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ON THE INITIAL BOUNDARY VALUE PROBLEM FOR THE LINEARIZED MHD EQUATIONS

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1. Introduction and results

We will consider the well-posedness of the initial boundary value problem for the linearized equations of ideal MHD. The original system of equations takes the following form.

$$\begin{aligned} &\rho_p(\partial_t + (u,\nabla))p + \rho\operatorname{div} u = 0, \\ &\rho(\partial_t + (u,\nabla))u = -\nabla p + \mu_0(\nabla\times H)\times H, \\ &\partial_t H - \nabla\times(u\times H) = 0, \\ &(\partial_t + (u,\nabla))s = 0 \qquad \text{in } [0,T]\times\Omega. \end{aligned} \tag{1.1}$$

The boundary condition is

$$(\nu, u) = 0 \qquad \text{on } [0,T]\times\partial\Omega. \tag{1.2}$$

The constraint conditions

$$(\nu, H) = 0 \qquad \text{on } [0,T]\times\partial\Omega, \tag{1.3}$$

$$\operatorname{div} H = 0 \qquad \text{in } [0,T]\times\Omega \tag{1.4}$$

are also imposed. Here Ω is a bounded domain in $\mathbb{R}^3$, T is a positive constant and $\nu = \nu(x) = {}^t(\nu_1,\nu_2,\nu_3)$ denotes the unit outward normal to the boundary at $x\in\partial\Omega$. Pressure $p = p(t,x)$, velocity $u = u(t,x) = {}^t(u_1,u_2,u_3)$, magnetic field $H = H(t,x) = {}^t(H_1,H_2,H_3)$ and entropy $s = s(t,x)$ are unknown functions. We suppose that density $\rho = \rho(p,s)$ is a smooth known function of $p > 0$ and s satisfying $\rho > 0, \rho_p = \partial\rho/\partial p > 0$. The magnetic permeability μ_0 is a positive constant.

In order to employ a useful symmetrization of (1.1), we introduce the new unknown vector valued function $U = {}^t(q, {}^t u, {}^t H, s)$ in place of ${}^t(p, {}^t u, {}^t H, s)$, where

$q = p + \frac{1}{2}|H|^2$ is the total pressure. We linearize the equations (1.1) about $\overline{U}$ where $\overline{U} = {}^t(\overline{q}, {}^t\overline{u}, {}^t\overline{H}, \overline{s}) \in C^{l+1}([0,T]\times\overline{\Omega})$ is a solution of (1.1) which satisfies (1.2)–(1.4) with $\overline{p} > 0$ in $[0,T]\times\overline{\Omega}$. The concrete form of the linearized equations will be given later in Section 2.

Definition. The initial boundary value problem for the linearized equations is said to be well posed in $H^l(\Omega)$, for an integer $l \geq 1$, if the following conditions are satisfied:

For any initial data $U_0 \in H^l(\Omega)$ satisfying

$$(\nu, H_0) = 0 \qquad \text{on } \partial\Omega, \tag{1.5}$$

$$\mathrm{div} H_0 = 0 \qquad \text{in } \Omega, \tag{1.6}$$

and the compatibility conditions of order $l-1$ for the linearized equations and the boundary condition (1.2), there exists a unique solution $U \in C([0,T_1]; H^l(\Omega))$ of the linearized equations such that it satisfies (1.2), (1.3), (1.4) with $T = T_1$ and the estimate

$$||U(t)||_{H^l(\Omega)} \leq C||U_0||_{H^l(\Omega)} \tag{1.7}$$

holds for any $t \in [0,T_1]$. Here C and $T_1(\leq T)$ are positive constants independent of U_0. (For $\partial_t U$, see, e.g., R. Temam [16], ch. II.3.)

Let $\partial\Omega \in C^{l+3}$, $l \geq 1$, then main results of the present paper are the following two theorems.

Theorem I. *The initial boundary value problem for the linearized equations* (2.2) *with* (1.2), (1.3), *and* (1.4) *is well posed in* $H^1(\Omega)$.

Theorem II. *Let* $\overline{H} \not\equiv 0$ *on* $[0,T]\times\partial\Omega$. *Then the above problem is not well posed in* $H^l(\Omega)$ *for* $l \geq 2$.

