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無限人非協力ゲームの均衡点の唯一性

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1 Introduction

A non-cooperative game with a continuum of players is an ideal representation of strategic situations where each player's strategy is relatively negligible but aggregated strategies affect on his payoff. However, if non-atomic games implied the same result as the corresponding finite game, it would be sufficient to study non-cooperative game with many but finite players and the formulation with a continuum of players would be less attractive for the researchers.

One of the appealing features of non-atomic games is existence of a pure strategy equilibrium. This result is obtained in several formulations of a game with a continuum of players. Schmeidler (1973) shows that there exists a pure strategy equilibrium if every player's payoff depends on his own strategy and the integral of the strategy profile. Rath (1992) reformulates this case and shows the direct proof of the existence.

In this paper, we show sufficient conditions of the uniqueness of the equilibrium in Schmeidler and Rath's formulation. We show the conditions of players' payoffs for the uniqueness of the equilibrium. In the game with finite players, these conditions of payoffs does not always imply the uniqueness of the equilibrium. Thus, this uniqueness of the equilibrium can be regarded as another appealing feature of the game with a continuum of players.

This paper is an intoroduction paper to the results of Watanabe (1997). In this paper we focus to sufficient conditions of uniqness for the interior equilibrium on case of n strategies and show the sketch of the proof for main theorems. However proofs of lemmas are omitted. Watanabe (1997) shows the all proofs.

2 Notations and Definitions

Let $(T, \mathcal{B}, \lambda)$ be a player space where T is a uncountable set in a complete separable metric space, $\mathcal{B}$ is a $\sigma-$algebra on T and λ is an atomless probability measure on $\mathcal{B}$. Let $E = \{e^1, \ldots, e^n\}$ be the finite set of strategies where e^i is the ith unit vector in $\mathcal{R}^n$. A strategy profile is a measurable function from T to E. The set of all strategy profiles is denoted by F. Let $s(f)$ be an average strategy for a strategy profile $f \in F$ defined by

$$s(f) = \int_T f d\lambda = (\int_T f_1 d\lambda, \ldots, \int_T f_n d\lambda).$$

Then $S = \{s(f) | f \in F\}$ is the unit simplex in $\mathcal{R}^n$. A payoff function is a real valued function defined on $E \times S$ which is continuous on S. Let $\mathcal{U}$ be the set of all payoff functions. We introduce sup norm topology on $\mathcal{U}$. A game g is defined as a measurable function from T to $\mathcal{U}$. Thus, for a given g, $g(t)(e^i, q)$ means a payoff of player $t \in T$ when his strategy is $e^i \in E$ and an average strategy is $q \in S$.

Definition 2.1 *A $f \in F$ is said to be a Nash equilibrium of a game g, if and only if,*

$$\lambda(\{t \in T | \; g(t)(f(t), f) \geq g(t)(e^j, f) \quad \textit{for all} \quad e^j \in E \;\}) = 1.$$

The existence of pure strategy equilibria shown in the sequential studies (e.g. Schmeidler (1972) and Rath (1992)) with the unit interval is easily extended to our model with an uncountable set in a complete separable metric space, since preserving upperhemicontinuity of integrations shown by Aumann(1976), which is a key of the proof, can be extended to a set in a complete separable metric space endowed with an atomless measure (see Hildenbrand (1974)).

Theorem 2.1 (Schmeidler(1973) and Rath(1992)) *There exists pure strategy equilibria for any game.*

Hence, in the following we only consider about pure strategies. As the definition of the equilibrium, two strategy profiles which is different only on the nullsets are the same strategy profiles in the game with a continuum of players. Thus, we consider that there exists the unique equilibrium if any equilibrium has identical value outside the null sets. Formally, we define the uniqueness of the equilibrium as follows.

Definition 2.2 *For any game g, we say that the equilibrium of g is unique if for any equilibrium f and f' in g,*

$$\lambda(\{t \in T | f(t) \neq f'(t)\}) = 0.$$

Rath (1992) defined a best response correspondence from the set of average strategies to the set of average strategies and showed the excellent proof of existence of the equilibrium. Considering this correspondence makes analysis of the game easier than using the correspondence from the set of the strategy profiles as finite games. We also use this best response correspondence.

Let Γ be a correspondence from S to S defined by

$$\Gamma(q) = \{\int f d\lambda | \ f(t) \in B(t,q), \quad \text{for almost all} t \in T \ \}$$

where

$$B(t,q) = \{e^i \in E | \ g(t)(e^i,q) \geq g(t)(e^j,q) \quad \text{for any} e^j \in E \ \}$$

Thus, Γ is the best response correspondence for an average strategy. q is said to be a fixed point of Γ if and only if $q \in \Gamma(q)$. Rath (1992) shows that a strategy profile f is an equilibrium if $s(f)$ is a fixed point of Γ. However, there may be several several strategy profiles which have the same average strategy. The following condition implies that the strategy profile is uniquely determined outside the null sets for the fixed point of Γ.

