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Branch locus of polynomial maps

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Abstract

In the paper of [NF97b] we studied the geometrical and topological properties of the moduli space of polynomial maps of degree 3 from a viewpoint of complex dynamical systems. Making use of the discussion of [FN97] and [NF97a], we decide the branch locus of the moduli space of polynomial maps of degree 4.

1 Polynomials of degree 4

1.1 Coefficient coordinate on polynomials of degree 4

Let $\mathrm{Poly}_4(\mathbf{C})$ be the space of all polynomial maps of the form

$$\begin{aligned} &p : \mathbf{C} \to \mathbf{C}, \\ &p(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \quad (a_4 \neq 0). \end{aligned}$$

The group $\mathfrak{A}(\mathbf{C})$ of all affine transformations acts on $\mathrm{Poly}_4(\mathbf{C})$ by conjugation:

$$g \circ p \circ g^{-1} \in \mathrm{Poly}_4(\mathbf{C}) \quad \text{for} \quad g \in \mathfrak{A}(\mathbf{C}),\ p \in \mathrm{Poly}_4(\mathbf{C}).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_4(\mathbf{C})$ are *holomorphically conjugate* if and only if there exists $g \in \mathfrak{A}(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_4(\mathbf{C})$ under this action will be denoted by $M_4(\mathbf{C})$, and called the *moduli space* of holomorphic conjugacy classes $\langle p\rangle$ of polynomial maps p of degree 4.

Under the conjugacy of the action of $\mathfrak{A}(\mathbf{C})$, it can be assumed that any map in $\mathrm{Poly}_4(\mathbf{C})$ is "monic" and "centered", i.e.,

$$p(z) = z^4 + c_2 z^2 + c_1 z + c_0.$$

This p is determined up to the action of the group $G(3)$ of cubic roots of unity, where each $\eta \in G(3)$ acts on $p \in \mathrm{Poly}_4(\mathbf{C})$ by the transformation $p(z) \mapsto p(\eta z)/\eta$.

Let $\mathcal{P}_1(4)$ be the affine space of all monic and centered polynomials of degree 4 with coordinate (c_0, c_1, c_2). Then we have a three-to-one canonical projection $\Phi : \mathcal{P}_1(4) \to M_4(\mathbf{C})$. Thus $\mathcal{P}_1(4)$ serves as a coordinate space for $M_4(\mathbf{C})$ though there remains the ambiguity up to the group $G(3)$.

We introduce one more coordinate system in $M_4(\mathbf{C})$ after Milnor in [Mil93]: for each $p(z) \in \mathrm{Poly}_4(\mathbf{C})$, let $z_1, \cdots, z_4, z_5(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \le i \le 4)$, and $\mu_5 = 0$. Consider the elementary symmetric functions of the five multipliers,

$$\begin{aligned}
\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4,\\
\sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4\\
\sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4,\\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4\\
\sigma_5 &= 0.
\end{aligned}$$

Note that these are well-defined on the moduli space $M_n(\mathbf{C})$, since μ_i's are invariant under an affine conjugacy. Applying the Fatou index theorem, we have a linear relation ([NF97b]):

$$4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0. \tag{1}$$

Let $\Sigma(4)$ be an affine space with coordinates $(\sigma_1, \sigma_2, \sigma_4)$, so-called multipliers' coordinates. We have a natural projection $\Psi : M_4(\mathbf{C}) \to \Sigma(4)$.

Definition 1 $\mathrm{Per}_1(\mu)$ is the locus of all classes in $M_4(\mathbf{C})$ having a fixed point with multiplier μ. Similarly, $\mathrm{Preper}_{(n)}1$ is the locus of all classes having a pre-fixed critical orbit with tale-length $n \neq 0$.

2 Summary of properties of $\mathrm{Poly}_4(\mathbf{C})$

Now we summarize the properties of the $\mathrm{Poly}_4(\mathbf{C})$ given by [NF97b] and [FN97].

Moduli space: The number of the inverse images of the space $\Sigma(4)$ under the map Ψ is 1, 2, or ∞. The space $\mathcal{P}_1(4)$ is a three-sheeted ramified covering of $\mathbf{C}^3$

Coordinates: $(\sigma_1, \sigma_2, \sigma_4)$ with linear relation $4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0$

Normal Forms : $\mathcal{P}_1(4) = \{f(z) = z^4 + c_2 z^2 + c_1 z + c_0\}$

Transformation formula:

$$\begin{aligned} \sigma_1 &= -8c_1 + 12 && (2) \\ \sigma_2 &= 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c + 1 + 48 && (3) \\ \sigma_4 &= 16c_0c_2^4 + (-4c_1^2 + 8c_1)c_2^3 - 128c_0^2c_2^2 + (144c_0c_1^2 - 288c_0c_1 \\ &\quad +128c_0)c_2 - 27c_1^4 + 108c_1^3 - 144c_1^2 + 64c_1 + 256c_0^3 && (4) \end{aligned}$$

Dynamical curves:

$$\Psi\left(\mathrm{Per}_1(\mu)\right) \,:\, \mu^4 - \sigma_1\mu^3 + \sigma_2\mu^2 + (3\sigma_1 - 2\sigma_2 - 4)\mu + \sigma_4 = 0$$

Symmetry locus: The symmetry locus is a proper subspace of the envelope of the plane family $\{\mathrm{Per}_1(\mu)\}_\mu$. The symmetry locus $\mathcal{S}_4$ in $\mathrm{M}_4(\mathbf{C})$ forms the following algebraic curve:

$$\begin{cases} \sigma_1 = s \\ \sigma_2 = 3(3s-4)(s+4)/32 \\ \sigma_4 = -(3s-4)^3(s-12)/4096. \end{cases}$$

And its normal form is given by a one parameter family $\{z^4 + az\}_a$.

