

## AN EXPLICIT BOUND FOR UNIFORM PERFECTNESS OF THE JULIA SETS OF RATIONAL MAPS

TOSHIYUKI SUGAWA

ABSTRACT. A compact set  $C$  in the Riemann sphere is called uniformly perfect if the moduli of annuli separating  $C$  are bounded. Mañé-da Rocha and Hinkkanen showed independently uniform perfectness of the Julia sets of rational maps of degree  $\geq 2$ , but they presented no explicit bounds for uniform perfectness. In this note, we shall provide such an explicit bound and, as a result, we give another proof of uniform perfectness of the Julia sets. As an application, we refer to a lower estimate of the Hausdorff dimension of the Julia sets. We also give a concrete bound for the family of quadratic polynomials  $f_c(z) = z^2 + c$  in terms of the parameter  $c$ .

### 1. INTRODUCTION

Let  $C$  be a closed set in the Riemann sphere  $\widehat{\mathbb{C}}$  with  $\#C \geq 3$  and  $\Omega$  its complement. We say  $C$  is *uniformly perfect* if there exists a constant  $0 < c < 1$  such that  $C \cap \{z \in \mathbb{C}; cr < |z - a| < r\} \neq \emptyset$  for any  $a \in C$  and  $0 < r < \text{diam}(C)$ , where  $\text{diam}$  denotes the Euclidean diameter.

The notion of uniform perfectness first appeared in Beardon-Pommerenke [3], and was investigated more deeply by Pommerenke [12] and [13], and afterwards by many authors (see [15] and its references). By definition, the sets with some kind of self-similarities are expected to have uniform perfectness. In fact, the limit sets are known to be uniformly perfect for a wide class of Kleinian groups (cf. Sugawa [16]). On the other hand, Pommerenke [13] first showed the uniform perfectness of the Julia sets of hyperbolic rational maps. Later, Mañé-da Rocha [11] and Hinkkanen [7] proved in the case of general rational maps of degree  $\geq 2$ , independently. For a simpler proof, see the textbook [4] by Carleson and Gamelin. But, their proofs are done by contradiction, thus no explicit bounds for uniform perfectness are given. In this note, we shall present such an explicit bound and also exhibit some applications of this result. Our proof employs the hyperbolic geometry, and hence is different from ones of the above authors. As Hinkkanen remarked in [7], we should note that there exists an entire function whose Julia set is not uniformly perfect. We also note that Hinkkanen and Martin [8] recently proved uniform perfectness for the Julia sets of finitely generated rational semigroups.

We will state the main result in the next section as well as fundamental definitions and notation. In Section 3, we shall discuss the connection between branched coverings and uniform perfectness, which will be a key to prove our main theorem. Section 4 is devoted to prove our main theorem, and we make an essential use of Sullivan's No Wandering Domains Theorem. We shall give applications of the main result to estimations of Hausdorff

---

*Date:* April 6, 1998.

*1991 Mathematics Subject Classification.* Primary 30D05, Secondary 30F45.

*Key words and phrases.* uniformly perfect, Julia set.

dimension, the (logarithmic) capacity density of the Julia set and the Poincaré metric of the Fatou set in Section 5. In the last section, as a special case, we investigate the quadratic family of polynomials  $f_c(z) = z^2 + c$  with  $c \in \mathbb{C}$  outside the Mandelbrot set. We have rather satisfactory result in the case  $c < -2$ . For a general  $c$ , we shall explain a method using a holomorphic motion of the Julia set.

Finally, the author would like to express his hearty gratitude to Professors M. Taniguchi and K. Matsuzaki for helpful comments.

## 2. MAIN RESULT

Let  $C$  be a closed set in the Riemann sphere containing at least three points and  $\Omega$  its complement. We denote by  $\mathcal{A}_\Omega$  and  $\mathcal{A}_\Omega^\circ$  the sets of annuli and round annuli, respectively, in  $\Omega$  separating  $C$ , where annuli mean doubly connected domains and round annuli do special annuli of the form  $\{z \in \mathbb{C}; r_1 < |z - a| < r_2\}$  for some  $a \in \mathbb{C}$  and  $0 \leq r_1 < r_2 \leq \infty$ , and we say that an annulus  $A$  separates  $C$  if  $A \cap C = \emptyset$  and if each component of  $\widehat{\mathbb{C}} \setminus A$  intersects  $C$ . The modulus  $m(A)$  of an annulus  $A$  is the number  $m$  such that  $A$  is conformally equivalent to the round annulus  $\{z; 1 < |z| < e^m\}$ . We set

$$M_\Omega = \sup_{A \in \mathcal{A}_\Omega} m(A), \quad M_\Omega^\circ = \sup_{A \in \mathcal{A}_\Omega^\circ} m(A),$$

and call them the modulus and the round modulus of  $\Omega$ . (If  $\mathcal{A}_\Omega$  and/or  $\mathcal{A}_\Omega^\circ$  is empty, then we define  $M_\Omega = 0$  and/or  $M_\Omega^\circ = 0$ , respectively.) For these constants, it is known that  $\frac{1}{2}M_\Omega - 1.7332 \dots \leq M_\Omega^\circ \leq M_\Omega$ , and that if  $\Omega \subset \mathbb{C}$ ,  $M_\Omega - 2.8911 \dots \leq M_\Omega^\circ \leq M_\Omega$  (cf. [15]).

It is easily verified that  $C$  is uniformly perfect if and only if  $M_\Omega^\circ < \infty$ , equivalently  $M_\Omega < \infty$ .

Next, for the later use, we consider quantities determined by the hyperbolic geometry of  $\Omega$ . Let  $C$  be a closed set in the Riemann sphere with  $\#C \geq 3$  and  $\Omega = \widehat{\mathbb{C}} \setminus C$ . Then each component  $D$  of  $\Omega$  is hyperbolic, i.e., there exists a holomorphic universal covering map  $p: \mathbb{H} \rightarrow D$  from the upper half plane onto  $D$ . Thus  $D$  can be regarded as the quotient space  $\mathbb{H}/\Gamma$  of  $\mathbb{H}$  over the covering transformation group  $\Gamma = \{\gamma \in \text{PSL}(2, \mathbb{R}); p \circ \gamma = p\}$ . Since the hyperbolic (or Poincaré) metric  $\rho_{\mathbb{H}}(z)|dz| = \frac{|dz|}{2\text{Im}z}$  is invariant under the action of  $\text{PSL}(2, \mathbb{R})$ ,  $D$  inherits the hyperbolic metric  $\rho_D(z)|dz|$  so that  $p: \mathbb{H} \rightarrow D$  is a local isometry with respect to the hyperbolic metric, i.e.,  $\rho_{\mathbb{H}} = p^*\rho_D$ . Therefore, we can define the hyperbolic metric  $\rho_\Omega$  of  $\Omega$  componentwise.

The hyperbolic distance  $d_\Omega(z_0, z_1)$  of a pair of points  $z_0, z_1$  in the same component of  $\Omega$  can be defined by  $\inf_\alpha \int_\alpha \rho_\Omega(z)|dz|$ , where the infimum is taken over all paths joining  $z_0$  to  $z_1$  in  $\Omega$ . We also define  $d_\Omega(z_0, z_1) = +\infty$  if  $z_0$  and  $z_1$  do not belong to the same component of  $\Omega$ . For  $z \in \Omega$  we denote by  $\iota_\Omega(z)$  the injectivity radius of  $\Omega$  at  $z$ , that is,  $\iota_\Omega(z)$  is the maximal radius  $r$  so that the hyperbolic disk  $\{w \in \Omega; d_\Omega(z, w) < r\}$  is simply connected.