Theorem I has the following significance. First it release us from troubles with compatibility conditions, since one of order zero is the boundary condition itself and also it follows the well posedness in $H^0(\Omega)$ in more precise sense than J. Rauch's result (cf. [10]) under the condition of Theorem I. As a special case, where $\overline{U}$ is a static equilibrium defined over $\overline{\Omega}$ whose boundary is a magnetic surface, i.e., a surface where $(\nu, \overline{H}) = 0$, contained in plasma region, these facts above mentioned will be useful to the linearized internal (local) stability second–order system. (See I. B. Bernstein et al. [1] and J. P. Freidberg [3]. For equilibrium, see R. Temam [13], [15] and A. Friedman & Y. Liu [4]. For the existence of solutions, see R. Temam [16], ch. II.4.)

We note also that we can present estimates based upon (1.7) in Theorem I. By using them it is able to obtain the well posedness of (1.1)–(1.4) in a function space whose elements have regularities of order less than that of $H^l_*(\Omega)$ ($l \geq 8$). (For the latter see T.Yanagisawa & A. Matsumura [18] or P. Secchi [12].) But here we do not enter into detail.

Theorem II implies that, for any Ω with smooth boundary, the regularity loss of solutions of the linearized problem always arises in $H^l(\Omega)$ ($l \geq 2$). The initial data are to be taken in a way such that their supports are sufficiently small and intersects with $\partial\Omega$. Obviously Theorem II is also valid in case where the linearized equations are such that the equation

$$\partial_t H + (\overline{u}, \nabla)H - (\overline{H}, \nabla)u + \overline{H}(\mathrm{div} u) = \text{a certain terms of lower order}$$

guarantees that $(\nu, H)|_{\partial\Omega}(t) = 0$ for $t \in [0, T_1]$ whenever $(\nu, H_0)|_{\partial\Omega} = 0$ and where the condition $\mathrm{div} H = 0$ in Ω is neglected as usual. (The iteration scheme using such a linearization was noticed by the second author. See [18].)

Theorem I, which proves the non–existence of "loss of regularity" of solutions in $H^1(\Omega)$, has been found by us after the completion of the proof of Theorem II (cf. [13]).

This paper presents the detailed proof of Theorem I. For the proof of Theorem II see [8].

2. Linearized problem

Using the unknown vector valued function $U = {}^t(q, {}^tu, {}^tH, s)$ we rewrite (1.1) as follows.

$$\begin{aligned}
&\alpha(\partial_t + (u, \nabla))q - \alpha(H, \partial_t H + (u, \nabla)H) + \mathrm{div}\, u = 0, \\
&\rho(\partial_t + (u, \nabla))u + \nabla q - (H, \nabla)H = 0, \\
&\partial_t H + (u, \nabla)H - (H, \nabla)u + H(\mathrm{div} u) - (\mathrm{div} H)u = 0, \\
&(\partial_t + (u, \nabla))s = 0 \qquad \text{in } [0, T] \times \Omega.
\end{aligned} \tag{2.1}$$

Here we put $\mu_0 = 1$, for simplicity and $\alpha = \rho_p/\rho$. Then we linearize (2.1) about a solution $\overline{U} \in C^{l+1}([0, T] \times \overline{\Omega})$ to (2.1) with (1.2)–(1.4). The resulting equations are the following.

$$\begin{aligned}
&\overline{\alpha}(\partial_t + (\overline{u}, \nabla))q - \overline{\alpha}(\overline{H}, \partial_t H + (\overline{u}, \nabla)H) + \mathrm{div}\, u = l_1, \\
&\overline{\rho}(\partial_t + (\overline{u}, \nabla))u + \nabla q - (\overline{H}, \nabla)H = l_2, \\
&\partial_t H + (\overline{u}, \nabla)H - (\overline{H}, \nabla)u + \overline{H}(\mathrm{div} u) = l_3, \\
&(\partial_t + (\overline{u}, \nabla))s = l_4 \qquad \text{in } [0, T] \times \Omega.
\end{aligned} \tag{2.2}$$

