation class, so the straightening map for quadratic-like maps with connected Julia sets takes well-defined images in the Mandelbrot set.

2.4 Renormalization

A quadratic polynomial P_c is called *n-renormalizable* if there are neighborhoods U, V of the critical point such that the restriction $P_c^n: U \to V$ is a quadratic-like map with connected filled-in Julia set K. This set K is often referred to as the "little filled-in Julia set" of the renormalization. Obviously, $P_c^n(K) = K$; for $j = 1, 2, \ldots, n-1$, $P_c^j(K)$ is different from K: if $P_c^j(K) = K$ for some $j < n$, then the critical orbit would visit U at least twice during the first n iterations of P_c, and P_c^n restricted to U would not be quadratic-like.

It is known [McM, Theorem 7.3] that K can meet any $P_c^j(K)$ (for $1 \le j \le n-1$) at most at a single point, which is necessarily periodic and repelling, and the period strictly divides n. Such a point will be a fixed point of P_c^n within the little Julia set. Since the little Julia set is connected, the straightening theorem turns it into the Julia set of a polynomial $P' \in \mathbf{M}$ and it makes sense to ask whether the corresponding fixed point of P' is the α or β fixed point. The set K can never meet its forward images both at α and at β. Accordingly, we have the following distinction [McM].

Definition 3 (Types of Renormalization)
An n-renormalization of a quadratic polynomial is said to have

disjoint type *if the little Julia set is disjoint from all its images under at most $n-1$ iterations;*

β-type *if the little Julia set meets some of its first $n-1$ forward images only at its β-fixed point;*

α-type *if the little Julia set meets some of its first $n-1$ forward images only at its α-fixed point.*

The first two types are also known as simple renormalizations*, while the last type is known as* crossed renormalization. *If the little Julia set does meet some of its forward images, then the renormalization is called* immediate *if the intersection point is a fixed point.*

Examples of simple renormalizations are shown in Figure 4 (disjoint case) and in Figure 1 (β-case). An example of a crossed renormalization is given in Figure 2. Further examples are shown and discussed in [McM, Chapter 7].

It is known from the work of Douady and Hubbard [DH] that the locus of simple n-renormalization consists of finitely many connected components, each of which is homeomorphic to the entire Mandelbrot set (to be precise, the renormalization locus is homeomorphic to the entire Mandelbrot set only for disjoint type; for β-type, the renormalization locus is homeomorphic to the Mandelbrot set without its root $c = 1/4$, but the homeomorphism can still be extended to the entire Mandelbrot set). A homeomorphism from such a connected component to $\mathbf{M}$ is given by the straightening maps for the little Julia sets arising in the renormalization process. The details of this construction have never been published, but there is a manuscript of P. Haïssinski [Ha]. Every connected component of the simple n-renormalization locus is based at a hyperbolic component of $\mathbf{M}$ of period n which is the image of the period 1 component, and every hyperbolic component of period n is contained in a connected component of the n-renormalization locus.

Any such homeomorphism from $\mathbf{M}$ to a connected component of the simple n-renormalization locus is called a *tuning map* of period n; a tuning map is a branch of the inverse of the straightening map. A connected component of the simple n-renormalization locus corresponding to disjoint type is based at a primitive hyperbolic component, and by recent work of Lyubich, the tuning map is known to be quasiconformal. In the β-case, a connected component of the simple n-renormalization locus is based at a hyperbolic component which bifurcates from a hyperbolic component of some period strictly dividing n. Such hyperbolic components have smooth boundaries and are lacking their characteristic cusps. In the immediate case, these hyperbolic components are immediate bifurcations from the main cardioid (the period 1 hyperbolic component) of $\mathbf{M}$.

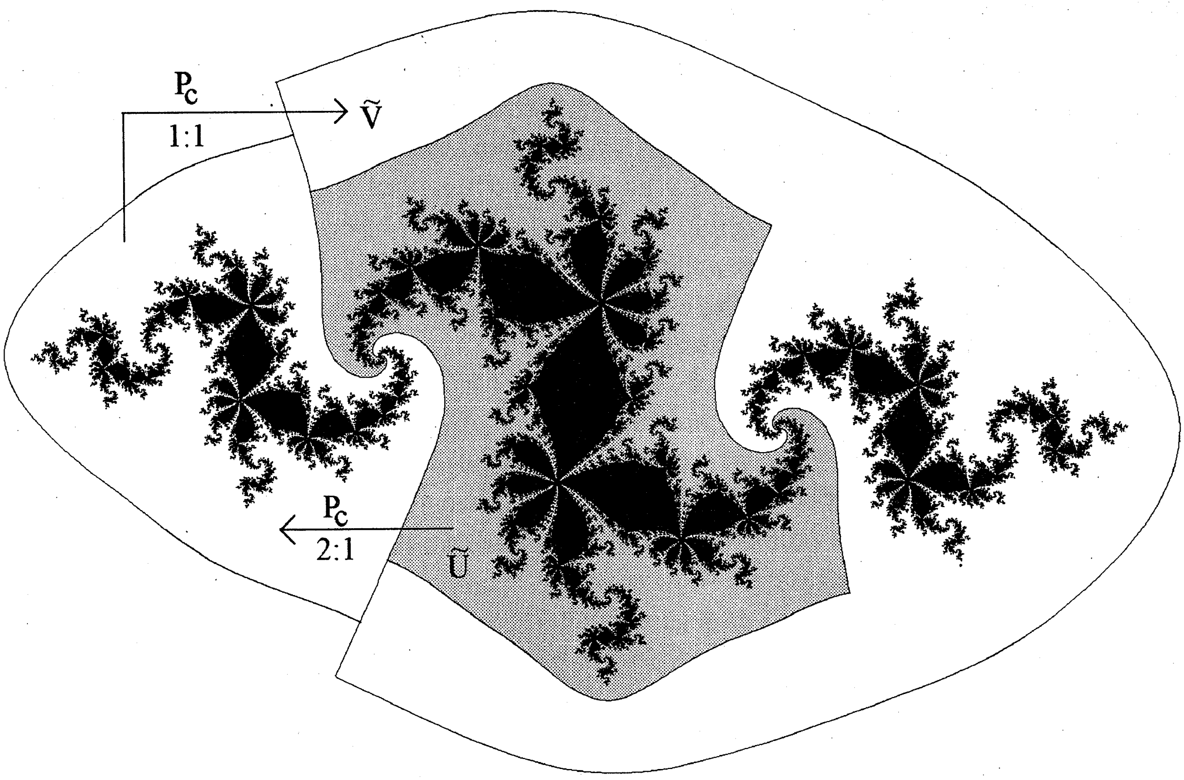

Figure 1: An example of a simple renormalization of β-type.

Further properties of the simple renormalization loci will be mentioned throughout the paper in comparison with the crossed renormalization loci.