Remark There are significant relations between symmetries of Julia sets and the symmetry locus ([FN]). A. F. Beardon [Bea90] studies symmetries of Julia sets. He gave a sufficient and necessary condition for the Julia set of two polynomials P and Q are same.

Let P and Q be polynomials, P having degree at last two. Then $J(P) = J(Q)$ if and only if there is some σ in $\Sigma(P)$ with $PQ = \sigma QP$: thus

$$\mathcal{F}(P) = \{Q \;:\; QP = \sigma PQ \text{ for some } \sigma \text{ in } \Sigma(P)\}$$

where $\mathcal{F}(P)$ is the class of polynomials with the same Julia sets as P, and $\Sigma(P)$ is the group of symmetries of J.

3 Branch locus

In the case of cubic polynomials, the envelope of the line family $\{\mathrm{Per}_1(\mu)\}_\mu$ coincides with the symmetry locus ([NF97b]). But, in the case of polynomials of degree 4, the symmetry locus is the proper subspace of the envelope ([NF97a]).

In fact, the images of the surfaces $\mathrm{Per}_1(\mu)$ are easily obtained by using the linear relation (1):

$$\Psi(\mathrm{Per}_1(\mu)) \ : \ \mu^4 - \sigma_1\mu^3 + \sigma_2\mu^2 + (3\sigma_1 - 2\sigma_2 - 4)\mu + \sigma_4 = 0.$$

And a defining equation of the envelope of $\{\Psi(\mathrm{Per}_1(\mu))\}_\mu$ is

ENV :

$$54\sigma_1^5 + (-81\sigma_2 - 27\sigma_4 - 135)\sigma_1^4 + (36\sigma_2^2 - 144\sigma_2 - 1008)\sigma_1^3 + (-4\sigma_2^3 + 360\sigma_2^2 + (144\sigma_4 + 2976)\sigma_2 + 576\sigma_4 + 4192)\sigma_1^2 + (-160\sigma_2^3 - 2176\sigma_2^2 + (-384\sigma_4 - 6400)\sigma_2 - 1280\sigma_4 - 5376)\sigma_1 + 16\sigma_2^4 + 448\sigma_2^3 + (-128\sigma_4 + 2176)\sigma_2^2 + (256\sigma_4 + 3840)\sigma_2 + 256\sigma_4^2 + 768\sigma_4 + 2304 = 0.$$

This defining equation is obtained by seeking the common factor of $\Psi(\mathrm{Per}_1(\mu))$ and $\frac{\partial}{\partial\mu}\Psi(\mathrm{Per}_1(\mu))$ where the singular factor $\Psi(\mathrm{Per}_1(1))$ is removed.

A defining equation of the symmetry locus satisfies a defining equation of ENV.

To say more intuitively, the symmetry locus corresponds with the condition that the equation $\mathrm{Per}_1(\mu)$ has triple root, while the envelope corresponds with the condition of double root.

In the case of polynomials of degree 4, the envelope deeply concerns the branch locus.

In this paper, branch locus is defined the locus where the number of inverse images of Ψ is not two.

Theorem 1 *The branch locus is characterized as follows;*

$$\textit{branch locus} = \{\sigma_1 - 4 = 0\} \cup ENV$$

Before proving this theorem, we need "inverse problem" described in [NF97a] (Proposition 2):

> *The composition $\Psi \circ \Phi \ : \ \mathcal{P}_1(4) \to \Sigma(4)$ is not surjective: this map has no inverse image for any point on the "punctured" curve $\mathcal{E}$:*
>
> $$(\sigma_1, \sigma_2, \sigma_4) = (4, s, s^2/4 - 2s + 4), s \neq 6.$$

Proof of outline of "inverse problem" Fix a point $(\sigma_1, \sigma_2, \sigma_4) \in \Sigma(4)$. The following equation is obtained by substituting the equation (2) to (3) of transformation formula:

$$4c_2^3 - 16c_0c_2 = -\sigma_2 - \frac{9}{32}\sigma_1^2 - \frac{3}{4}\sigma_1 + \frac{3}{2} \tag{5}$$

Let V be the value of the right hand of the relation (5):

$$V = \frac{1}{32}(-32\sigma_2 + 9\sigma_1^2 + 24\sigma_1 - 48) \tag{6}$$

First we start the case of $V=0$. We put $c_1 = \frac{12-\sigma_1}{8}$ and $c_2 = 0$. Then c_0 is a one of the solutions of the equation given by (4):

$$1048576c_0^3 - 4096\sigma_4 - 27\sigma_1^4 + 432\sigma_1^3 - 1440\sigma_1^2 + 1792\sigma_1 - 768 = 0.$$

It is important that the coefficient of the c_0^3 term does not vanish.