We observe that the terms of lower order l_i, $i = 1, \ldots, 4$, are linear combinations of the components of U with coefficients depending smoothly on the components of $\overline{U}$ and their derivatives of the first order with respect to x and t. In particular, we have

$$l_3 = -(u, \nabla)\overline{H} + (H, \nabla)\overline{u} - H(\mathrm{div}\overline{u})$$

and $\overline{\alpha} = \alpha(\overline{q}, \overline{H}, \overline{s})$, etc. We obtain $(2.2)_3$ by subtracting $\overline{u}(\mathrm{div} H) + u(\mathrm{div}\overline{H})$ from the third equations of the linearizetion of (2.1). For simplicity of the description we omit s in (2.2) without loss of generality, although we can not do so if we are discussing the theory of stability. Note that unknowns in the principle part of $(2.2)_1$–$(2.2)_3$ and one of $(2.2)_4$ are independent of each other and in addition only derivatives tangential to $\partial\Omega$ appears in $(2.2)_4$. In the following, we set U and $\overline{U}$ to be ${}^t(q, {}^tu, {}^tH)$ and ${}^t(\overline{q}, {}^t\overline{u}, {}^t\overline{H})$, respectively, which may be all real vector valued functions.

Adding $(2.2)_1 \times (-\overline{H})$ to $(2.2)_3$, we get the following system which is a symmetization of (2.2).

$$\begin{aligned}
&\overline{\alpha}(\partial_t + (\overline{u}, \nabla))q - \overline{\alpha}(\overline{H}, \partial_t H + (\overline{u}, \nabla)H) + \operatorname{div} u = l_1, \\
&\overline{\rho}(\partial_t + (\overline{u}, \nabla))u + \nabla q - (\overline{H}, \nabla)\overline{H} = l_2, \\
&\partial_t H + (\overline{u}, \nabla)H - (\overline{H}, \nabla)u - \overline{\alpha}\overline{H}\{(\partial_t + (\overline{u}, \nabla))q - (\overline{H}, \partial_t H + (\overline{u}, \nabla)H)\} \\
&= l_3 - l_1\overline{H} \qquad \text{in } [0,T] \times \Omega.
\end{aligned} \tag{2.3}$$

We write equations of our problem in the following form.

$$\begin{aligned}
&A_0(\overline{U})\partial_t U + \sum_{j=1}^{3} A_j(\overline{U})\partial_j U + B(\overline{U})U = 0 && \text{in } [0,T] \times \Omega, \\
&MU = 0 && \text{on } [0,T] \times \partial\Omega, \\
&NU = 0 && \text{on } [0,T] \times \partial\Omega, \\
&\operatorname{div} H = 0 && \text{in } [0,T] \times \Omega, \\
&U(0,x) = U_0(x) && \text{for } x \in \Omega,
\end{aligned} \tag{2.4}$$

where $\partial_j = \partial/\partial_{x_j}$, $j = 1, 2, 3$,

$$A_0(\overline{U}) = \left(\begin{array}{c|ccc|ccc}
\overline{\alpha} & 0 & 0 & 0 & -\overline{\alpha}\overline{H}_1 & -\overline{\alpha}\overline{H}_2 & -\overline{\alpha}\overline{H}_3 \\
\hline
0 & \overline{\rho} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \overline{\rho} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{\rho} & 0 & 0 & 0 \\
\hline
-\overline{\alpha}\overline{H}_1 & 0 & 0 & 0 & 1 + \overline{\alpha}\overline{H}_1^2 & \overline{\alpha}\overline{H}_1\overline{H}_2 & \overline{\alpha}\overline{H}_1\overline{H}_3 \\
-\overline{\alpha}\overline{H}_2 & 0 & 0 & 0 & \overline{\alpha}\overline{H}_1\overline{H}_2 & 1 + \overline{\alpha}\overline{H}_2^2 & \overline{\alpha}\overline{H}_2\overline{H}_3 \\
-\overline{\alpha}\overline{H}_3 & 0 & 0 & 0 & \overline{\alpha}\overline{H}_1\overline{H}_3 & \overline{\alpha}\overline{H}_2\overline{H}_3 & 1 + \overline{\alpha}\overline{H}_3^2
\end{array}\right),$$

$$B(\overline{U})U = -\begin{pmatrix} l_1 \\ l_2 \\ l_3 - l_1\overline{H} \end{pmatrix},$$

$$A_\nu(\overline{U}) = \sum_{j=1}^{3} \nu_j A_j(\overline{U}) = \left(\begin{array}{c|ccc|ccc}
0 & & {}^t\nu & & 0 & 0 & 0 \\
\hline
 & 0 & 0 & 0 & 0 & 0 & 0 \\
\nu & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \quad \text{on } \partial\Omega,$$

$M =$ all elements are equal to zero except that the (2,2),(2,3),(2,4) entries are equal to ${}^t\nu$,

$N =$ all elements are equal to zero except that the (5,5),(5,6),(5,7) entries are equal to ${}^t\nu$,

and $B(\overline{U}) = B(\overline{U}, \partial_t\overline{U}, \partial_j\overline{U}; 1 \leq j \leq 3)$.