3 Crossed Renormalization: The Immediate Case

If a quadratic polynomial is crossed n-renormalizable, then the little Julia set meets some of its images at periodic points which separate the little Julia sets. We call this renormalization *immediate* if this periodic point has period 1, i.e. it is a fixed point.

Fix an integer $n \geq 2$ and consider two positive integers p, q which are relatively prime with $0 < p < q$. We will be interested in the subset of the p/nq-limb of $\mathbf{M}$ consisting of parameters which are immediately n-renormalizable of crossed type.

To this end, we will first construct a quadratic-like map for every parameter in the p/nq-wake of $\mathbf{M}$ (Section 3.1). Such a map is renormalizable if the Julia set of this polynomial-like map is connected. We will denote the locus of such polynomials by $C^n_{p,q}$. The straightening map defines a map from $C^n_{p,q}$ to the p/q-limb which turns out to be a homeomorphism (Section 3.3). We will then show that $C^n_{p,q}$ contains every immediately n-renormalizable parameter of crossed type in the p/nq-limb (Section 3.4) and that $C^n_{p,q}$ can be obtained from the entire p/nq-limb by cutting off subsets bounded by certain pairs of preperiodic parameter rays landing at Misiurewicz points (Section 3.2). Finally, we will show how to tell whether a parameter in $C^n_{p,q}$ is immediately n-renormalizable of crossed type in terms of internal addresses (Section 3.5). The general (non-immediate) case of crossed renormalization will be dealt with in Section 4.

3.1 The Principal Construction

For every parameter c in the p/nq-wake of the Mandelbrot set, we will now construct a quadratic-like map from a restriction of the n-th iterate of the original polynomial to an appropriate dynamically defined subset. We do not require $c \in \mathbf{M}$.

Let $s_0 \geq 0$ be the potential of the critical value (with $s_0 = 0$ iff $c \in \mathbf{M}$). Any equipotential $s > s_0$ bounds a simply connected region which we will call $R(s)$. It contains the filled-in Julia set. From now on, fix a potential $s > 2^n s_0$.

For all parameters c in the p/nq-wake, the α-fixed point is the landing point of exactly nq dynamic rays which are permuted transitively by the dynamics of the polynomial. This permutation has combinatorial rotation number p/nq, i.e. every ray jumps over $p-1$ rays onto its image ray, counting counterclockwise. Similarly, the point $-\alpha$ is the landing point of equally many rays in a symmetric configuration to the rays landing at α, and the polynomial maps $-\alpha$ and its rays onto α with its rays. None of these rays contains the critical point (or the rays could not land), and obviously α or $-\alpha$ cannot be the critical point, either. In particular, $\alpha \neq -\alpha$.

The nq rays landing at α, together with the nq rays landing at $-\alpha$, cut $R(s)$ into $2nq-1$ sectors in a symmetric way (see Figure 2). The sector containing the critical point is itself symmetric and meets both α and $-\alpha$ on its boundary; call its closure Y_0. The closures of the remaining $nq-1$ sectors at α will be called $Y_1, \ldots, Y_{nq-1}$ in the same cyclic order as the dynamics on the bounding rays, such that Y_1 contains the critical value. This way, the restriction of Y_j to $R(s/2)$ will be mapped onto Y_{j+1} for $j = 0, 1, \ldots, nq-2$. Finally, we label the closures of the remaining $nq-1$ sectors at $-\alpha$ by $Z_1, \ldots, Z_{nq-1}$ such that $Z_j = -Y_j$. Then the restriction of Z_j to $R(s/2)$ maps to Y_{j+1} for $j \leq nq-2$. Since they have the same image as the Y_j and the global degree of P_c is two, the restriction of P_c onto any Y_j or Z_j is injective for $j \neq 0$. Let $Z := Z_1 \cup Z_2 \cup \ldots \cup Z_{nq-1}$. Then the common image of Y_{nq-1} and Z_{nq-1}, restricted to $R(s/2)$, is $Y_0 \cup Z$; again, the map is injective on both sectors. Finally, Y_0 contains the critical point and maps onto Y_1 in a two-to-one fashion.

All the Y_j and Z_j together cover $R(s)$, with overlaps only at their boundaries. The restriction of P_c to $R(s/2)$ is a quadratic-like map with range $R(s)$. We will now identify a smaller subset of $R(s)$ such that the n-th iterate of P_c will be a quadratic-like map with range contained in $R(s)$.

As a first step for our quadratic-like maps, define

$$\tilde{U} := \left(Y_0 \cup \bigcup_{j=1}^{q-1} (Y_{jn} \cup Z_{jn}) \right) \cap R(s/2^n) \quad \text{and} \quad \tilde{V} := Y_0 \cup \bigcup_{j=1}^{q-1} Y_{jn} \cup \bigcup_{j=1}^{nq-1} Z_j$$

(see Figure 2). Then $P_c^n \colon \mathrm{int}(\tilde{U}) \to \mathrm{int}(\tilde{V})$ is a proper map between the interiors and has degree 2. However, there are two problems: $\mathrm{int}(\tilde{U})$ and $\mathrm{int}(\tilde{V})$ are disconnected; and $\tilde{U}$ and $\tilde{V}$ have common boundary points along entire ray segments. The first problem can be cured by adding a small disk around α to $\tilde{V}$ (for example a round disk with respect to linearizing coordinates of α), and adding to $\tilde{U}$ two disks around α and $-\alpha$ which are appropriately smaller so that they map to the added disk in $\tilde{V}$ under P_c^n. The second problem can be cured by thickening the boundaries slightly along the bounding rays, for example along dynamic rays at slightly different angles. These two problems and their solutions are quite standard; for details, see for example Milnor [M2].

We will work with regions U and V which are the interiors of the enlarged and thickened regions $\tilde{U}$ and $\tilde{V}$, respectively. Then $P_c^n: U \to V$ is indeed a quadratic-like map in the sense of Douady and Hubbard. The polynomial P_c becomes n-renormalizable whenever the filled-in Julia set of this quadratic-like map is connected. We will show in the next section that this renormalization is indeed immediate and of crossed type. It may not be clear at this point that for a parameter to be n-renormalizable it is necessary that our particular construction yields a connected Julia set. We will argue in Section 3.4 that this is indeed the case.

Of course, there is a considerable amount of freedom in the choice of s and in the two thickening steps. However, all choices will yield hybrid equivalent quadratic-like maps. In particular, the (filled-in) Julia set is independent of all the choices. This implies that this Julia set is completely contained in $\tilde{U}$. We will call the (filled-in) Julia set of the quadratic-like map just constructed the "little Julia set". Even when P_c is not renormalizable, the little Julia set is well-defined. Let $C_{p,q}^n$ be the subset of the p/nq-limb for which the little Julia set is connected. Obviously, $C_{p,q}^n \subset \mathbf{M}$ because for $c \notin \mathbf{M}$, the critical orbit will eventually leave the domain of the polynomial-like map so that the little Julia set is disconnected.