Let  $\mathcal{C}_\Omega$  denote the set of free homotopy classes of non-trivial loops in  $\Omega$ , where a loop (=closed curve) is called *non-trivial* if this is not null-homotopic (=contractible) in  $\Omega$ . For a loop  $\alpha$  in  $\Omega$ , we define the length of it by

$$\ell_\Omega(\alpha) = \int_\alpha \rho_\Omega(z)|dz|,$$

and for the free homotopy class  $[\alpha]$  represented by  $\alpha$ , we define

$$\ell_\Omega[\alpha] = \inf_{\alpha' \in [\alpha]} \ell_\Omega(\alpha').$$

Finally, we set

$$L_\Omega = \inf_{[\alpha] \in \mathcal{C}_\Omega} \ell_\Omega[\alpha].$$

(If  $\mathcal{C}_\Omega$  is an empty set, we set  $L_\Omega = +\infty$ .) We remark that the injectivity radius  $\iota_\Omega(z)$  is equal to half the infimum of lengths of non-trivial loops in  $\Omega$  passing through  $z$ . In particular,  $L_\Omega$  is nothing other than twice the (global) injectivity radius  $\inf_{z \in \Omega} \iota_\Omega(z)$  of  $\Omega$ .

Concerning the constant  $L_\Omega$ , the following estimate is fundamental.

**Proposition 2.1** ([15]).

$$L_\Omega \leq \frac{\pi^2}{M_\Omega} \leq \min\{L_\Omega e^{L_\Omega}, \frac{L_\Omega^2}{2} \coth^2(L_\Omega/2)\}.$$

In particular,  $M_\Omega < \infty$  if and only if  $L_\Omega > 0$ .

In order to estimate  $M_\Omega$  from above, by this proposition, we have only to do  $L_\Omega$  from below. Now let us state the main theorem. For basic definitions and results about the complex dynamics of the rational maps, we refer to the textbook [2] by Beardon as a general reference.

Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a rational map of degree  $d \geq 2$ . We denote by  $J = J_f$  and  $\Omega = \Omega_f$  the Julia set and the Fatou set of  $f$ , respectively. (In other words,  $\Omega_f$  is the domain of normality of the iteration family  $\{f^n\}_{n=1,2,\dots}$  of  $f$  and  $J_f = \widehat{\mathbb{C}} \setminus \Omega_f$ .) Note that  $\Omega_f$  is completely invariant under  $f$ , precisely,  $f(\Omega_f) = \Omega_f = f^{-1}(\Omega_f)$ .

We denote by  $\text{Crit}(f)$  the set of critical points of  $f$  in the Fatou set  $\Omega_f$  and let  $U_1, \dots, U_s$  be the complete list of the components of  $\Omega_f$  which contains at least one critical point of  $f$  and is not simply connected. And we set  $W_j = f(U_j)$  and  $C_j = \text{Crit}(f) \cap U_j$  for  $j = 1, \dots, s$ . Note here that  $\#\text{Crit}(f) \leq 2d-2$ , so  $s \leq 2d-2$ . Now we introduce two kinds of curve family:  $\mathcal{S}(v_1, v_2)$  and  $\mathcal{T}(v)$ , for  $v_1, v_2, v \in f(C_j)$  with  $v_1 \neq v_2$ . Let  $\mathcal{S}(v_1, v_2)$  and  $\mathcal{T}(v)$  consist of the loops  $\beta : S^1 \rightarrow W_j$ , where  $S^1$  denotes the unit circle  $\{z \in \mathbb{C}; |z| = 1\}$ , satisfying the conditions (a), (b), (c) and (a), (b'), (c), respectively, in the following:

- (a)  $\beta$  is contractible in  $W_j$ .
- (b)  $\beta$  passes through  $v_1$  and  $v_2$ .
- (b')  $\beta$  passes through  $v$  essentially two times, at least.
- (c) There exists a non-trivial loop  $\alpha$  in  $U_j$  such that  $f_*(\alpha) = \beta$ .

More precisely, the condition (b') says that there exist distinct points  $\zeta_0$  and  $\zeta_1$  in  $S^1$  with  $\beta(\zeta_0) = \beta(\zeta_1) = v$  such that the restrictions  $\beta|_{I_1}$  and  $\beta|_{I_2}$  of the loop  $\beta$  are both non-trivial closed curves in  $W_j$ , where  $I_1$  and  $I_2$  are the connected component of  $S^1 \setminus \{\zeta_0, \zeta_1\}$ . In particular,  $\mathcal{T}(v)$  is empty if  $W_j$  is simply connected. And we set

$$a_j(v_1, v_2) = \inf_{\beta \in \mathcal{S}(v_1, v_2)} \ell_\Omega(\beta), \quad b_j(v) = \inf_{\beta \in \mathcal{T}(v)} \ell_\Omega(\beta) \text{ and}$$

$$a_j = \min_{v_1, v_2 \in f(C_j), v_1 \neq v_2} a_j(v_1, v_2), \quad b_j = \min_{v \in f(C_j)} b_j(v),$$

where we set  $a_j = +\infty$  if  $\#f(C_j) = 1$ .

Finally, let  $A_1, \dots, A_t$  be all of the cycles of Herman rings of  $f$ . We note here that, by Shishikura's theorem,  $0 \leq t \leq d - 2$ , in particular, if  $d = 2$  there are no Herman rings. And, since the Julia set has no isolated points, the Herman rings have finite moduli, so  $L_{A_k} > 0$  for all  $k$ .

Now we are ready to state our main theorem.

**Theorem 2.2 (Main Theorem).** *For an arbitrary rational map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  of degree  $d \geq 2$ , the following holds.*

$$L_{\Omega_f} \geq \min\{a_1, \dots, a_s, b_1, \dots, b_s, L_{A_1}, \dots, L_{A_t}\}.$$

The proof of this theorem will be given in Section 4.

For any  $\beta \in \mathcal{S}(v_1, v_2)$ , it is clear by definition that  $\ell_{\Omega}(\beta) \geq 2d_{\Omega}(v_1, v_2)$ . Similarly, for  $\beta \in \mathcal{T}(v)$ , we have  $\ell_{\Omega}(\beta) \geq 4\iota_{\Omega}(v)$ . Thus, we conclude that  $a_j(v_1, v_2) \geq 2d_{\Omega}(v_1, v_2)$  and  $b_j(v) \geq 4\iota_{\Omega}(v)$  and hence have the following

**Corollary 2.3.** *Under the same situation as the Main Theorem, it follows that*

$$L_{\Omega_f} \geq \min\{C_1, C_2, C_3\} (> 0),$$

where

$$C_1 = \min_{v_1 \neq v_2 \in f(\text{Crit}(f))} 2d_{\Omega_f}(v_1, v_2),$$

$$C_2 = \min_{v \in f(\text{Crit}(f))} 4\iota_{\Omega_f}(v), \quad \text{and}$$

$$C_3 = \min_{k=1, \dots, t} L_{A_k}.$$

*In particular, the Julia set  $J_f$  is uniformly perfect.*

*Remark 1.* As is well-known, any polynomial has no Herman rings. In general, if there exists a Herman ring  $A$ , it is known that the boundary of  $A$  is contained in the closure of forward orbits of the critical points of  $f$ . Therefore, if each critical point of  $f$  is (pre)periodic or contained in a (super)attracting or parabolic basin, then we can conclude that  $f$  has no Herman rings. We also note that a cycle of (super)attracting or parabolic components always contains a critical point, thus a component of it appears as a member of the list  $U_1, \dots, U_s$ .