The resulting system $(2.4)_1$, $(2.4)_2$ and $(2.4)_4$ is a symmetric hyperbolic system with characteristic boundary of constant multiplicity in the sense of J. Rauch [10].

Note that $A_0(\overline{U})$ is positive definite, although $A_0(\overline{U}) \neq I$. The boundary condition $(2.4)_2$ is maximal nonnegative. Actually, the boundary matrix $A_\nu = \sum_{j=1}^n \nu_j A_j$ is of a constant rank 2 on $\partial\Omega$ and Ker $A_\nu \subset$ Ker M on $\partial\Omega$ which is maximal nonnegative subset of A_ν. Now we give a lemma which will be useful in the proofs of theorems.

Lemma 2.1.

(i) *Let* $\overline{U}$ *be a solution* $\in C^{l+1}([0,T]\times\overline{\Omega})$ *of* (1.1)-(1.4). *Then the assumption in* Theorem II, *i.e.,* $\overline{H} \not\equiv 0$ *on* $[0,T]\times\partial\Omega$, *implies that* $\overline{H} \not\equiv 0$ *on* $\{t=0\}\times\partial\Omega$.

(ii) *Assume that* $\overline{U} \in C^{l+1}([0,T]\times\overline{\Omega})$ *satisfies* (1.2), (1.3) *and* $\overline{p} > 0$ *in* $[0,T]\times\overline{\Omega}$. *This implies that* $\overline{U}$ *satisfies neither* (1.1) *nor* (1.4). *Then, if* (1.5) *holds for* $U(0)$ *the solution* $U(t)$ *of* $(2.4)_1$ *that belongs to* $\in C([0,T_1],H^2(\Omega))$ *of* $(2.4)_1$ *satisfies* (1.3) *in* $[0,T_1]\times\partial\Omega$.

(iii) *Let* $\overline{U} \in C^{l+1}([0,T]\times\overline{\Omega})$ *satisfy* (1.2)-(1.4). *Then, if* (1.6) *holds for* $U(0)$, *the solution* $U(t)$ *of* $(2.4)_1$ *that belongs to* $U(t) \in C([0,T_1];H^1(\Omega))$ *also satisfies* (1.4), *i.e.,* $(2.4)_4$, *in* $[0,T_1]\times\Omega$.

Proof. Under the condition in the assertion (i), we have:

$$\partial_t\overline{H} + (\overline{u},\nabla)\overline{H} - (\overline{H},\nabla)\overline{u} - \overline{H}\mathrm{div}\overline{u} = 0 \qquad \text{on } [0,T]\times\partial\Omega.$$

Since $(\overline{u},\nabla)$ is a differential operator on $\partial\Omega$, $\overline{H}$ may be regarded as a solution to the symmetric hyperbolic system of equations defined on the surface manifold $\partial\Omega$. This proves the conclusion of (i).

Next, for a solution $U \in C([0,T_1];H^2(\Omega))$ of $(2.4)_1$, i.e., (2.3), it holds that

$$\partial_t(H,\nu) + (\overline{u},\nabla)(H,\nu) + \mathrm{div}\overline{u}(H,\nu) - \{(\nu,\nabla)(\overline{u},\nu)\}(H,\nu) = 0 \qquad \text{on } [0,T]\times\partial\Omega,$$

since $(H-(H,\nu)\nu,\nabla)$ is tangential to $[0,T]\times\partial\Omega$ and for example

$$\begin{aligned}((\overline{u},\nabla)H,\nu) &= (\overline{u},\nabla)(H,\nu) - ((\overline{u},\nabla)\nu,H)\\ &= (\overline{u},\nabla)(H,\nu) - \sum_{i,j=1}^{3}(\overline{u}_i,H_j\frac{\partial^2\varphi}{\partial x_i\partial x_j}) \qquad \text{on } [0,T]\times\partial\Omega.\end{aligned}$$

Here $\varphi \in C^{l+2}$ is a definition function of $\partial\Omega$ and ${}^t\nu = (\frac{\partial\varphi}{\partial x_i}/\sqrt{|\nabla\varphi|^2},\ i=1,2,3,)$ in a neighborhood of a point on $\partial\Omega$. Therefore the local uniqueness of the solution (H,ν) of the above equation proves the assertion of (ii).