Condition N A game g satisfies Condition N if for any $e^i, e^j \in E$, $e^i \neq e^j$ and any $q \in S$, $\lambda(\{t \in T | g(t)(e^j,q) = g(t)(e^i,q)\}) = 0$

Condition N means that the set of players who have two indifferent strategies is an null set for any average strategy.

Lemma 2.1 *If a game g satisfies condition N and the fixed point of Γ of g is unique, then the equilibrium of the game g is unique.*

proof. Let us consider that two equilibrium $f, f' \in F$. Since Γ has the unique fixed point and an average strategy of an equilibrium is the fixed point of Γ, we have $s(f) = s(f')$. Suppose $s(f) = s(f') = q$ and we define the subset of T, T_a by $T_a = \{t \in T | f(t) \neq f'(t)\}$

We have to show $\lambda(T_a) = 0$. Let T_b be the set of players whose strategies are the best response of q. This can be written as $T_b = \{t \in T | t \in B(t,q)\}$ and by definition of the equilibrium we have $\lambda(T \setminus T_b) = 0$.

For any $t \in T_a \cap T_b$ and any $e^j \in E$, we have $g(t)(f(t),q) \geq g(t)(e^j,q)$ and $g(t)(f'(t),q) \geq g(t)(e^j,q)$. This implies $g(t)(f(t),q) = g(t)(f'(t),q)$. From condition N, we have $\lambda(\{t \in T | g(t)(f(t),q) = g(t)(f'(t),q)\}) = 0$. Since $(T_a \cap T_b) \subset \{t \in T | g(t)(f(t),q) = g(t)(f(t),q)\}$, $T_a \cap T_b$ has zero measure. Since $T_a \subset (T_a \cap T_b) \cup (T \setminus T_b)$, we have $\lambda(T_a) = 0$. **Q.E.D.**

3 Case of n Strategies for Normalized Games

In this section 5, we consider only a normalized game in which payoff of the nth strategy is always zero for any average strategy.

Definition 3.1 *A game g is said to be a normalized game if $g(t)(e^n,q) = 0$ for any $t \in T$ and $q \in S$.*

Any game $\hat{g}$ can be normalized to the game g by

$$g(t)(e^j,q) = \hat{g}(t)(e^j,q) - \hat{g}(t)(e^n,q).$$

Since any positive affine transformation does not change the best response structure between two games, any game also have the unique equilibrium if its normalized game have the unique equilibrium. Thus, we can use the uniqueness condition for any game by the normalization, not only for normalized games, though our conditions is mainly applicable to the class of the games which is originally a normalized game itself.

In this section, we consider the case where each player has n strategies. In the case we can only show the uniqueness of an interior equilibrium.

Definition 3.2 *For any game g, the interior equilibrium of g is said to be unique if for any equilibrium f and f' in g, satisfying that $s(f)_i > 0$ and $s(f')_i > 0$ for any $i \in \{1, \dots, n\}$,*

$$\lambda(\{t \in T | f(t) \neq f'(t)\}) = 0.$$

$q \in S$ is said to be an interior fixed point of Γ if $q \in \Gamma(q)$ and $q_i > 0$ for any $i \in \{1, \dots, n\}$. We find that the following lemma holds (see, Watanabe (1997))

Lemma 3.1 *Let g be a normalized game. If a game g satisfies condition N and the interior fixed point of Γ of g is unique, then the interior equilibrium of g is unique.*

In normalized games, nth strategy is a special strategy in compare to the other strategies. To describe conditions of uniqueness, we consider the following two operations. In the first operation, we add θ to ith $(i = 1, \dots, n-1)$ average strategy and subtract θ from nth average strategy. We denote this operation by $\Delta^i(\theta)$. Formally, for any $\theta \geq 0$ and $i \in \{1, \dots, n-1\}$, we define $\Delta^i(\theta)$ by

$$\Delta^i(\theta) = \theta(e^i - e^n).$$

The second operation makes $n-1$ average strategies multiplied by θ and nth average strategy decreased to adjust the sum of all average strategies to one. We denote this operation by $\otimes$. Formally, for a given $\theta > 0$ and $q \in S$, we define $\theta \otimes q$ by

$$\theta \otimes q = (\theta q_1, \theta q_2, \dots, \theta q_{n-1}, 1 - \theta \sum_{j=1}^{n-1} q_j)$$

Condition R: Rivalry Condition For any $t \in T$, $q \in S$, $i, k \in \{1, \dots n-1\}, i \neq k$, $j \in \{1, \dots n\}$ and $\theta > 0$ satisfying $q + \Delta^k(\theta) \in S$, if $g(t)(e^i, q) \geq g(t)(e^j, q)$, then $g(t)(e^i, q + \Delta^k(\theta)) \geq g(t)(e^j, q + \Delta^k(\theta))$.