*Remark 2.* Let  $B_1$  and  $B_2$  be connected components of a cycle of Herman rings  $A_j$ . Then  $B_2 = f^l(B_1)$  for some  $l \in \mathbb{N}$ . Since  $f^m : B_1 \rightarrow B_1$  is known to be analytically conjugate to an irrational rotation of a round annulus, where  $m$  is the period of  $A_j$ , we can see that  $f^l : B_1 \rightarrow B_2$  is biholomorphic. Hence,  $L_{A_j}$  is equal to the hyperbolic length of the core curve of any component of  $A_j$ .

*Remark 3.* A pair of critical values  $v_1, v_2$  can accidentally be very close to each other in  $\Omega$ , i.e.,  $d_{\Omega}(v_1, v_2)$  is very small, while  $a_j(v_1, v_2)$  is not so small. (The phenomenon  $\mathcal{T}(v) \neq \emptyset$  can be considered as a limiting case of the above situation.) So, the formulation in the above corollary does not always provide a good estimate for uniform perfectness.

## 3. BRANCHED COVERINGS AND UNIFORM PERFECTNESS

In this section, we shall investigate the connection between branched coverings and uniform perfectness. Let  $f : U \rightarrow W$  be a holomorphic (possibly branched) covering map from a (connected) hyperbolic Riemann surface  $U$  onto another  $W$ . Precisely speaking, for each point  $w \in W$  there exists an open neighborhood  $V$  of  $w$  satisfying the condition: For each component  $\tilde{V}$  of  $f^{-1}(V)$  there exist a natural number  $n \geq 1$  and conformal homeomorphisms  $\varphi : \tilde{V} \rightarrow \Delta_r$  and  $\psi : V \rightarrow \Delta_{r^n}$  with  $\psi(w) = 0$  such that  $\psi \circ f \circ \varphi^{-1}(\zeta) = \zeta^n$ , where  $\Delta_r$  denotes the disk  $\{|\zeta| < r\}$ .

When a loop  $\alpha$  is freely homotopic to another  $\alpha'$  in  $U$ ,  $f_*\alpha := f \circ \alpha$  is freely homotopic to  $f_*\alpha'$ . Therefore, the natural homomorphism  $f_* : \mathcal{C}_U \rightarrow \mathcal{C}_W$  can be defined by  $f_*[\alpha] = [f_*\alpha]$ .

First suppose that  $f$  is unbranched, then by the homotopy lifting property we can see that the induced map  $f_*$  is injective. And moreover  $\ell_U[\alpha] = \ell_W[f_*\alpha]$  because  $f$  is a local isometry, therefore we have the next

**Proposition 3.1.** *If  $f : U \rightarrow W$  is an unbranched holomorphic covering map, then  $L_U \geq L_W$ .*

In the case when  $f$  is branched, we need more efforts to estimate  $L_U$  from below. In fact, for any finitely connected planar Jordan domain  $U$ , it is known that there exists a branched holomorphic covering map from  $U$  onto the unit disk (so-called the Ahlfors map), thus  $L_U$  cannot be estimated from below by only the data of  $W$  (in this case,  $L_W = +\infty$ ).

Let  $\text{Crit}(f)$  be the set of critical points of  $f$  and for  $v_1, v_2, v \in f(\text{Crit}(f))$  with  $v_1 \neq v_2$  define the curve families  $\mathcal{S}(v_1, v_2)$  and  $\mathcal{T}(v)$  by the same way as in the previous section. And we set

$$a(v_1, v_2) = \inf_{\beta \in \mathcal{S}(v_1, v_2)} \ell_W(\beta), \quad b(v) = \inf_{\beta \in \mathcal{T}(v)} \ell_W(\beta), \quad \text{and}$$

$$a = \inf_{v_1 \neq v_2 \in f(\text{Crit}(f))} a(v_1, v_2), \quad b = \inf_{v \in f(\text{Crit}(f))} b(v).$$

Then, the following lemma is a key step to our proof of the Main Theorem.

**Lemma 3.2.** *For a non-trivial loop  $\alpha$  in  $U$  such that  $\beta = f_*\alpha$  is contractible in  $W$ , it follows that*

$$\ell_U[\alpha] \geq \min\{a, b\}.$$

*Proof.* First we show that  $\ell_U[\alpha] > 0$ . In fact, if  $\ell_U[\alpha] = 0$  then  $\alpha$  surrounds a puncture of  $U$ , in other words, there exists a holomorphic injection  $g : \Delta^* = \Delta \setminus \{0\} \rightarrow U$  such that  $\alpha$  is freely homotopic to  $\varepsilon^n$  for some integer  $n \neq 0$ , where  $\varepsilon = g(\{|\zeta| = 1/2\})$ . As is easily seen,  $f(g(\Delta^*))$  is a neighborhood of a puncture of  $W$  and thus  $\beta$  is freely homotopic to non-zero multiple of a simple loop around the puncture in  $W$ . On the other hand,  $\beta$  is contractible in  $W$ , therefore  $W$  must be conformally equivalent to the complex plane  $\mathbb{C}$ , but this is impossible because  $W$  is hyperbolic. Hence we have  $\ell_U[\alpha] > 0$ . In particular, we see that  $\ell_U[\alpha] = \ell_U(\alpha_0)$  for the closed geodesic  $\alpha_0$  freely homotopic to  $\alpha$  in  $U$ . So, for the proof, it suffices to that  $\ell_U(\alpha) \leq \min\{a, b\}$  in the case  $\alpha$  is a smooth curve. Approximating  $\alpha$  by another smooth curve if necessary, we may further assume that  $\alpha$  does not pass any critical point. Here, we should observe  $\ell_U(\alpha) \geq \ell_W(\beta)$  by the Schwarz-Pick lemma:  $f^*\rho_W \leq \rho_U$ .

Let  $p : \Delta \rightarrow W$  be a holomorphic universal covering map of  $W$  from the unit disk  $\Delta$  and set  $C = p^{-1}(f(\text{Crit}(f)))$ . Since  $\beta$  is contractible, a lift  $\tilde{\beta} : S^1 \rightarrow \Delta$  of  $\beta$  via  $p$  is closed. Let  $K$  be the holomorphically convex hull of  $\tilde{\beta}(S^1)$  in  $\Delta$ . In other words,  $K = \Delta \setminus D_0$ , where  $D_0$  is the relatively non-compact component of  $\Delta \setminus \tilde{\beta}(S^1)$  in  $\Delta$ .