To prove (iii), we observe that $(2.2)_3$ implies

$$\partial_t H - \nabla\times(\overline{u}\times H) - \nabla\times(u\times\overline{H}) + \overline{u}\mathrm{div}H = 0 \qquad \text{in } [0,T]\times\Omega.$$

Hence we see that in the sense of distribution

$$\partial_t\mathrm{div}H + (\overline{u},\nabla)\mathrm{div}H + \mathrm{div}\overline{u}\cdot\mathrm{div}H = 0 \qquad \text{in } [0,T]\times\Omega,$$

where $\mathrm{div}H \in C([0,T_1],L^2(\Omega))$ and $\overline{u} \in C^2([0,T_1]\times\overline{\Omega})$. Setting $\dot{x} = \overline{u}(t,x)$, $x(t,\alpha) = \alpha$ at $t = 0$, we obtain a trajectory transformation $x(t,\alpha)$ whose Jacobian determinant $\left|\dfrac{Dx(t,\alpha)}{D\alpha}\right| > 0$ for $t \in [0,T_1]$. Using molifier and the transformation $x(t,\alpha)$, we see that first $\mathrm{div}H = 0$ on $\{x(t,\alpha); \alpha \in \Omega^\delta,\ t \in [0,T_1]\}$, where $\Omega^\delta = \{x|\mathrm{dist}(\alpha,\partial\Omega) > \delta\}$. By letting $\delta \longrightarrow 0$, we get the assertion of (iii). □

Here we remark that the argument in proof of Lemma 2.1 (ii) does not apply to the case where $U \in C([0,T_1];H^1(\Omega))$. The reason is that $(\overline{H},\nabla)(u,\nu)|_{\partial\Omega}$ and $(\overline{H},\nu)\mathrm{div}u|_{\partial\Omega}$ are not always meaningful.

Taking account of the finiteness of the speed of propagation for the solution, we use a suitable finite partition of unity $\{\phi_\alpha\}$ of $\overline{\Omega}$ where $\sum_\alpha \phi_\alpha = 1$ and diffeomorphisms. Then we are reduced to the problem in the half space. We fix $p \in \partial\Omega$ arbitrarily. We assume that $\partial\Omega \in C^{l+3}$. Then there exists a C^{l+2}-admissible boundary coordinate system $(y(x))$ which maps p to the origin. We have

$$\mathbb{P} = \mathbb{P}(y) = \left(\frac{\partial x_i}{\partial y_j}\right)(y), \quad {}^t\mathbb{P}\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{G} \end{pmatrix} \quad \text{on } \{y_1 = 0\}, \tag{2.5}$$

$\mathbb{P} = (\delta_{i,j})$ at the origin,
where the G is a certain 2×2 matrix (cf. p301 of [6]).

Let us denote the inverse map of $y(x)$ by ψ. Then the known and unknown functions are changed as follows: for $x = \psi(y)$

$$\begin{aligned} &\tilde{u}(t,y) = \mathbb{P}^{-1}u(t,x), \quad \tilde{H}(t,y) = \mathbb{P}^{-1}H(t,x), \quad \tilde{q}(t,y) = q(t,x), \\ &\tilde{\rho}(t,y) = \rho(t,x), \quad \tilde{\overline{u}}(t,y) = \mathbb{P}^{-1}\overline{u}(t,x), \quad \tilde{\overline{H}}(t,y) = \mathbb{P}^{-1}\overline{H}(t,x), \\ &\tilde{\overline{q}}(t,y) = \overline{q}(t,x), \quad \tilde{\overline{\alpha}}(t,y) = \overline{\alpha}(t,x), \quad \tilde{\overline{\rho}}(t,y) = \overline{\rho}(t,x). \end{aligned}$$