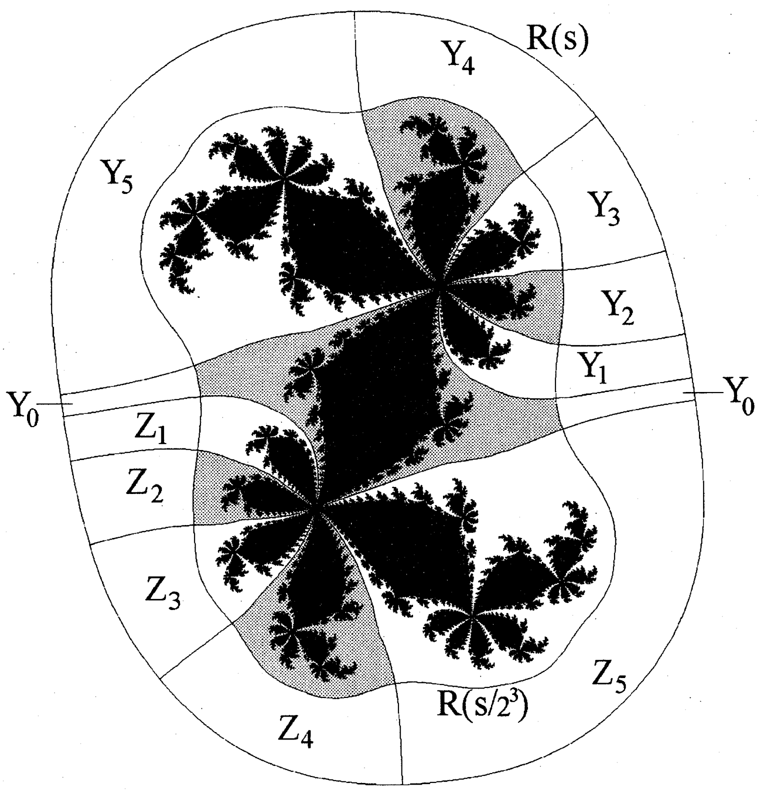

Figure 2: The construction of the quadratic-like maps. The domain $\tilde{U}$ is shaded, and $\tilde{V}$ is $R(s) - (Y_1 \cup Y_3 \cup Y_5)$.

3.2 The Boundary of the Renormalization Locus

The loci of simple renormalization are finitely many homeomorphic copies of the Mandelbrot set within itself. Boundary points of these "little Mandelbrot sets" which disconnect it from the rest of the Mandelbrot set are certain Misiurewicz points, and the subset of $\mathbf{M}$ which is disconnected by such a point can be chopped off by a pair of parameter rays landing at this Misiurewicz point. These facts are well known and have analogues for crossed renormalization. We will describe them in this section, partly by way of examples.

We want to find $C^n_{p,q}$ within the limb $\mathbf{M}_{p/nq}$. For any parameter within $\mathbf{M}_{p/nq}$, and even within the entire p/nq-wake of $\mathbf{M}$, we can construct the quadratic-like map as in Section 3.1. Such a parameter c is crossed n-renormalizable iff the entire critical orbit remains within $\tilde{U} = Y_0 \cup \bigcup_j (Y_{jn} \cup Z_{jn})$ under P^n_c.

Since $c \in \mathbf{M}$, the critical orbit will always remain within the filled-in Julia set of P_c, so it can escape from $\tilde{U}$ only through the set

$$Z' := Z - \bigcup_{j=1}^{q-1} Z_{jn} \; .$$

We need only be concerned with the first time the critical orbit enters Z'. This will obviously happen after sqn steps for some integer $s \geq 1$.

Let us first consider the case $s = 1$. Under $qn - 1$ iterations, the sector Y_1 maps homeomorphically onto $(Y_0 \cup Z)$ (extended out to an appropriate equipotential. In this section, we will be rather sloppy about equipotentials and ignore necessary restrictions of the sectors at equipotentials because that will be irrelevant for our considerations). There is a unique point $z_1 \in Y_1$ which maps onto $-\alpha$ under P_c^{qn-1}, and it is the landing point of qn strictly preperiodic dynamic rays (compare Figure 3). These rays cut Y_1 into qn sub-sectors. For $j = 1, 2, \ldots, q-1$, there is exactly one sub-sector which maps onto Z_{jn} under P^n_c, and one sub-sector will map onto Y_0. These q sub-sectors are distributed evenly around z_1, and if the critical point is contained in one of them, then the critical orbit will survive qn iterations in $\tilde{U}$. However, if the critical value is in one of the remaining $qn - q$ sub-sectors, then it will leave $\tilde{U}$ after $n - 1$ iterations.

Now we transfer this configuration into parameter space. The external angles of the qn dynamic rays bounding these sub-sectors are also the external angles of qn parameter rays of the Mandelbrot set which land at a common Misiurewicz point (compare for example [S1, Section 4]) and cut the complex parameter plane into qn regions. The region containing the parameter c is bounded by exactly the same external angles as the sub-sector within Y_1 containing the critical value. Of the qn parameter regions, $qn - q$ of them do not intersect $C^n_{p,q}$, so that the renormalization locus $C^n_{p,q}$ is contained in the q-star formed by the remaining q regions. This is the first step of chopping away subsets of $\mathbf{M}_{p/nq}$ in order to approximate $C^n_{p,q}$, and it describes for which parameters the critical orbit survives the first qn iterations within $\tilde{U}$.

If $P_c^{qn}(0)$ is in Y_0 or some Z_{jn}, then the critical orbit survives another $qn-1$ (respectively $(q-j)n-1$) iterations within $Y_1 \cup Y_2 \cup \ldots \cup Y_{qn-1}$. In the next step, the critical orbit can again visit Y_0 or Z, and it escapes whenever it hits a wrong sector Z_j. The qn dynamic rays landing at $-\alpha$ can be transported back for $(q-j)n-1$ iteration steps into the sector Z_{jn} or Y_0 containing $P_c^{qn}(0)$. Transporting these rays back another $qn-1$ steps, we obtain qn preperiodic dynamic rays within the sub-sector of Y_1 containing the critical value. These qn rays cut the sub-sector into qn sub-sub-sectors. If the critical value is contained in q of these, then the critical orbit will survive $qn+(q-j)n$ iteration steps within $\tilde{U}$; otherwise, it will not. The index j depends of course on which of the sectors Z_j the critical orbit visits first. These qn new preperiodic dynamic rays have again qn counterparts in parameter space which land at a common Misiurewicz point, and they further subdivide the parameter region containing $C_{p,q}^n$. Of the qn sub-sub-regions around this Misiurewicz point, only q will intersect $C_{p,q}^n$.