Now we show that  $\#(K \cap C) \geq 2$ . If  $K \cap C$  is an empty set, then it is clear that  $\beta$  is homotopic to a point with a homotopy in  $W \setminus f(\text{Crit}(f))$ . Since  $f : U \setminus f^{-1}(f(\text{Crit}(f))) \rightarrow W \setminus f(\text{Crit}(f))$  is an unbranched covering map, this homotopy can be lifted via  $f$  to a homotopy from  $\alpha$  to a point, but this contradicts the assumption that  $\alpha$  is non-trivial. Next, suppose that  $K \cap C$  consists of one point  $\zeta_0$ . By assumption, we note that  $\zeta_0 \in K \setminus \tilde{\beta}(S^1)$ . Then it is not difficult to see that the loop  $\tilde{\beta}$  is freely homotopic to  $\varepsilon^n$  in  $\Delta \setminus C$ , where  $\varepsilon$  is a sufficiently small simple loop around  $\zeta_0$  in  $\Delta \setminus C$  and  $n$  is the winding number of  $\tilde{\beta}$  around  $\zeta_0$ . This implies that  $\beta$  is freely homotopic to  $p_*(\varepsilon^n)$  in  $W \setminus f(\text{Crit}(f))$ , therefore  $\alpha$  is freely homotopic to a loop  $\delta$  with  $f_*\delta = p_*(\varepsilon^n)$ . In particular,  $\ell_U[\alpha] \leq \ell_U(\delta)$  and the length of  $\delta$  can be arbitrarily small, therefore  $\ell_U[\alpha] = 0$ , this is not the case. Now we have proved that  $\#(K \cap C) \geq 2$ .

Here we recall that  $\tilde{\beta}$  is parametrized by  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ . For each  $\theta \in \mathbb{R}$ , we denote by  $S_\theta$  the hyperbolic segment joining  $\tilde{\beta}(1)$  and  $\tilde{\beta}(e^{i\theta})$  in  $\Delta$ . Now we define positive numbers  $\theta_+$  and  $\theta_-$  by

$$\theta_\pm = \max\{\theta \geq 0; S_{\pm u} \cap C = \emptyset \text{ for all } u \in [0, \theta]\}.$$

Then we see that  $\theta_+ + \theta_- \leq 2\pi$  and if the equality occurs we have  $\#(S_{\theta_\pm} \cap C) \geq 2$  since  $K \subset \bigcup_{t \in \mathbb{R}} S_t$  and  $\#(K \cap C) \geq 2$ . In any case, there exist distinct two points  $\tilde{v}_+$  and  $\tilde{v}_-$  such that  $\tilde{v}_\pm \in S_{\pm\theta_\pm} \cap C$ . We put  $v_\pm = p(\tilde{v}_\pm)$ .

Let  $u_n^\pm$  ( $n = 1, 2, \dots$ ) be an increasing sequence of positive numbers which converges to  $\theta_\pm$  for each signature, and  $\tilde{\beta}_n$  the curve obtained from  $\tilde{\beta}$  by replacing its subarcs  $\tilde{\beta}|_{I_n^+}, \tilde{\beta}|_{I_n^-}$  by the hyperbolic segments  $S_{u_n^+}, S_{-u_n^-}$ , respectively, where  $I_n^\pm$  denotes the subinterval  $\{e^{i\theta}; \pm\theta \in [0, u_n^\pm]\}$  of  $S^1$ . And set  $\beta_n = p_*\tilde{\beta}_n$  for each  $n = 1, 2, \dots$ . By construction,  $\beta_n$  is freely homotopic to  $\beta$  in  $W \setminus f(\text{Crit}(f))$ , and it holds that  $\ell_W(\beta_n) = \ell_\Delta(\tilde{\beta}_n) \leq \ell_\Delta(\tilde{\beta}) = \ell_W(\beta)$ . Let  $\alpha_n$  be the lift of  $\beta_n$  via  $f$  determined by  $\alpha_n(1) = \alpha(1)$ , then  $\alpha_n$  is closed and homotopic to  $\alpha$ .

Let  $\alpha' = \lim \alpha_n$  and  $\beta' = f_*\alpha'$ . Then, we note that  $\ell_W(\beta') = \lim \ell_W(\beta_n) \leq \ell_W(\beta)$ . Further, we can see that  $\beta' \in \mathcal{S}(v_+, v_-)$  or  $\beta' \in \mathcal{T}(v_+)$  according to that  $v_+ \neq v_-$  or not. Therefore, we can compute as follows.

$$\ell_U(\alpha) \geq \ell_W(\beta) \geq \ell_W(\beta') \geq \min\{a(v_+, v_-), b(v_+)\} \geq \min\{a, b\}.$$

If  $\beta = f_*\alpha$  is not contractible in  $W$ , then  $\ell_U(\alpha) \geq \ell_W(\beta) \geq L_W$ . Whence we have the following

**Corollary 3.3.** *Let  $f : U \rightarrow W$  be a holomorphic branched covering between hyperbolic Riemann surfaces  $U$  and  $W$ . Then it follows that*

$$L_U \geq \min\{L_W, a, b\},$$

where the constants  $a$  and  $b$  are as in the above.

## 4. PROOF OF THE MAIN THEOREM

Let  $\alpha$  be a non-trivial closed curve in  $\Omega = \Omega_f$ . In order to prove our main theorem, we should show that  $\ell_\Omega(\alpha) \geq C$ , where  $C = \min\{a_1, \dots, a_s, b_1, \dots, b_s, L_{A_1}, \dots, L_{A_t}\}$ . We denote by  $\alpha_n$  the image  $f^n \circ \alpha = (f^n)_*(\alpha)$  of  $\alpha$  under the  $n$ -th iterate of  $f$ . We note here that  $\ell_\Omega(\alpha) \geq \ell_\Omega(\alpha_1) \geq \ell_\Omega(\alpha_2) \dots$  by the Schwarz-Pick lemma. Let  $U$  be the component of  $\Omega$  containing  $\alpha$ . Then, by Sullivan's No Wandering Domains Theorem,  $U$  is eventually periodic, i.e.,  $D = f^k(U)$  is a periodic component for some integer  $k$ . As is well-known, a periodic component  $D$  is one of the following:

1. a (super)attracting immediate basin. In this case, the sequence of curves  $\alpha_n$  is attracted to a (super)attracting cycle (in  $\Omega$ ), in particular,  $\alpha_n$  is contractible in  $\Omega$  for sufficiently large  $n$ .
2. a parabolic immediate basin. In this case, a subsequence of  $\alpha_n$  is absorbed by a simply connected attracting petal (in  $\Omega$ ), therefore  $\alpha_n$  is contractible in  $\Omega$ , too, for sufficiently large  $n$ .
3. a Siegel disk. In this time,  $D$  is simply connected itself, thus  $\alpha_k$  is of course contractible in  $D$ .
4. a Herman ring.

Hence, we can conclude that if  $\alpha_n$  is non-trivial for any  $n$ , then  $\alpha_n$  is contained in a cycle of Herman rings  $A_j$  for sufficiently large  $n$ . In this case,  $\alpha_n$  is freely homotopic to a non-zero multiple of the core curve  $\beta$  of a component of  $A_j$ , thus  $\ell_\Omega(\alpha_n) \geq \ell_{A_j}(\beta) = L_{A_j}$ .

In particular, we have  $\ell_\Omega(\alpha) \geq L_{A_j} \geq C$ .

Otherwise, there exists an integer  $n \geq 0$  such that  $\alpha_n$  is non-trivial while  $\alpha_{n+1}$  is trivial in  $\Omega$ . Since  $\ell_\Omega(\alpha) \geq \ell_\Omega(\alpha_n)$ , we may assume that  $n = 0$ , in other words,  $\alpha$  is non-trivial but  $\beta = f_*(\alpha)$  is contractible in  $\Omega$ . If  $f : U \rightarrow W := f(U)$  is unbranched covering, a homotopy connecting  $\beta$  with a constant curve in  $W$  can be lifted to a homotopy connecting  $\alpha$  with a constant curve in  $U$  via  $f$ , thus  $\alpha$  is contractible in  $U$ , this is a contradiction. Therefore,  $f : U \rightarrow W$  must be branched, i.e.,  $U = U_j$  for some  $j = 1, \dots, s$ . Now we can apply the key lemma in the previous section! By Lemma 3.2, we have  $\ell_\Omega(\alpha) \geq \min\{a_j, b_j\} \geq C$ , thus the proof is now completed.