Our problem in Theorem I is reduced to find the solutions to the following localized system of equations. For $T_1 << 1$,

$$\begin{aligned} &\tilde{A}_0(\tilde{\overline{U}})\partial_t\tilde{U} + \sum_{j=1}^{3}\tilde{A}_j(\tilde{\overline{U}})\partial_j\tilde{U} + \tilde{B}(\tilde{\overline{U}})\tilde{U} = 0 && \text{in } [0,T_1] \times \{y_1 > 0\}, \\ &\widetilde{M}\tilde{U} = 0 && \text{on } [0,T_1] \times \{y_1 = 0\}, \\ &\widetilde{N}\tilde{U} = 0 && \text{on } [0,T_1] \times \{y_1 = 0\}, \\ &\tilde{U}(0) = \widetilde{\phi_\alpha U_0} \quad \text{for a certain } \alpha && \text{in } \{y_1 > 0\}, \end{aligned} \tag{2.6}$$

where

$$\mathcal{P} = \mathcal{P}(y) = \begin{pmatrix} 1 & & \\ & \mathbb{P}(y) & \\ & & \mathbb{P}(y) \end{pmatrix}, \quad (\tilde{A}_0(\tilde{\overline{U}}))(t,y) = {}^t\mathcal{P}(y)(A_0(\overline{U}))(t,\psi(y))\mathcal{P}(y),$$

$$(\tilde{A}_j(\tilde{\overline{U}}))(t,y) = {}^t\mathcal{P}(y)\left(\sum_{l=1}^{3}(A_l(\overline{U}))(t,\psi(y))\left(\frac{\partial y_j}{\partial x_l}\right)(\psi(y))\right)\mathcal{P}(y), \ j = 1,2,3.$$

From (2.5), we see that

$$\widetilde{M} = -M\mathcal{P} = \begin{array}{l}\text{all elements are equal to zero except}\\ \text{that the (2,2) entry is equal to 1,}\end{array} \tag{2.7}$$

$$\widetilde{N} = -N\mathcal{P} = \begin{array}{l}\text{all elements are equal to zero except}\\ \text{that the (5,5) entry is equal to 1,}\end{array} \tag{2.8}$$

$$\widetilde{A}_0(\widetilde{\overline{U}}) = \left(\begin{array}{c|ccc|ccc} \widetilde{\overline{\alpha}} & 0 & 0 & 0 & & G_{1,3} & \\ \hline 0 & \widetilde{\overline{\rho}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & 0 & 0 & 0 \\ 0 & 0 & G_{2,2} & & 0 & 0 & 0 \\ \hline & 0 & 0 & 0 & & & \\ G_{3,1} & 0 & 0 & 0 & & G_{3,3} & \\ & 0 & 0 & 0 & & & \end{array}\right) \quad \text{on } \{y_1 = 0\}, \tag{2.9}$$

where the $G_{i,j}$ are $i \times j$ matrices,

$$\widetilde{A}_1(\widetilde{\overline{U}}) = \begin{array}{l}\text{all elements are equal to zero except}\\ \text{that the (1,2) and (2,1) entries are equal to 1 on } \{y_1 = 0\}.\end{array} \tag{2.10}$$

The concrete form of $(2.6)_1$ is as follows.

$$\begin{aligned} &\widetilde{\overline{\alpha}}\{\partial_t\widetilde{q} + (\widetilde{\overline{u}}, \nabla_y)\widetilde{q} - ({}^t\mathbb{P}\mathbb{P}\widetilde{\overline{H}}, \partial_t\widetilde{H} + (\widetilde{\overline{u}}, \nabla_y)\widetilde{H})\} + \mathrm{div}\widetilde{u} = \widehat{l}_1, \\ &\widetilde{\overline{\rho}}\,{}^t\mathbb{P}\mathbb{P}(\partial_t\widetilde{u} + (\widetilde{\overline{u}}, \nabla_y)\widetilde{u}) + \nabla_y\widetilde{q} - {}^t\mathbb{P}\mathbb{P}(\widetilde{\overline{H}}, \nabla_y)\widetilde{H} = \widehat{l}_2, \\ &{}^t\mathbb{P}\mathbb{P}[\partial_t\widetilde{H} + (\widetilde{\overline{u}}, \nabla_y)\widetilde{H} - (\widetilde{\overline{H}}, \nabla_y)\widetilde{u} \\ &+\widetilde{\overline{\alpha}}\widetilde{\overline{H}}\{-\partial_t\widetilde{q} - (\widetilde{\overline{u}}, \nabla_y)\widetilde{q} + ({}^t\mathbb{P}\mathbb{P}\widetilde{\overline{H}}, \partial_t\widetilde{H} + (\widetilde{\overline{u}}, \nabla_y)\widetilde{H})\}] = \widehat{l}_3, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0, T_1] \times \{y_1 > 0\}, \end{aligned} \tag{2.11}$$