Condition H: Homogeneity For any $t \in T$, $q \in S$, $e^i, e^j \in E$ and $\theta > 0$ satisfying $\theta \otimes q \in S$, if $g(t)(e^i, q) > g(t)(e^j, q)$, then $g(t)(e^i, \theta \otimes q) > g(t)(e^j, \theta \otimes q)$

Some useful class of functions satisfies the above conditions. The following condition describes the class of functions.

Condition G If g can be written as $g(t)(e^i, q) = \bar{h}_t(q_1, \ldots, q_{n-1})h_t^i(q_i) \quad i = 1, \ldots, n-1$ where $\bar{h}_t(q_1, \ldots, q_{n-1})$ is a positive function and $h_t^i(q_i)$ $(i = 1, \ldots, n-1)$ is a non-increasing and homogeneous of degree m function, then g is said to be satisfying condition G.

Lemma 3.2 *If g satisfies condition G, then it satisfies condition H and R.*

Two main theorems in this section are shown as follows.

Theorem 3.1 *If normalized game g satisfies condition N, H and R and an interior equilibrium exists, then the interior equilibrium exists uniquely.*

This theorem and lemma 3.2 implies the following corollary.

Corollary 3.1 *If normalized game g satisfies condition N and G, and an interior equilibrium exists, then the interior equilibrium exists uniquely.*

To prove the theorem we have to show three lemmas. As I mentioned in the introduction, all proofs of the lemmas are omitted.

First lemma asserts that a correspondence Γ is single-valued if condition N holds.

Lemma 3.3 *If g satisfies condition N, then the best response correspondence Γ is single valued.*

Since Γ is single valued, we rewrite a correspondence Γ as a function γ. In other words, we define a function γ from S to S by $\gamma(q) \in \Gamma(q)$ for any $q \in S$ and γ is uniquely determined.

We define $\bar{S}$ by $\bar{S} = \{(x_1, \ldots, x_{n-1}) \in R^{n-1} | x_i \geq 0 \ \sum_{i=1}^{n-1} x_i \leq 1.\}$ Let $\bar{\gamma}$ be a function from $\bar{S}$ to $\bar{S}$ such that $\gamma_i(\bar{q}) \in \Gamma_i(q)$ for any $\bar{q} \in \bar{S}$ and $i \in \{1, \ldots, n-1\}$, where $q = (\bar{q}, 1 - \sum_{j=1}^{n-1} \bar{q}_j)$. Thus, $\bar{\gamma}$ is a projection of Γ to the $n-1$ dimensional real space and lemma 3.3 implies that $\bar{\gamma}$

is uniquely determined. Hence we find that $x \in \bar{S}$ is a interior fixed point of $\bar{\gamma}$, if and only if $\hat{q}(x) \in R^n$ is a interior fixed point of Γ where $\hat{q}(x)$ is defined by

$$\hat{q}_i(x) = \begin{cases} x_i & 1 \leq i \leq n-1 \\ 1 - \sum_{j=1}^{n-1} x_j & i = n \end{cases}$$

Hence, we have only to show the uniqueness of interior fixed points $\bar{\gamma}$ to prove the uniqueness of interior fixed points Γ.

If an average strategy is q, the measure of the set of players whose best response strategy is e^i equals to $\Gamma_i(q)$. But, the measure of the set of players whose best response strategy is only e^i may be less than $\Gamma_i(q)$ because some non-null players whose best response strategy are e^i has other best response strategies. Condition N excludes this possibility and the following lemma asserts this fact, described as $\bar{\gamma}$.

Lemma 3.4 *If a normalized game g satisfies condition N, then for any $x \in \bar{S}$ and $i \in \{1, \ldots, n-1\}$,*

$$\bar{\gamma}_i(x) = \lambda(B_i(x)) \tag{1}$$

holds where $B_i(x) = \{t \in T| \ g(t)(e^i, \hat{q}(x)) > g(t)(e^j, \hat{q}(x)) \ \ \text{for all} \ \ j. \}$ In other words, the measure of the players whose best response for $\hat{q}(x)$ is only e^i is equal to $\bar{\gamma}_i(x)$.