## 5. APPLICATIONS

We now present some applications of the main result. For a rational map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  of degree  $d \geq 2$ , we have seen that  $L_{\Omega_f} \geq C$ , where  $\Omega_f$  denotes the Fatou set  $\widehat{\mathbb{C}} \setminus J_f$  and  $C > 0$  is the constant which appears in Theorem 2.2 or Corollary 2.3.

First of all, we state a result concerning Hausdorff dimension. The following theorem is essentially due to Järvi-Vuorinen [9], while a quantitative version as in the following can be found in [15].

**Theorem 5.1.** *The Hausdorff dimension of the Julia set  $J_f$  of a rational map  $f$  can be estimated as*

$$\text{H-dim}(J_f) \geq \frac{\log 2}{\log(2e^{M_{\Omega_f}^2} + 1)} \geq \frac{\log 2}{M_{\Omega_f}^2 + \log 3} \geq \frac{\log 2}{\pi^2/L_{\Omega_f} + \log 3}.$$

In particular, any rational map of degree  $\geq 2$  has always the Julia set of positive Hausdorff dimension. This is a well-known fact and is also shown in [4] by uniform perfectness of the Julia set in another context.

The next theorem ensures the regularity of the Julia set in the sense of Dirichlet (cf. [17]) by Wiener's criterion.

**Theorem 5.2** (Pommerenke [12]. See also [15]). *Let  $f$  be a rational map of degree at least two. Then, for each point  $a \in J_f$  and  $0 < r < \text{diam} J_f$ , it holds that  $\text{Cap}(J_f \cap B(a, r)) \geq cr$ , where  $c \leq 1$  is a constant satisfying  $\log 1/c \leq M_{\Omega_f}^0 + 7 \log 2$ ,  $\text{diam}$  stands for the Euclidean diameter,  $\text{Cap}$  the logarithmic capacity and  $B(a, r)$  is the closed disk centered at  $a$  with radius  $r$ .*

In fact, the above property characterizes uniform perfectness of  $J_f$  (see [12]). Similarly, we can state a characterization of uniform perfectness of the closed set in terms of Hausdorff contents [9] (see also [15]).

Finally, we mention the estimate of the hyperbolic (or Poincaré) metric  $\rho(z)|dz| = \rho_{\Omega}(z)|dz|$  of  $\Omega = \Omega_f$  in terms of the distance function  $\delta(z) = \delta_{\Omega}(z) = \text{dist}(z, J_f) = \inf_{a \in J_f} |z - a|$ , provided that  $\infty \in J_f$ . It is always true that  $\rho(z) \leq 1/\delta(z)$ . On the other hand, if  $\Omega$  is simply connected, it is well-known that  $\rho(z) \geq 1/4\delta(z)$ , while this kind of inequality need not hold in general, even in the case  $\partial\Omega$  is a perfect set. But this is true in our situation, indeed the validity of this inequality characterizes uniform perfectness.

**Theorem 5.3** (cf. [15]). *For a rational map  $f$  of degree at least two, we set  $L = L_{\Omega_f}$ . If  $\infty \in J_f$  then we have*

$$\frac{1}{4} \tanh L/2 \leq \inf_{z \in \Omega_f} \rho_{\Omega_f}(z) \delta_{\Omega_f}(z) \leq \frac{\sqrt{3}L}{\sqrt{\pi^2 + 4L^2}}.$$

For other applications and characterizations of uniform perfectness, see [15] and its references.

## 6. QUADRATIC POLYNOMIALS

In this section, as the simplest example, we shall consider the quadratic polynomials  $f(z) = f_c(z) = z^2 + c$  and attempt to give concrete lower and upper bounds for the uniform perfectness constant  $L_{\Omega_f}$  (abbreviated by  $L_c$ ) of the Julia set  $J_c$  of  $f_c$  in terms of the parameter  $c$ . (For a general rational map  $f$ , we may estimate  $L_{\Omega_f}$  in the similar way as below, in principle.) For general results of the dynamics of quadratic polynomials, the reader will find a good account in the book [4] by Carleson and Gamelin.

Since  $f_c$  is a polynomial, the point at infinity is a superattracting fixed point of  $f_c$ . And 0 is a unique finite critical point of  $f_c$  and  $c$  is the corresponding critical value. Let  $\mathcal{M}$  denote the Mandelbrot set  $\{c \in \mathbb{C}; (f_c^n(0))_{n=1,2,\dots}$  is a bounded sequence $\}$ . As is well-known,  $c \in \mathcal{M}$  if and only if the Julia set  $J_c$  is connected, in which case  $L_c = +\infty$  since  $\Omega = \Omega_c := \widehat{\mathbb{C}} \setminus J_c$  is simply connected, thus we have nothing to do. So we assume that  $c \notin \mathcal{M}$  in the sequel. In this case, the Julia set  $J_c$  is a Cantor set, therefore the Fatou set  $\Omega_c$  is connected. In order to estimate  $L_c$  from below, by Corollary 2.3, it is sufficient to estimate  $d_{\Omega}(c, \infty)$ ,  $\iota_{\Omega}(c)$  and  $\iota_{\Omega}(\infty)$  from below. To accomplish it, we may utilize the monotonicity property of the hyperbolic metrics. If we find a hyperbolic domain  $\tilde{\Omega}$



containing  $\Omega$  which is easier to estimate its hyperbolic metric, then  $\rho_\Omega \geq \rho_{\tilde{\Omega}}$  by the Schwarz-Pick lemma. Therefore, it holds that  $d_\Omega(a, b) \geq d_{\tilde{\Omega}}(a, b)$  for any  $a, b \in \Omega$ . On the other hand, it is not always true that  $\iota_\Omega(a) \geq \iota_{\tilde{\Omega}}(a)$ , but we can avoid this difficulty as follows. Fix  $a \in \Omega$ . Let  $D$  be an arbitrary simply connected subdomain of  $\Omega$  containing  $a$ , then we have

$$\iota_\Omega(a) \geq \inf_{w \in \partial D} d_\Omega(a, w) \geq \inf_{w \in \partial D} d_{\tilde{\Omega}}(a, w).$$

The most useful (but not necessarily sufficient) domain  $D$  is thought to be a thrice punctured sphere, since it has been studied for a long time and its hyperbolic metric can be expressed almost explicitly (see, for example, [1] and [3]). Any thrice punctured sphere is conformally (indeed, Möbius) equivalent to the canonical one:  $D_0 = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ . The following is the precise version of Landau's theorem due to Hempel [6]: The hyperbolic metric  $\rho_0(z)|dz|$  of  $D_0$  satisfies

$$(6.1) \quad \rho_0(z) \geq \frac{1}{2|z|(|\log|z|| + K)},$$

where  $K = \Gamma(\frac{1}{4})^4/4\pi^2 = 4.3768796 \dots$ , and the equality occurs if  $z = -1$ . Note that this estimate is efficient only on the half plane  $\operatorname{Re} z \leq \frac{1}{2}$ , otherwise we have only to use the functional equation  $\rho_0(1-z) = \rho_0(z)$ .