where $\widehat{l}_i$, $i = 1, 2, 3$, denote terms of lower order. Here we use the relations such that for $x = \psi(y)$

$$\begin{aligned} &\nabla_x = {}^t\mathbb{P}^{-1}\nabla_y, \ (\overline{u}, \nabla_x) = (\widetilde{\overline{u}}, \nabla_y), \ u = \mathbb{P}\widetilde{u}, \\ &(\overline{u}, \nabla_x)H = \mathbb{P}(\widetilde{\overline{u}}, \nabla_y)\widetilde{H} - \mathbb{P}\{(\widetilde{\overline{u}}, \nabla_y)\mathbb{P}^{-1}\}\mathbb{P}\widetilde{H}, \\ &(\nabla_x, u) = (\nabla_y, \widetilde{u}) - ({}^t({}^t\nabla_y{}^t\mathbb{P}^{-1}), \mathbb{P}\widetilde{u}), \text{ etc.} \end{aligned}$$

The resulting system (2.11) is again a symmetric hyperbolic system having the same properties as (2.4).

In the following we always assume that $\overline{U}$ and $\overline{U}^\delta$ satisfy the assumption of Lemma 2.1 (ii). By virtue of Lemma 2.1 (iii), we consider solutions omitting (1.4) and (1.6) in the localized problem of Section 3.

3. Proof of Theorem I

First we show the existence of approximate systems and approximate initial data which satisfy the compatibility conditions of order 4.

Assuming that $\partial\Omega \in C^4$, we consider the approximate problem: for $T_1 << 1$ and for sufficiently small $\delta > 0$

$$
\begin{aligned}
&\widetilde{A}_0(\widetilde{\overline{U}^\delta})\partial_t \widetilde{U}^\delta + \sum_{j=1}^{3} \widetilde{A}_j(\widetilde{\overline{U}^\delta})\partial_j \widetilde{U}^\delta + \widetilde{B}(\widetilde{\overline{U}^\delta})\widetilde{U}^\delta = 0 && \text{in } [0,T_1] \times \{y_1 > 0\}, \\
&\widetilde{M}\widetilde{U}^\delta = 0 && \text{on } [0,T_1] \times \{y_1 = 0\}, \\
&\widetilde{N}\widetilde{U}^\delta = 0 && \text{on } [0,T_1] \times \{y_1 = 0\}, \\
&\widetilde{U}^\delta(0) = \widetilde{f}^\delta && \text{with compact support on } \{y_1 \geq 0\}.
\end{aligned} \tag{3.1}
$$

Here $\overline{U}^\delta$ enjoys the following properties: Let p be a point on $\partial\Omega$ and let $y = \psi^{-1}(x) \in C^3$ be an admissible coordinate system defined on a boundary patch with center p. For some $r_0 > 0$ we set $\mathcal{B}(0) = \{y; \text{dist}(0,y) < r_0 \text{ and } y_1 \geq 0\} \subset \overline{\mathbb{R}^3_+}$. Then there exist maps $\psi^\delta \in C^{11}$ defined over the half ball $\mathcal{B}(0)$ satisfying the following properties.

(i) $\Omega^\delta(p) = \psi^\delta(\mathcal{B}(0))$ has an admissible boundary coordinate system $(\psi^\delta)^{-1}$.

(ii) $(\psi^\delta)^{-1}(p) = 0$.

(iii) $\psi^\delta \longrightarrow \psi$ in $C^3(\overline{\mathcal{B}(0)})$ as $\delta \longrightarrow 0$ (cf. C. Morrey [6]).