This argument can be repeated: in order for the critical orbit to survive one more turn within $\tilde{U}$, there is another collection of qn sub-sub-…-sectors, and only q of them may contain the critical value. We get a countable collection of further necessary conditions which translate into a countable collection of cuts in parameter space along pairs of parameter rays at preperiodic angles. Conversely, when a parameter c is not cut off by such a parameter ray pair, then the critical orbit will remain in $\tilde{U}$ forever, and $c \in C_{p,q}^n$. The first few cuts in the dynamic plane and in parameter space are indicated in Figure 3. All the Misiurewicz points at which the bounding parameter rays land have the property that the critical orbit will terminate at the α-fixed point after finitely many iterations.

From this discussion, we can conclude the following lemma.

Lemma 4 (The Loci $C_{p,q}^n$ are Connected and Almost Compact)
Any set $C_{p,q}^n$ is connected and full, and its union with the root of the limb $\mathbf{M}_{p/nq}$ is compact.

PROOF. Let r be the root of $\mathbf{M}_{p/nq}$, so that $\overline{\mathbf{M}}_{p/nq} = \mathbf{M}_{p/nq} \cup \{r\}$. This set is compact, connected and full. Starting with this set, every cut by a pair of parameter rays as described above leaves a compact, connected and full set, and the countable nested intersection of compact connected and full sets is compact, connected and full. Removing the root again, the remaining set $C_{p,q}^n$ is still connected and full. $\square$

The situation is similar to that for simple renormalizations in the immediate case, say of period n: in that case, any connected component of the renormalization locus is a little Mandelbrot set which is separated from the rest of its limb within the Mandelbrot set by a countable collection of parameter ray pairs landing at Misiurewicz points, and these Misiurewicz points again have the property that the critical orbit will terminate at the α-fixed point. The difference is that in the simple case, the renormalization locus does not extend over such a Misiurewicz point, while it does extend in the crossed case in q of the qn directions. Any limb at angle p/qn contains a connected component of $C_{p,q}^n$ (the immediate n-renormalization locus) and of the simple qn-renormalization locus. Every Misiurewicz point bounding the latter is also the landing point of parameter rays restricting $C_{p,q}^n$, but $C_{p,q}^n$ is larger and extends further, all the way out to "antenna tips" of $\mathbf{M}$; compare Figure 3. The crossed n-renormalization locus $C_{p,q}^n$ is of course homeomorphic to $\mathbf{M}_{p/q}$, and the locus of simple qn-renormalization corresponds to the tuned copy of $\mathbf{M}$ of period q within $\mathbf{M}_{p/nq}$ under this homeomorphism.

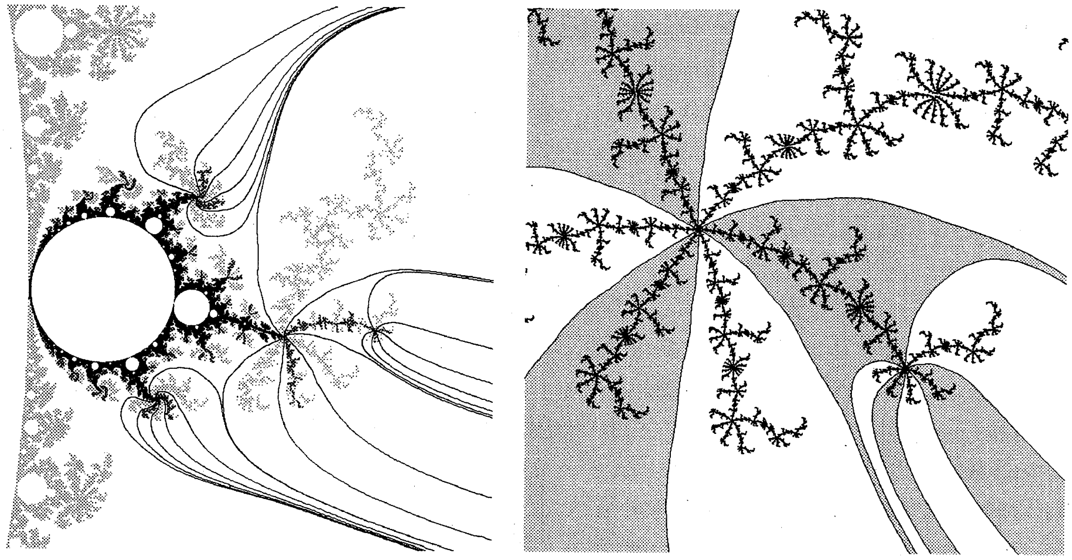

Figure 3: Left: the renormalization locus $C^n_{p,q}$ in $\mathbf{M}_{p/q}$, displayed for $n = 2$ and $p/q = 1/3$: the entire Mandelbrot set is shown in light grey; the crossed 2-renormalization locus is shown in dark grey, and the simple 6-renormalization locus is shown in black. Right: some dynamic rays corresponding to parameter rays bounding the renormalization locus.

3.3 A Homeomorphism from $C^n_{p,q}$ to the p/q-Limb

For every parameter in the p/nq-wake of the Mandelbrot set, we have constructed a quadratic-like map and we have defined $C^n_{p,q}$ to be the subset of $\mathbf{M}_{p/nq}$ where this quadratic-like map has connected Julia set, so that the corresponding polynomials are n-renormalizable. The straightening theorem supplies a well-defined map $\chi: C^n_{p,q} \to \mathbf{M}$. In this section, we will argue that χ is a homeomorphism onto $\mathbf{M}_{p/q}$, the p/q-limb of $\mathbf{M}$.

Recall that for every $c \in \mathbf{M}_{p/nq}$, the α-fixed point of the polynomial P_c is the landing point of exactly nq dynamic rays. These rays are permuted transitively by the dynamics of the polynomial, and the combinatorial rotation number of this cyclic permutation is p/nq. Between any pair of adjacent rays, there is a part of the Julia set of P_c. Our region U contains q of these nq sectors between adjacent rays (plus a small neighborhood around α extending into all sectors). The map P_c^n permutes these sectors transitively with combinatorial rotation number $np/nq = p/q$. The straightening map χ on $C^n_{p,q}$ thus takes images within $\mathbf{M}_{p/q}$. The root of that limb cannot be in the image of χ because the α-fixed point is rationally indifferent at the root, while we started with a repelling α-fixed point and a topological conjugation can never turn a repelling periodic point into an indifferent one.

The little Julia set is contained in $\tilde{U} \subset Y_0 \cup Y_n \cup Y_{2n} \cup \ldots \cup Y_{(q-1)n} \cup Z$, and the critical orbit of the little Julia set first visits the sectors Y_0, Y_n, Y_{2n}, $\ldots$, $Y_{(q-1)n}$. Therefore, the little Julia set meets the interiors of all the sectors Y_0, Y_n, Y_{2n}, $\ldots$, $Y_{(q-1)n}$, but of no other Y_i. The image of the little Julia set will then be contained in $Y_1 \cup Y_{n+1} \cup Y_{2n+1} \cup \ldots \cup Y_{(q-1)n+1}$. Therefore, the little Julia set and its image are different. The α-fixed point is their only common point and disconnects both of them. The renormalization is hence indeed of crossed type, and it is obviously immediate.