In order to find out a thrice punctured sphere containing  $\Omega$ , we have only to specify three points  $a_1, a_2, a_3$  in the Julia set, for example, repelling periodic points or their inverse images. In our present case, any periodic point is repelling since  $c \notin \mathcal{M}$ . For example, fixed points of  $f_c$  are solutions of the equation:  $z^2 + c = z$ , thus  $(1 \pm \sqrt{1-4c})/2$ . We note that if  $\alpha$  is in  $J_c$ , so is  $-\alpha$ . If we selected the three points  $a_1, a_2, a_3$  in the Julia set, let  $T$  be the Möbius transformation mapping  $a_1, a_2$  and  $a_3$  to  $0, 1$  and  $\infty$ , respectively. Then,  $d_{\Omega_c}(c, \infty) \geq d_{\widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3\}}(c, \infty) = d_0(T(c), T(\infty))$ , where  $d_0$  denotes the hyperbolic distance in  $D_0$ , however it seems impossible to estimate  $\iota_{\Omega_c}(c)$  and  $\iota_{\Omega_c}(\infty)$  by only the data  $a_j$ .

For simplicity, we further assume that  $c < -2$  for a moment. Set  $\alpha = (1 + \sqrt{1-4c})/2$  and  $\beta = 1 - \alpha$ , then these are fixed points of  $f = f_c$ . Then we see that  $S = \bigcup_{n=1}^{\infty} f^{-n}(\alpha) \subset [-\alpha, \alpha]$ , hence  $J_c \subset [-\alpha, \alpha]$  since  $\bar{S} = J_c$ .

For the later convenience, we set  $t = \sqrt{1-4c} - 3 > 0$ . Let  $T(z) = \frac{(\beta-\alpha)(z+\alpha)}{(\beta+\alpha)(z-\alpha)} = (3+t)\frac{\alpha+z}{\alpha-z}$ . Then  $T(\infty) = -(3+t)$  and  $T(c) = -\frac{t(3+t)}{4+t}$ . We also note that  $T(J_c) \subset [0, \infty]$ . Using (6.1), we can calculate as

$$d_{\Omega_c}(c, \infty) \geq d_0(T(c), T(\infty)) = \int_{T(\infty)}^{T(c)} \rho_0(x) dx \geq \frac{1}{2}Q(t),$$

where

$$Q(t) = \begin{cases} \log \frac{\log(3+t) + K}{K} + \log \frac{\log \frac{4+t}{t(3+t)} + K}{K} & \text{if } t \leq t_0 \\ \log \frac{\log(3+t) + K}{\log \frac{4+t}{t(3+t)} + K} & \text{if } t > t_0 \end{cases}$$

and  $t_0 = 0.38297 \dots$  is the positive root of the equation  $3+t = (4+t)/t(3+t)$ .

Next, we shall estimate the injectivity radii of  $\Omega_c$  at  $c$  and  $\infty$ . As a preparation, we consider the quantity  $h(a) = \iota_{D_0}(-a)$  for  $a > 0$ . First we assume that  $0 < a \leq 1$ . Let

$\Delta$  be the domain defined by  $\{\tau \in \mathbb{H}; 0 < \operatorname{Re}\tau < 1, |\tau - \frac{1}{2}| > \frac{1}{2}\}$ , and  $\lambda : \Delta \rightarrow \mathbb{H}$  the conformal homeomorphism from  $\Delta$  onto the upper half plane  $\mathbb{H}$  which maps  $0, 1, \infty$  to  $1, \infty, 0$ , respectively. We denote by  $g : \mathbb{H} \rightarrow \Delta$  the inverse map of  $\lambda$ . Then, as is well-known,  $\lambda$  is analytically continued to the universal covering map of  $D_0$  from  $\mathbb{H}$  by the reflection principle, in particular,  $1/2\operatorname{Im}\tau = \rho_0(\lambda(\tau))|\lambda'(\tau)|$ . The map  $\lambda$  is nothing but the classical elliptic modular function. For the point  $\tau_0 = (e^{i\theta} + 1)/2 = g(1+a) \in g((1, 2])$  ( $\pi/2 \leq \theta < \pi$ ) we can see that  $d_{\mathbb{H}}(\tau_0, g((0, 1))) \leq d_{\mathbb{H}}(\tau_0, g((-\infty, 0)))$  and that the shortest hyperbolic segment  $\gamma$  connecting  $\tau_0$  and  $g((0, 1)) = \{ti; t > 0\}$  is contained in  $\{\tau \in \Delta; \operatorname{Re}\tau \leq \frac{1}{2}\}$ . Noting that  $\lambda(\{\tau \in \Delta; \operatorname{Re}\tau = \frac{1}{2}\}) = \{z \in \mathbb{H}; |z-1| = 1\}$ , we have  $h(a) = \iota_{D_0}(-a) = \iota_{D_0}(1+a) = \int_{\lambda_*\gamma} \rho_0(z)|dz|$  and  $\lambda_*\gamma$  is contained in  $\{z \in \mathbb{H}; |z-1| \leq 1\}$ . Denote by  $\beta$  the closed curve obtained as the union of  $1 - \lambda_*\gamma$  and its complex conjugate, then  $|\beta| \leq 1$  and  $2h(a) = \int_{\beta} \rho_0(z)|dz|$ . Note that  $|dz| \geq (|dr| + r|d\theta|)/\sqrt{2}$ , where  $z = re^{i\theta}$ . Put  $a_0 = \min |\beta|$ , then by (6.1) we have

$$\begin{aligned} 2h(a) &\geq \int_{\beta} \frac{|dz|}{2|z|(-\log|z| + K)} \geq \int_{\beta} \frac{|dr| + r|d\theta|}{2r\sqrt{2}(-\log r + K)} \\ &\geq \frac{2}{2\sqrt{2}} \log \left( \frac{-\log a_0 + K}{-\log a + K} \right) + \frac{1}{2\sqrt{2}} \frac{2\pi}{-\log a_0 + K} \\ &\geq \frac{\pi/\sqrt{2}}{-\log a + K}, \end{aligned}$$

because  $K > \pi$ .

Next, we consider the case  $a > 1$ . Since the Möbius transformation  $z \mapsto 1/z$  preserves  $D_0$ , we have  $h(a) = h(1/a)$ , thus  $h(a) \geq \pi/2\sqrt{2}(\log a + K)$ . Therefore, for any  $a > 0$ , we have

$$\iota_{D_0}(-a) \geq \frac{\pi}{2\sqrt{2}(|\log a| + K)}.$$

Letting  $D = \mathbb{C} \setminus [0, \infty)$ , we can estimate the injectivity radius of  $\Omega_c$  at  $T^{-1}(-a)$  as follows:

$$\iota_{\Omega_c}(T^{-1}(-a)) = \iota_{T(\Omega_c)}(-a) \geq \inf_{w \in \partial D} d_0(-a, w) = \iota_{D_0}(-a) \geq \frac{\pi}{2\sqrt{2}(|\log a| + K)}.$$

Hence,

$$\begin{aligned} &\min\{2d_{\Omega_c}(c, \infty), 4\iota_{\Omega_c}(c), 4\iota_{\Omega_c}(\infty)\} \\ (6.2) \quad &\geq R(t) := \min \left\{ Q(t), \frac{\sqrt{2}\pi}{\log(3+t) + K}, \frac{\sqrt{2}\pi}{\log \frac{4+t}{t(3+t)} + K} \right\} \\ &= \begin{cases} \frac{\sqrt{2}\pi}{\log \frac{4+t}{t(3+t)} + K} & \text{if } 0 < t < 0.12626\dots, \\ \log \frac{\log(3+t) + K}{K} + \log \frac{\log \frac{4+t}{t(3+t)} + K}{K} & \text{if } 0.12626\dots < t \leq t_0, \\ \log \frac{\log(3+t) + K}{\log \frac{4+t}{t(3+t)} + K} & \text{if } t_0 = 0.38297\dots < t, \end{cases} \end{aligned}$$

where  $K = \Gamma(\frac{1}{4})^4/4\pi^2 = 4.3768796\dots$ .