Lemma 3.5 *If a normalized game g satisfies condition N and condition R, then for any $x \in \bar{S}$ $i \neq k, \in \{1, \ldots, n-1\}$, and any $\theta > 0$ satisfying $x + \theta\bar{e}^k$, we have $\bar{\gamma}_i(x + \theta\bar{e}^k) \geq \bar{\gamma}_i(x)$ where $\bar{e}^k \in R^{n-1}$ is a kth unit vector, that is kth element is one and the other elements are zero.*

Now we show that $\bar{\gamma}_i$ is a homogeneous function of degree zero.

Lemma 3.6 *If a normalized game g satisfies condition N and H, then for any $i \in \{1, \ldots, n-1\}$, $\bar{\gamma}_i$ is a homogeneous function of degree zero, that is, for any $x \in \bar{S}$ and $\theta > 0$ satisfying $\theta x \in \bar{S}$, $\bar{\gamma}_i(x)$ is equal to $\bar{\gamma}_i(\theta x)$.*

proof of theorem 3.1 To prove theorem 3.1, we have only to show that an interior fixed point of Γ is at most one. Condition N and lemma 3.3 implies that we have only to show that an interior fixed point of $\bar{\gamma}$ is at most one.

Suppose that there exists two different interior fixed points y, y'. Then, there exists j such that $y_j \neq y'_j$ Without loss of generality, we can assume $y_j < y'_j$. Since y and y' is interior fixed points, for any $i \in \{1, \dots, n\}$, we have $y_i \neq 0$ and $y'_i \neq 0$. Hence, there exists $\bar{\theta}$ such that $\bar{\theta} = \max_j \frac{y'_j}{y_j}$. Let $j0$ be the index which gives the maximum of the above equation, so that $\bar{\theta} = \frac{y'_{j0}}{y_{j0}}$. Since $y_j < y'_j$ holds for some j, we have

$$y_{j0} < y'_{j0}. \tag{2}$$

Choose a sufficiently small $\epsilon > 0$ such that $\epsilon\bar{\theta}y \in \bar{S}$ and $\epsilon\bar{\theta}y' \in \bar{S}$. and let z be $\epsilon\bar{\theta}y$ and z' be $\epsilon\bar{\theta}y'$. For any $i \in \{1, \dots, n-1\}$, we have $z_i \geq z'_i$ because $z_i - z'_i = \epsilon(\bar{\theta}y_i - y'_i) \geq \epsilon(\frac{y'_i}{y_i}y_i - y'_i) = 0$. Moreover $y \neq y'$ implies that $z_{i0} > z'_{i0}$ holds at least for some $i0$. Note that $z_{j0} = z'_{j0}$ from the definition of $\bar{\theta}$

We define $\{w_1, \dots, w_{n-1}\}$ by

$$w^0 = z' \quad w^k = w^{k-1} + (z_k - z'_k)\bar{e}^k \quad (k = 1, \dots, n-1).$$

and $\Delta\bar{\gamma}^k_{j0} \in \bar{S}$ $(k = 1, \dots, n-1)$ by $\Delta\bar{\gamma}^k_{j0} = \bar{\gamma}_{j0}(w^k) - \bar{\gamma}_{j0}(w^{k-1})$. Lemma 3.5 implies $\Delta\bar{\gamma}^k_{j0} \geq 0$ for any $k \in \{1, \dots, n-1\}, k \neq j0$, and we find that $\bar{\gamma}_{j0}(z) - \bar{\gamma}_{j0}(z') = \sum_{k=1,k\neq i}^{n-1} \Delta\bar{\gamma}^k_{j0}$. Therefore we have

$$\bar{\gamma}_{j0}(z) - \bar{\gamma}_{j0}(z') \geq 0. \tag{3}$$

Now consider δ defined by $\delta = (\bar{\gamma}_{j0}(z) - y_{j0}) - (\bar{\gamma}_{j0}(z') - y'_{j0})$. (2) and (3) implies $\delta > 0$. However, since y and y' are fixed points of $\bar{\gamma}$ and lemma 3.6 implies $\bar{\gamma}$is a homogenous function of degree zero, $\delta = 0$ should be zero: $\begin{array}{l} \bar{\gamma}_{j0}(z) = \bar{\gamma}_{j0}(\epsilon\bar{\theta}y) = \bar{\gamma}_{j0}(y) = y_{j0} \\ \bar{\gamma}_{j0}(z') = \bar{\gamma}_{j0}(\epsilon\bar{\theta}y') = \bar{\gamma}_{j0}(y') = y'_{j0}. \end{array}$ This leads a contradiction, so that an interior fixed point of $\bar{\gamma}$ is at most one. **Q.E.D.**

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