Our map χ has the following important properties:

Proposition 5 (The Straightening Map)
The straightening map χ is a homeomorphism from $C^n_{p,q}$ onto $\mathbf{M}_{p/q}$. Moreover, its restriction to any hyperbolic component is a biholomorphic map onto another hyperbolic component and extends homeomorphically to the closures, and the map reduces periods of hyperbolic components exactly by a factor n. In particular, all periods of hyperbolic components in $C^n_{p,q}$ are divisible by n. Conversely, only hyperbolic components will map onto hyperbolic components.

PROOF. We begin with the second statement. Within hyperbolic components, there are attracting periodic orbits, the multipliers of which are preserved by hybrid conjugations. Since hyperbolic components are well known to be conformally parametrized by the multipliers of the attracting orbits, and these parametrizations extend homeomorphically onto the closures ([DH1], [M3] or [S1]), the straightening map inherits the same properties. (In fact, if there are any non-hyperbolic components of the interior of $\mathbf{M}$, then the straightening map will still be holomorphic there.) Conversely, only hyperbolic parameters can map onto hyperbolic parameters. Since any little Julia set can meet its images (other than itself) only at repelling periodic points, it follows that the straightening map reduces periods of hyperbolic components exactly by a factor of n.

It is not hard to check that our construction is "continuous" in the following sense: nearby parameters in $C^n_{p,q}$ yield nearby domains U and V for our quadratic-like maps, and the quadratic-like maps themselves are close on the intersection of their domains. Douady and Hubbard [DH] show that under these circumstances the straightening map is continuous (in degree two only!). Moreover, they show that it is an open mapping over the p/q-limb. This requires some non-trivial arguments, which are now completely standard: we have to turn our quadratic-like maps into an analytic family defined in a neighborhood of $C^n_{p,q}$ and define the straightening map in this neighborhood. In order to obtain a well-defined straightening map even when the Julia set is disconnected, one has to construct a tubing. With a chosen tubing, the image of the map χ is well-defined, but in the disconnected case it depends on the tubing. All these constructions are straightforward in our case, so that we indeed have a continuous open mapping $\chi\colon C^n_{p,q} \to \mathbf{M}_{p/q}$.

First we show that χ is surjective onto $\mathbf{M}_{p/q}$ by showing that its image is both open and closed in $\mathbf{M}_{p/q}$. Since the limb $\mathbf{M}_{p/q}$ is well known to be connected and the image is non-empty, this makes χ surjective.

The image of $C^n_{p,q}$ is obviously open because χ is an open map. That the image is closed follows almost from continuity: let $c \in \mathbf{M}_{p/q}$ be a boundary point of the image. Then there is a sequence $c_1, c_2 \ldots$ in the image converging to c, and there are points $c'_1, c'_2, \ldots \in C^n_{p,q}$ with $\chi(c'_i) = c_i$. Let c' be a limit point of the sequence (c'_i) within $\mathbf{M}$. If c' is not the root of $\mathbf{M}_{p/nq}$, then $c' \in C^n_{p,q}$ because the union of $C^n_{p,q}$ with its root is compact, and $\chi(c') = c$ by continuity. However, if c' is the root, then the c_i must converge to the root of $\mathbf{M}_{p/q}$: this follows from the proof of local connectivity of $\mathbf{M}$ at the roots of the p/q-wake and the p/nq-wake (see Hubbard [H, Theorem I] or Schleicher [S2]). More precisely, if the sequence (c'_i) has an infinite subsequence on the closure of the hyperbolic component of period qn in $C^n_{p,q}$, then the image sequence (c_i) will also have an infinite subsequence on the closure of the hyperbolic component of period q in $M_{p/q}$, and this sequence will converge to the root of the component. Otherwise, there must be an infinite subsequence in sublimbs of these components, and the diameters of these sublimbs tend to zero as the parameter tends to the root (the Yoccoz inequality in [H] makes this precise; a qualitative version of this same statement can be found in [S2], and both are essential ingredients in the proof of local connectivity of $\mathbf{M}$ at parabolic parameters). This finishes the argument that χ is surjective from $C^n_{p,q}$ to $M_{p/q}$, and it extends to a continuous surjective map between the closures of both sets (which are the same sets with the roots added).

Douady and Hubbard [DH] show that χ is not only an open mapping, but it is what they call *topological holomorphic* over $M_{p/q}$. This implies that χ has a local mapping degree over every $c \in \mathbf{M}_{p/q}$, and it is easy to conclude that it inherits a global mapping degree over $\mathbf{M}_{p/q}$. But since $C^n_{p,q}$ contains a single hyperbolic component of period qn which maps biholomorphically onto the unique component of period q in $M_{p/q}$, and this image component has no further inverse images, the mapping degree is 1. The inverse map χ^{-1} is continuous as well, and χ is a homeomorphism. □

3.4 Our Construction is Complete

In the last sections, we have identified subsets $C^n_{p,q}$ of $\mathbf{M}_{p/nq}$ containing immediately n-renormalizable polynomials of crossed type. We need to show that $C^n_{p,q}$ contains every polynomial with these renormalization properties, so that the sets $C^n_{p,q}$ and the construction above completely describe the n-renormalization locus within $\mathbf{M}_{p/nq}$. This will be done in this section.

Proposition 6 (Completeness of the Construction)
Any polynomial in $\mathbf{M}_{p/nq}$ which is immediately n-renormalizable of crossed type is contained in $C^n_{p,q}$.

PROOF. Let $c \in \mathbf{M}_{p/q}$ be such that P_c is immediately n-renormalizable of crossed type. Then the restriction of P_c^n to appropriate open simply connected domains U, V defines a quadratic-like map $P_c^n\colon U \to V$. Let K' be its filled-in Julia set; it is connected. Since the renormalization is immediate and of crossed type, K' and $P_c(K')$ intersect exactly in α, and α separates K'. Let $q \geq 2$ be the number of connected components of $K' - \{\alpha\}$.

The α-fixed point of P_c is repelling and the landing point of at least 2 periodic dynamic rays; let k be the number of these rays. By [McM, Theorem 7.11], we have $k \geq qn$. These rays, together with the k rays landing at $-\alpha$, cut the complex plane into $2k-1$ simply connected closed sectors $Y_0, Y_1, \ldots, Y_{k-1}, Z_1, \ldots, Z_{k-1}$, labeled similarly as above in Figure 2. Since K' is connected, it contains the critical point, so K' intersects the interior of Y_0. It follows that K' intersects $\mathrm{int}(Y_{jn})$ for $j = 1, 2, \ldots, q-1$. This accounts for at least q connected components of $K' - \{\alpha\}$, and since q was defined as the total number of connected components of $K' - \{\alpha\}$, these are all the connected components. If k was greater than qn, then $K' - \{\alpha\}$ would have more than q connected components because P_c permutes the k rays landing at α and thus the sectors between them transitively. This is not the case, so $k = qn$. It follows that

$$K' \subset Y_0 \cup \bigcup_{j=1}^{q-1} Y_{jn} \cup \bigcup_{j=1}^{q-1} Z_{jn} = \tilde{U}$$

(the restriction for the Z_i follows from the symmetry of the map). Since the entire critical orbit of P_c^n is contained in K', it is contained in the sets Y_{jn} and Z_{jn} as specified above.