In contrast, the estimation of  $L_c$  from above is rather easy. In the same assumption as the above, we set  $\gamma = \sqrt{-c - \alpha} > 0$ . Note that  $\gamma \in J_c \subset [-\alpha, \alpha]$  since  $f_c(\gamma) = -\alpha \in J_c$ . Then, for  $x \in (-\gamma, \gamma)$ , we see that  $f_c(x) = x^2 + c < \gamma^2 + c = -\alpha$ , hence  $(-\gamma, \gamma) \subset \Omega_c$ . This implies that the annulus  $A = \widehat{\mathbb{C}} \setminus ([-\alpha, -\gamma] \cup [\gamma, \alpha])$  separates the Julia set  $J_c$ . The Möbius transformation  $T(z) = \frac{\gamma+z}{\alpha-z}$  maps  $A$  onto Teichmüller's extremal domain  $\widehat{\mathbb{C}} \setminus ([-r_1, 0] \cup [r_2, +\infty])$ , where  $r_1 = (\alpha - \gamma)/2\alpha$  and  $r_2 = 2\gamma/(\alpha - \gamma)$ . Thus we have  $m(A) = 2\mu(\sqrt{r_1/(r_1 + r_2)})$ , where  $\mu(r)$  denotes the modulus of Grötzsch's extremal domain  $\mathbb{D} \setminus [0, r]$  for  $0 < r < 1$ , where  $\mathbb{D}$  denotes the unit disk (see [10]). The behaviour of the function  $\mu(r)$  is well understood. Amongst them, it will be useful to record the following (cf. [10]):

$$\log \frac{(1 + \sqrt{1 - r^2})^2}{r} < \mu(r) < \log \frac{2(1 + \sqrt{1 - r^2})}{r} < \log \frac{4}{r} \quad \text{and}$$

$$\mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}.$$

In particular, we can see that

$$m(A) = 2\mu\left(\frac{\alpha - \gamma}{\alpha + \gamma}\right) = \frac{\pi^2}{\mu(\gamma/\alpha)}.$$

Therefore, we have  $L_c \leq \pi^2/M_{\Omega_c} \leq \pi^2/2\mu((\alpha - \gamma)/(\alpha + \gamma)) = \mu(\gamma/\alpha)$ . Noting Corollary 2.3, we summarize the results obtained above.

**Theorem 6.1.** *For  $c < -2$ , the Fatou set  $\Omega_c$  of  $f_c(z) = z^2 + c$  satisfies*

$$(6.3) \quad R(t) \leq L_{\Omega_c} \leq \frac{\pi^2}{2\mu\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right)} = \mu\left(\frac{\gamma}{\alpha}\right),$$

where  $t = \sqrt{1 - 4c} - 3 > 0$ ,  $\alpha = (1 + \sqrt{1 - 4c})/2 > 2$ ,  $\gamma = \sqrt{-c - \alpha} > 0$ ,  $R(t)$  is the function defined by (6.2) and  $\mu(r)$  denotes the modulus of Grötzsch's extremal domain  $\mathbb{D} \setminus [0, r]$ .

*Remark .* Since  $(\alpha - \gamma)/(\alpha + \gamma) = 2\alpha/(\alpha + \gamma)^2 \sim 1/t$ , the upper bound in (6.3) behaves like  $\pi^2/2 \log t$  as  $t \rightarrow \infty$ , while  $R(t) \sim 4/t \log t$ .

When  $t \rightarrow +0$ , the upper bound in (6.3) is  $\frac{1}{2} \log 1/t + O(1)$  since  $\gamma/\alpha = \sqrt{t/8}(1 + O(t))$ , however  $R(t) \sim \sqrt{2\pi}/\log 1/t$ .

The badness of the lower bound  $R(t)$  is mainly caused by having replaced the Julia set with only three points in the estimation when the critical values are very close to the Julia set.

In order to get a result for any  $c \in \mathbb{C} \setminus \mathcal{M}$ , we can use the following fundamental property of the uniform perfectness constants (cf. [15]).

**Proposition 6.2.** *The constants  $M_{\Omega}$  and  $L_{\Omega}$  are quasi-invariant, in other words, if  $f : \Omega \rightarrow \Omega'$  is a  $K$ -quasiconformal homeomorphism ( $K \geq 1$ ) then*

$$M_{\Omega}/K \leq M_{\Omega'} \leq KM_{\Omega} \quad \text{and} \quad L_{\Omega}/K \leq L_{\Omega'} \leq KL_{\Omega}.$$

Next we consider a holomorphic motion of the Julia set, which is an important tool introduced by Mañé-Sad-Sullivan. For a subset  $E$  of  $\widehat{\mathbb{C}}$  and a pointed hyperbolic Riemann surface (or, more generally, complex hyperbolic manifold)  $(X, x_0)$  with hyperbolic distance  $d_X$ , a map  $F : X \times E \rightarrow \widehat{\mathbb{C}}$  is called a *holomorphic motion* of  $E$  parametrized by  $(X, x_0)$  if the following holds:

1.  $F(\cdot, a) : X \rightarrow \widehat{\mathbb{C}}$  is holomorphic for each  $a \in E$ ,
2.  $F_x := F(x, \cdot) : E \rightarrow \widehat{\mathbb{C}}$  is injective for each  $x \in X$ , and
3.  $F_{x_0}$  is the identity map of  $E$ .

We can state the optimal  $\lambda$ -lemma proved by Ślodkowsky [14] (see also [5]) as in the following form.

**Theorem 6.3** (Optimal  $\lambda$ -lemma). *Let  $F$  be a holomorphic motion of a subset  $E$  of the Riemann sphere parametrized by a simply connected hyperbolic Riemann surface  $X$  with basepoint  $x_0$ . Then,  $F$  can be extended to a holomorphic motion  $\widetilde{F}$  of the whole sphere  $\widehat{\mathbb{C}}$  parametrized by  $(X, x_0)$  with the following properties.*

1.  $\widetilde{F} : X \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is (jointly) continuous,
2. for each  $x \in X$ , the map  $\widetilde{F}_x : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is a quasiconformal homeomorphism with Beltrami coefficient  $\mu_x = \partial_{\bar{z}} \widetilde{F}_x / \partial_z \widetilde{F}_x$  satisfying  $d_T(\mu_x, 0) \leq d_X(x, x_0)$ , where  $d_T$  denotes the Teichmüller distance

$$d_T(\mu, \nu) = \operatorname{ess. sup}_{z \in \mathbb{C}} d_{\mathbb{D}}(\mu(z), \nu(z)) = \operatorname{arctanh} \left( \left\| \frac{\mu - \nu}{1 - \bar{\nu}\mu} \right\|_{\infty} \right).$$

Now we construct a holomorphic motion of the Julia set by the standard method (see, for instance, [5]). As is well-known (cf. [4]), the functions  $\Phi_n(c) := (f_c^n(c))^{2^{-n}}$  converges locally uniformly in  $\mathbb{C} \setminus \mathcal{M}$  to a holomorphic function  $\Phi(c)$ , which is, in turn, a conformal mapping of  $\mathbb{C} \setminus \mathcal{M}$  onto  $D := \{z \in \mathbb{C}; |z| > 1\}$ , where we take the branch  $\Phi_n$  so as to  $\Phi_n(c) = c + O(1)$  as  $c \rightarrow \infty$ . By the symmetry of the Mandelbrot set  $\mathcal{M}$ , we note that  $\overline{\Phi(\bar{z})} = \Phi(z)$ , in particular,  $\Phi((-\infty, -2)) = (-\infty, -1)$ .