Second, we assume that $V \neq 0$. From the relation (5), if there exists inverse images then we have $c_2 \neq 0$. Therefore dividing (3) by c_2, and substituting it into (4) we obtain the following equation:

$$Ac_2^6 + Bc_2^3 + C = 0 \tag{7}$$

where

$$\begin{aligned} A &= 262144(\sigma_1-4)^2, \\ B &= 1024(128\sigma_2 + (-144\sigma_1^2 + 384\sigma_1 - 256)\sigma_2 - 512\sigma_4 + 27\sigma_1^4 \\ &\quad -576\sigma_1^2 + 1280\sigma_1 - 768), \\ C &= -(32\sigma_2 - 9\sigma_1^2 - 24\sigma_1 + 48)^3. \end{aligned}$$

Here, we will make sure that the above equation (7) have solution(s) c_2 in the cases of $A \neq 0$ or $B \neq 0$. Now we note that $C = (32V)^3 \neq 0$.

1. If $A \neq 0$ or $B \neq 0$ then the equation (7) has solution(s) c_2. Substituting these c_2 to (3), c_0 is also obtained. The parameter c_1 depends only on σ_1.

2. If $A = 0$ and $B = 0$, then we have $\sigma_1 = 4$ and $\sigma_4 = (\sigma_2^2 - 8\sigma_2 + 16)/4$. Now, suppose the equation (7) has solution(s) c_2. Substituting above two conditions into the transformation formula, we have a relation $4c_0 - c_2^2 = 0$. As this relation is a factor of the left hand of the equation (5), it contradicts to the condition $C \neq 0$.

   Therefore there is not a solution c_2 satisfying the equation (7).

   We remark that if C is also 0 (that is $(\sigma_1, \sigma_2, \sigma_4) = (4, 6, 1)$) then there are infinitely many inverse images $(c_0, c_1, c_2) = (c_2^2/4, c_1, c_2)$. However, in this case, we mention again $V = 0$.

Therefore the equation (7) always has solution(s) c_2, except for $(\sigma_1, \sigma_2, \sigma_4) = (4, s, s^2/4 - 2s + 4)$, $s \neq 6$. If there is solution(s) c_2, substituting these c_2 to (3), c_0 is also obtained. The parameter c_1 depends only on σ_1. ∎

Making use of this proof, we prove Theorem 1 as below.

Proof of Theorem 1 If $V = 0$, then $c_2 = 0$ or $4c_0 - c_2^2 = 0$.

- In the case of $c_2 = 0$ and $4c_0 - c_2^2 = 0$:

  The points $(0, c_1, 0)$ correspond with the symmetry locus on $\Sigma(4)$ and the number of the inverse image is one. Hence these points (symmetry locus) belong to the branch locus and it is already known that the symmetry locus is a proper subspace of ENV.

- In the case that one of c_2 or $4c_0 - c_2^2$ is equal to zero:
  1. In the case of $c_2 = 0$ and $4c_0 - c_2^2 \neq 0$:
     We have $c_1 = (12 - \sigma_1)/8$ and c_0 is a root of the equation
     $$1048576c_0^3 - 4096\sigma_4 - 27\sigma_1^4 + 432\sigma_1^3 - 1440\sigma_1^2 + 1792\sigma_1 - 768 = 0.$$
     The above equation have three roots $c_0 = k, k\omega, k\omega^2$, however, these three maps $(c_0, c_1, c_2) \in \mathcal{P}_1(4)$ belong to same conjugacy class.
  2. In the case of $c_2 \neq 0$ and $4c_0 - c_2^2 = 0$:
     The one parameter family $\{(c_2^2/4, 1, c_2)\}_{c_2}$ corresponds to one point $(4, 6, 1) \in \Sigma(4)$. Only on this point, there are infinitely many inverse images.
     For the other points $(c_2^2/4, c_1, c_2)$, we know that there is only one inverse image (conjugacy class) by using the same argument as above case 1.

  Putting together above two cases, there are two inverse images except for the point $(4, 6, 1)$. The point $(4, 6, 1)$ belongs to the symmetry locus (of course it belongs to the ENV). Although this point does not belong to the "branch locus", we treat this point is an element of the branch locus in meaning that the number of inverse images is not two.

On the other hand, if $V \neq 0$ then the equation $Ac_2^6 + Bc_2^3 + C = 0$ is obtained from the inverse problem. This equation has multiple roots if and only if $A = 0$ or discriminant $= 0$. $A = 0$ means $\sigma_1 = 4$ and the discriminant $= 0$ coincides with the defining equation ENV.

At last, we note that the exceptional curve $\mathcal{E}$ is included in the plane $\sigma_1 = 4$. Therefore there are two inverse images except for $\sigma_1 = 0$ or on ENV. ∎

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