But this means that the critical orbit never leaves the domain of the quadratic-like map as constructed in Section 3.1, so we have $c \in C^n_{p,q}$ as claimed. The set $C^n_{p,q}$ contains indeed all parameters in $\mathbf{M}_{p/nq}$ which are immediately n-renormalizable of crossed type. □

The limb $\mathbf{M}_{p/nq}$ does not contain any polynomial which is n-renormalizable of simple type (because simple n-renormalization is organized in the form of embedded Mandelbrot sets based at hyperbolic components of period n, and the p/nq-limb contains only hyperbolic components of periods nq or greater; no such embedded Mandelbrot set can extend into any the p/nq-limb). We will see in Section 4 that crossed n-renormalization which is not immediate is always simply m-renormalizable for some $m > 1$ strictly dividing n; this renormalization type cannot occur within $\mathbf{M}_{p/nq}$, either. Therefore, $C^n_{p,q}$ is the locus of n-renormalization within $\mathbf{M}_{p/nq}$, and this renormalization is immediate and of crossed type.

3.5 Internal Addresses

The notion of *internal addresses* has been introduced in [LS] in order to efficiently describe the combinatorial structure of the Mandelbrot set. Formally, an internal address is a (finite or infinite) strictly increasing sequence of integers starting with 1, usually written as $1 \to n_2 \to n_3 \to n_4 \ldots$ with $1 < n_2 < n_3 < n_4 \ldots$. It is associated to a parameter $c \in \mathbf{M}$ as follows: all parameter rays of the Mandelbrot set at periodic angles land in pairs with equal periods, and the landing point of such a ray pair is the root of a hyperbolic component of the same period. This parameter ray pair separates the component from the origin. All these ray pairs which separate c from the origin are totally ordered. Then n_2 is the lowest period of parameter ray pairs separating c from the origin; n_3 is the lowest period of parameter ray pairs separating c from the ray pair of period n_2, and so on. (If the parameter c is exactly on such a periodic parameter ray pair, the corresponding period is still accepted.) The internal address is finite iff, after finitely many steps, there no longer is such a separating periodic parameter ray pair. This happens iff the parameter c is on the closure of a hyperbolic component, and the final entry in the internal address is the period of this component (if the parameter is a bifurcation point between two hyperbolic components, then their two periods show up as the last two entries in the internal address).

In order to distinguish all the parameters in $\mathbf{M}$, every entry n in an internal address has to encode in which sublimb of the corresponding component of period n the described parameter is. This sublimb is described by its internal angle p/q. The denominator q is redundant, while p can be arbitrary (coprime to q) and separates various combinatorially equivalent sublimbs of $\mathbf{M}$; for details, see [LS]. These "angled internal addresses" distinguish combinatorial classes (or "fibers", when taking extra care at hyperbolic components) of the Mandelbrot set completely; compare [S2].

A pleasant property of internal addresses is that they encode the combinatorics of the parameters they describe. It is shown in [LS, Proposition 6.7] that a parameter is simply n-renormalizable if and only if the internal address contains the entry n, and every subsequent entry is divisible by n. After n-renormalization, an internal address $1 \to n_2 \to n_3 \to \ldots \to n_j \to n \to k_2 n \to k_3 n \to \ldots$ turns into $1 \to k_2 \to k_3 \to \ldots$. The renormalization is immediate iff the entry n follows directly after the initial 1, so the internal address has the form $1 \to n \to k_2 n \to k_3 n \to \ldots$ with $2 \le k_2 \le k_3 \le \ldots$. After renormalization, the internal address becomes $1 \to k_2 \to k_3 \ldots$.

There is a corresponding statement for crossed renormalizations, which we state first for the immediate case.

Proposition 7 (Crossed Renormalization and Internal Addresses)
Let c be a parameter of the Mandelbrot set. Then c is immediately crossed n-renormalizable if and only its internal address is of the form

$$1 \to k_1 n \to k_2 n \to k_3 n \to \dots$$

with $2 \le k_1 < k_2 < k_3 \dots$. In this case, $c \in C^n_{p,q}$ with $q = k_1$ and some p coprime to q, and the internal address of $\chi(c)$ is $1 \to k_1 \to k_2 \to k_3 \to \dots$.

Within the space available, it is not possible to explain all the details of the combinatorial constructions and of the proof. For background, we refer to [LS] and in particular to the proof of the related Proposition 6.7. That proof can be modified to hold for the proposition at hand just as well, but here we will outline a different variant of the proof.

PROOF. Let $c \in \mathbf{M}$ be a parameter which is immediately n-renormalizable of crossed type and let $1 \to n_1 \to n_2 \to \dots$ be its internal address. Then c is in a p/q-limb of $\mathbf{M}$ for $q = n_1$ and some p coprime to q and thus $c \in C^n_{p,q}$. Since $C^n_{p,q}$ is connected, it connects c to the boundary of the main cardioid of $\mathbf{M}$. Any number n_i in the internal address corresponds to a hyperbolic component of period n_i, and the two parameter rays landing at its root disconnect $C^n_{p,q}$. The hyperbolic component of period n_i must then also be in $C^n_{p,q}$, and by Proposition 5, n_i is divisible by n. We have $n_1 \ge 2n$ because $1 \to n$ describes hyperbolic components of period n immediately bifurcating from the main cardioid, and these are simply n-renormalizable but not crossed n-renormalizable. It is easy to see that every entry in the internal address of $\chi(c)$ comes from an entry in the internal address of c. Since χ divides periods of hyperbolic components by n, every entry in the internal address of c is divisible by n and the internal address of $\chi(c)$ is as claimed.

Conversely, assume that some parameter c is not crossed n-renormalizable. To reach a contradiction, assume that the internal address has the form $1 \to k_1 n \to k_2 n \dots$ with $k_1 \ge 2$. To any internal address, there is a corresponding *kneading sequence* (compare [LS]; in the context of real kneading sequences, the associated internal address is the well-known sequence of *cutting times* investigated by many people). Such a kneading sequence is an infinite sequence of symbols 0 and 1 (and, when the internal address is finite, possibly of symbols $\star$), and it always starts with 1. When all the entries n_i are divisible by n and greater than n, then the only entries in the kneading sequence which may be different from 1 are at positions $2n$, $3n$, $4n$ ... (compare [LS, Algorithm 6.2]). However, from the fact that $c \notin C^n_{p,q}$, we will deduce that the internal address has an entry 0 at a position which is not divisible by n.