Now we define the function  $p : H = \{\zeta \in \mathbb{C}; \operatorname{Re} \zeta > 0\} \rightarrow D$  by  $p(\zeta) = -e^\zeta$ , then  $q := \Phi^{-1} \circ p$  is a universal covering map of  $\mathbb{C} \setminus \mathcal{M}$  from the right half plane  $H$ . Fix an arbitrary point  $c_1$  in  $\mathbb{C} \setminus \mathcal{M}$ . Then there exists a  $\zeta_1$  in  $H$  such that  $q(\zeta_1) = c_1$  and  $-\pi < \operatorname{Im} \zeta_1 \leq \pi$ . Set  $\zeta_0 = |\zeta_1|$ ,  $c_0 = q(\zeta_0)$  and let  $E_0$  be the set of repelling periodic points of  $f_{c_0}$ . Then, considering the roots of the equation  $f_c^n(z) = z$ , where  $c = q(\zeta)$  and  $n$  is taken over all positive integers, we can obtain a holomorphic motion of the set  $E_0$  parametrized by the right half plane  $H$  with basepoint  $\zeta_0$  because  $f_c$  has no parabolic periodic points for  $c \in \mathbb{C} \setminus \mathcal{M}$ , thus the roots do not collide. By the optimal  $\lambda$ -lemma, we have a holomorphic motion  $F : H \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that  $F_{\zeta_0} = \operatorname{id}$  and that  $F_\zeta(E_0)$  is the set of repelling periodic points of  $f_{q(\zeta)}$ . (We can take an  $F$  compatible with the dynamics, thus  $F_\zeta$  is a quasiconformal conjugate of  $f_{c_0}$  to  $f_{q(\zeta)}$ , but we do not this property here.) Since the set of repelling periodic points is dense in the Julia set,  $F_\zeta(J_{c_0}) = J_{q(\zeta)}$  holds. Hence, by Proposition 6.2 and the second property in Theorem 6.3, we have

$$1/K \leq L_{\Omega_{c_1}} / L_{\Omega_{c_0}}, M_{\Omega_{c_1}} / M_{\Omega_{c_0}} \leq K,$$

where  $\frac{1}{2} \log K = d_H(\zeta_1, \zeta_0)$ .

Now we compute  $K$ . We write  $\zeta_1 = |\zeta_1|e^{i\theta} = z_0e^{i\theta}$  with  $\theta \in (-\pi/2, \pi/2)$ . Then we can calculate as

$$d_H(\zeta_1, \zeta_0) = \operatorname{arctanh} \left( \left| \frac{\zeta_1 - \zeta_0}{\zeta_1 + \zeta_0} \right| \right) = \operatorname{arctanh} \left( \tan \frac{\theta}{2} \right) = \frac{1}{2} \log \left( \frac{1 + \sin \theta}{\cos \theta} \right).$$

Now we have shown the following.

**Theorem 6.4.** *For an arbitrary  $c \in \mathbb{C} \setminus \mathcal{M}$ , take a point  $\zeta = re^{i\theta}$  with  $\theta \in (-\pi/2, \pi/2)$  such that  $-e^\zeta = \Phi(c)$  and that  $-\pi < \operatorname{Im} \zeta = r \sin \theta \leq \pi$ . Then, we have the following estimates:*

$$L_{\Omega_{c_0}}/K \leq L_{\Omega_c} \leq KL_{\Omega_{c_0}} \quad \text{and} \quad M_{\Omega_{c_0}}/K \leq M_{\Omega_c} \leq KM_{\Omega_{c_0}},$$

where  $c_0 < -2$  is the number determined by  $\Phi(c_0) = -e^r$ , and

$$K = \frac{1 + \sin \theta}{\cos \theta}.$$

We remark that  $\log |\Phi(c)| = r \cos \theta$  is Green's function of the domain  $\widehat{\mathbb{C}} \setminus \mathcal{M}$  with pole at the infinity.

#### REFERENCES

- [1] AHLFORS, L. V. *Conformal Invariants*, McGraw Hill, New York (1973).
- [2] BEARDON, A. F. *Iteration of Rational Functions*, No. 132 in Grad. Texts Math., Springer-Verlag (1991).
- [3] BEARDON, A. F. and POMMERENKE, CH. The Poincaré metric of plane domains, *J. London Math. Soc. (2)*, **18** (1978), 475–483.
- [4] CARLESON, L. and GAMELIN, T. W. *Complex Dynamics*, Springer-Verlag (1993).
- [5] DOUADY, A. Prolongement de mouvements holomorphes [d'après Ślodkowski et autres], *Astérisque*, **227** (1995), 7–20.
- [6] HEMPEL, J. A. The Poincaré metric on the twice punctured plane and the theorems of Landau and Schottky, *J. London Math. Soc. (2)*, **20** (1979), 435–445.
- [7] HINKKANEN, A. Julia sets of rational functions are uniformly perfect, *Math. Proc. Cambridge Philos. Soc.*, **113** (1993), 543–559.
- [8] HINKKANEN, A. and MARTIN, G. J. Julia sets of rational semigroups, *Math. Z.*, **222** (1996), 161–169.
- [9] JÄRVI, P. and VUORINEN, M. Uniformly perfect sets and quasiregular mappings, *J. London Math. Soc.*, **54** (1996), 515–529.
- [10] LEHTO, O. and VIRTANEN, K. I. *Quasiconformal Mappings in the Plane, 2nd Ed.*, Springer-Verlag (1973).
- [11] MAÑÉ, R. and DA ROCHA, L. F. Julia sets are uniformly perfect, *Proc. Amer. Math. Soc.*, **116** (1992), 251–257.
- [12] POMMERENKE, CH. Uniformly perfect sets and the Poincaré metric, *Ark. Math.*, **32** (1979), 192–199.
- [13] POMMERENKE, CH. On uniformly perfect sets and Fuchsian groups, *Analysis*, **4** (1984), 299–321.
- [14] ŚLODKOWSKY, Z. Holomorphic motions and polynomial hulls, *Proc. Amer. Math. Soc.*, **111** (1991), 347–355.
- [15] SUGAWA, T. Various domain constants related to uniform perfectness, To appear in *Complex Variables*.
- [16] SUGAWA, T. Uniform perfectness of the limit sets of Kleinian groups, Preprint (1997).
- [17] TSUJI, M. *Potential Theory in Modern Function Theory*, Maruzen, Tokyo (1959).

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, 606-8502 KYOTO, JAPAN  
E-mail address: sugawa@kum.kyoto-u.ac.jp