After an initial string of entries 1 in the kneading sequence, the first 0 or $\star$ is at position $n_1 > 1$. Then c is in a p/n_1-limb of $\mathbf{M}$ for some p coprime to n_1. By assumption, $n_1 = nq$ for some $q \ge 2$ and $c \in \mathbf{M}_{p/nq}$. We can then construct a quadratic-like map as in Section 3.1. One can read off a substantial part of the kneading sequence from the dynamics of the critical orbit: if the k-th forward image of the critical point is contained in $Y_1 \cup Y_2 \cup Y_3 \dots Y_{qn-1}$, then the k-th entry in the kneading sequence is 1; if the image point is in $Z_1 \cup Z_2 \cup \dots Z_{qn-1}$, then the k-th entry in the kneading sequence is 0. Only if the image point is in Y_0, one cannot decide without looking more closely.

If any point z is in one of the sectors Y_0, Y_n, Y_{2n}, $\ldots$, $Y_{(q-1)n}$, then the next $n-1$ forward images on its orbit will still be in one of the sectors Y_1, $\ldots$, Y_{qn-1}; by symmetry, the same is true if z is in one of Z_n, Z_{2n}, $\ldots$, $Z_{(q-1)n}$. In particular, if the critical orbit falls into one of these sectors Y_{jn} or Z_{jn} (for $j = 0, 1, \ldots, q-1$), then we can predict a string of $n-1$ entries 1 in the kneading sequence.

If the critical orbit escapes from

$$\tilde{U} = \left(Y_0 \cup \bigcup_{j=1}^{q-1} (Y_{jn} \cup Z_{jn}) \right) \cap R(s/2^n)$$

under P_c^n, then it must do so through one of the Z_s with s not divisible by n: that means, there is a $k > 0$ such that $P_c^{kn}(0) \in Z_s$ (we can exclude that $P_c^{kn}(0) = -\alpha$ because $-\alpha$ is also in all the Z_{jn}). Now the critical orbit "loses its synchronization" with $\tilde{U}$: writing $s = jn - s'$ with $0 < s' < n$, it will spend $s' - 1$ steps within some Y_s for s not divisible by n, until at the s'-th step it will be back in Y_{jn} (or in Y_0 or Z if $j = q$). Unless it is in Z, we will have a string of $n-1$ symbols 1 as explained above, and in the next step the orbit reaches another Y_{jn} or Z. If it lands in Z, we have an entry 0 in the kneading sequence at a position which is not divisible by n (the position is congruent to s' modulo n), and this is the desired contradiction. The orbit under P_c^n must therefore remain in $\cup Y_{jn}$ forever, and when visiting Y_0, it must do it in such a way so as to generate an entry 1 in the kneading sequence. We will argue that this is possible only if the critical orbit escapes through the point $-\alpha$, and we had excluded that case above.

To do this, we have to look at kneading sequences somewhat differently: assume that the critical value is the landing point of some dynamic ray at angle ϑ. Instead of determining the kneading sequence by looking at the sectors Y_j or Z_j containing the critical orbit, we can simply look at the forward orbit of ϑ under doubling with respect to an appropriate partition. This partition is formed by the two inverses of ϑ under doubling: the angles $\vartheta/2$ and $(\vartheta+1)/2$. If $2^k\vartheta$ is in the interval $(\vartheta/2, (\vartheta+1)/2)$, then the k-th entry in the kneading sequence is 1; if $2^k\vartheta$ is on the boundary, this entry is $\star$, and the k-th entry is 0 in the remaining interval (containing the angle 0). This definition makes sense even if no dynamic ray lands at the critical value. We just have to associate an external angle ϑ to the parameter c, and we take the angle of any parameter ray which accumulates at c if $c \in \partial\mathbf{M}$; if $c \in \mathrm{int}(\mathbf{M})$, we can take any ray which accumulates at the boundary of the connected component of $\mathrm{int}(\mathbf{M})$ containing c. If there are several such rays, they will all "essentially" yield the same kneading sequence (except for an occasional symbol $\star$, which can be dealt with). For details, see [LS, Sections 5 and 6].

In our case, it is quite easy to see that the only external angles which generate only symbols 1 in the kneading sequence are those of dynamic rays landing at α, which means that the critical orbit "escapes" through the point $-\alpha$ and terminates at α. But we had excluded that case already, and this is the final contradiction. No critical orbit can thus escape from $\tilde{U}$ and still have a kneading sequence or an internal address of the type described above, which finishes the proof of the proposition. □

REMARK. The proposition specifies how internal addresses behave under crossed renormalization. We had mentioned above that an internal address is unique only when extended to an "angled internal address", specifying the internal angles of sublimbs containing the described parameter. Since the renormalization preserves the parametrization of hyperbolic components by multipliers and thus by internal angles, it also preserves the angles in the angled internal address: the angled internal address $1 \to k_1 n_{(p_1/q_1)} \to k_2 n_{(p_2/q_2)} \dots$ turns into $1 \to k_{1(p_1/q_1)} \to k_{2(p_2/q_2)} \dots$.

This result allows another approach to showing that the straightening map is a homeomorphism from $C^n_{p,q}$ to $M_{p/q}$: on a combinatorial level, this follows from internal addresses. If the Mandelbrot set is locally connected, then its topology is completely described by its combinatorics, and we get an actual homeomorphism. Without assuming local connectivity, the problem is reduced to a local one on every fiber and can easily be settled using well-known properties of the straightening map.

4 Crossed Renormalization: The General Case

In the last section, we have described the locus of crossed renormalization for the special case that the little Julia sets cross at a fixed point (the "immediate" case of crossed renormalization). The general case is that the little Julia sets cross at a periodic point of some period $m > 1$. It turns out that the general case can conveniently be reduced to the immediate case.

Theorem 8 (The General Case of Crossed Renormalization)
Let the polynomial P_c be crossed n-renormalizable so that the crossing point of the little Julia sets has exact period $m > 1$. Then P_c is simple m-renormalizable and the corresponding quadratic-like map is immediately n/m-renormalizable of crossed type.

Conversely, the image of any crossed m-renormalizable polynomial under a tuning map of period k is crossed m-renormalizable of period km, and the period of the intersection point of the little Julia sets is multiplied by k as well.

PROOF. Let x be a periodic point where the little Julia set crosses one of its forward images, and let $m > 1$ be its period. Obviously, m divides n, and $m < n$ because if the first return map of x was already the first return map of the little Julia set, then the little Julia set could not cross any of its forward images at x.

Since x disconnects the little Julia set and at least one of its forward images, it must be the landing point of at least four dynamic rays. The first return map of x must then permute these rays transitively (compare [M3] or [S1, Lemma 2.4]), so their period is a proper multiple of m.

All the dynamic rays of period m or dividing m which do not land alone cut the complex plane into some finite number of pieces. We will consider this situation from the point of view of the m-th iterate of P_c. Then this partition is formed by the fixed rays of P_c^m, and each piece is a "basic region" in the sense of Goldberg and Milnor [GM] (they exclude the case that some fixed points of P_c^m coincide; the finitely many paraabolic parameters where this happens can easily be dealt with at the end, removing punctures in tuned copies of the Mandelbrot set). The little Julia set and all of its images under some iterate are each contained within a single basic region: otherwise, the little Julia set would have to extend over a landing point of dynamic rays of period m for P_c, and such a landing point itself has period at most m. At such a point, the little Julia set must meet its m-th forward image, so this point must be x. But the rays landing at x have periods greater than m.

By [GM, Lemma 1.6], each basic region contains exactly one fixed point of f. Since crossing points of forward images of the little Julia set are fixed points of P_c^m, any two forward images of the little Julia set are in different basic regions, except those which cross at a point on the forward orbit of x. Let $\tilde{V}$ be the basic region containing the critical point and thus the little Julia set. Then $\tilde{V}$ contains $P_c^{jm}(0)$ for $j = 0, 1, 2, 3 \ldots$ because all these are contained in the little Julia set or those of its forward images which cross at x. All the other points on the critical orbit are contained within different images of the little Julia set and thus within different basic regions.

Let $\tilde{U}$ be the subset of $\tilde{V}$ which is mapped onto $\tilde{V}$ under P_c^m. We claim that $\tilde{U}$ and $\tilde{V}$ can be thickened slightly to two regions U, V so that $P_c^m: U \to V$ is a quadratic-like map with connected Julia set.

To see this, we first transport $\tilde{V}$ back m iterations of P_c along the critical orbit $0 \in \tilde{U}$, $P_c(0)$, ..., $P_c^m(0) \in \tilde{V}$. This pull-back will avoid $\tilde{V}$ except at the beginning and end, so $P_c^n: \tilde{U} \to \tilde{V}$ is a degree two map. Since the partition boundary forming $\tilde{V}$ consists of fixed rays of P_c^m, it is forward invariant, which implies $\tilde{U} \subset \tilde{V}$. And since all $P_c^{jm}(0) \in \tilde{V}$ within some forward image of the little Julia set, it follows that all $P_c^{(j-1)m}(0) \in \tilde{U}$, so the critical orbit of P_c^m will never leave $\tilde{U}$. It may happen that $\tilde{U}$ and $\tilde{V}$ have common boundary points, but this can be cured by a usual thickening procedure as in Section 3.1. Call the thickened regions U and V.

We then have a quadratic-like map $P_c^m: U \to V$ with connected Julia set, so P_c is m-renormalizable. None of the first $m - 1$ forward images of the little Julia set can meet the interior of $\tilde{V}$, so this renormalization is simple.

By construction, the little Julia set for the crossed n-renormalization is contained in $\tilde{V}$ and thus also in $\tilde{U}$ (by the same argument as above for the critical orbit). This construction preserves crossed renormalizability but reduces its period by m. Therefore, the renormalized map is still crossed m/n-renormalizable; the crossing point of this renormalization now has period one, so this crossed renormalization is immediate. This proves the first claim.

The proof of the converse statement is straightforward. □

Any crossed renormalization is thus either immediate, or it is the image of an immediate crossed renormalization under a simple renormalization. Crossed n-renormalization around a periodic point of period m can occur only if m strictly divides n, and the corresponding locus is then homeomorphic to countably many homeomorphic copies of limbs of the Mandelbrot set with denominator n/m. This homeomorphism is a restriction of a tuning map of period m (which reduces to the identity in the immediate case $m = 1$).

All the considerations from Section 3 can now be transferred easily to the general case. Any connected component of the crossed n-renormalization locus around a periodic point of period m can be obtained from the Mandelbrot set by chopping off subsets of $\mathbf{M}$ bounded by pairs of parameter rays at preperiodic angles. We can also state explicitly which internal addresses allow crossed renormalizations.

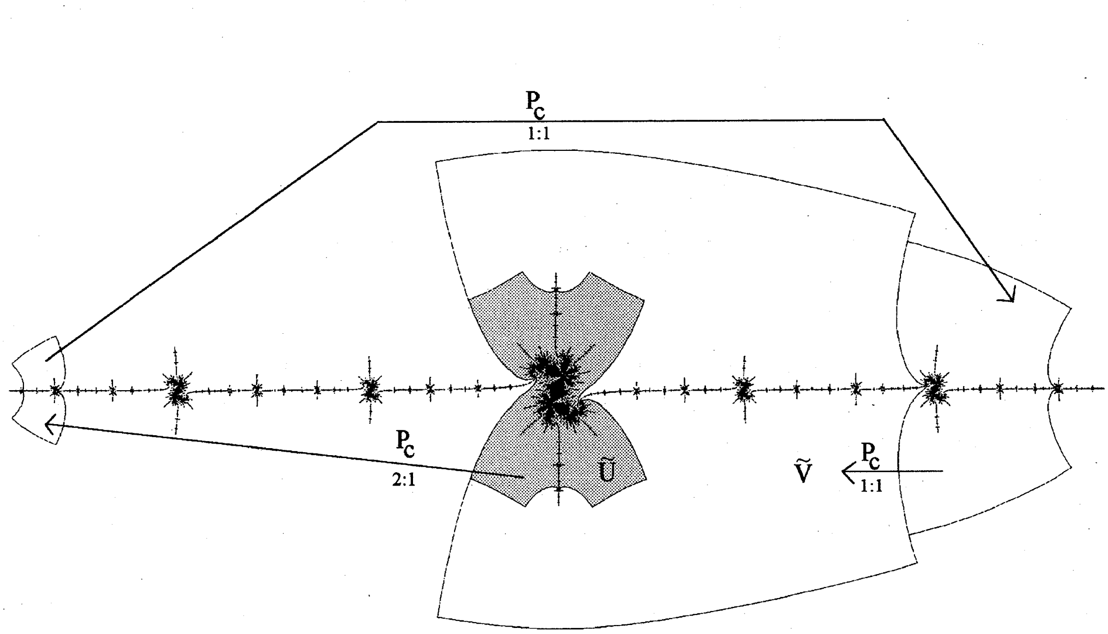

Figure 4: Crossed Renormalization in the general case can be reduced by a simple renormalization to an immediate crossed renormalization.

Corollary 9 (Internal Addresses for Crossed Renormalizations)
A parameter of the Mandelbrot set is n-renormalizable of crossed type around a periodic point of period m if and only if m strictly divides n and its internal address is of the form

$$1 \to n_1 \to \ldots \to n_j \to m \to k_1 n \to k_2 n \to k_3 n \to \ldots$$

with $1 \le k_1 < k_2 < k_3 \ldots$. *Crossed n-renormalization turns this internal address into* $1 \to k_1 \to k_2 \to k_3 \to \ldots$ *and preserves the angles in the angled internal address.* □

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