Furthermore, let $\widetilde{\widetilde{U}}^\delta$ be vector valued functions $\in C^{10}([0,T_1] \times \overline{\mathcal{B}(0)})$ such that

$$
\begin{aligned}
&\widetilde{\widetilde{U}}^\delta \longrightarrow \widetilde{\widetilde{U}} \quad \text{in } C^2([0,T_1] \times \overline{\mathcal{B}(0)}) \quad \text{as } \delta \longrightarrow 0, \\
&\widetilde{M}\widetilde{\widetilde{U}}^\delta = \widetilde{N}\widetilde{\widetilde{U}}^\delta = 0 \quad \text{on } [0,T_1] \times (\mathcal{B}(0) \cap \{y_1 = 0\}).
\end{aligned} \tag{3.2}
$$

Then setting $\mathbb{P}^\delta = \left(\dfrac{\partial\psi^\delta}{\partial y}\right)$ we define $\overline{U}^\delta \in C^{10}([0,T_1] \times \overline{\Omega^\delta(p)})$ as follows: for $x = \psi^\delta(y)$ $\overline{U}^\delta(t,x) = \mathcal{P}^\delta \widetilde{\widetilde{U}}^\delta(t,y)$.

We write $\mathbb{R}^3_+ = \{y_1 > 0\}$ hereafter.

Lemma 3.1. *There exist $\widetilde{f}^\delta$ having the following properties:*

(i) $\widetilde{f}^\delta \in H^5(\mathbb{R}^3_+)$.

(ii) $\widetilde{f}^\delta \longrightarrow \widetilde{\phi_\alpha U_0}$ *in* $H^1(\mathbb{R}^3_+)$ *as* $\delta \longrightarrow 0$ *and supp* $\widetilde{f}^\delta \subset\subset$ *a neighborhood of supp* $\widetilde{\phi}_\alpha \subset\subset \mathcal{B}(0)$, *where* $\mathbb{A} \subset\subset \mathbb{B}$ *means that* $\overline{\mathbb{A}} \subset \overset{\circ}{\mathbb{B}} \bigcup (\overline{\mathbb{B}} \bigcap \{y_1 = 0\})$ *and* $(\overline{\mathbb{A}} \bigcap \{y_1 = 0\}) \subset\subset (\overline{\mathbb{B}} \bigcap \{y_1 = 0\})$.

(iii) $\widetilde{f}^\delta$ *satisfies the compatibility condition of order* 4 *for* $(3.1)_1$ *and* $(3.1)_2$.

(iv) $\widetilde{N}\widetilde{f}^\delta = 0$ *on* $\{y_1 = 0\}$.

Here and hereafter we assume that for some α

$$\text{supp}\phi_\alpha \subset\subset \psi(\overline{\mathcal{B}(0)}).$$

Proof. In Lemma A.1 let $\varepsilon = \delta$ and let $l = 1$. Furthermore let $\widetilde{f} = \widetilde{\phi_\alpha U_0}$ and let $\widetilde{\widetilde{U}}^\varepsilon = \widetilde{\widetilde{U}}^\delta$, where $\widetilde{\widetilde{U}}^\delta$ satisfies (3.2) and belongs to $C^{10}([0,T_1] \times \overline{\mathcal{B}(0)})$. By this lemma we see $\widetilde{f}^\delta$ which satisfies the compatibility condition of order 0 for $(3.1)_1$, $(3.1)_2$ and also the condition $\widetilde{N}\widetilde{f}^\delta = 0$ on $\{y_1 = 0\}$. For fixed δ, setting $l = 1$, $m = 4$, $\widetilde{f} = \widetilde{f^\delta}$, $\widetilde{\widetilde{U}} = \widetilde{\overline{U}^\delta}$ and then applying Lemma A.2 we obtain $\widetilde{(\widetilde{f^\delta})}^\varepsilon$. Finally we choose a suitable subsequence $\{\widetilde{(\widetilde{f^\delta})}^{\varepsilon(\delta)}\}$. □

Combining Lemmas 2.1 (ii), 3.1 and Lemmas A.3, A.5 (i) we have the following lemma.

Lemma 3.2. *The initial boundary value problem* (3.1) *has a unique solution* $\widetilde{U}^\delta$ *in* $C([0,T_1]; H^2(\mathbb{R}^3_+))$.

Proof. For a fixed $\delta << 1$, let l, $\widetilde{f}$, $\widetilde{\widetilde{U}}$ in Lemma A.3 be 5, $\widetilde{f}^\delta$ in Lemma 3.1 and $\widetilde{\widetilde{U}}^\delta$ in (3.2), respectively. Then from Lemma A.3 we have a sequence $\{\widetilde{f}^{\delta,\varepsilon}\} \subset H^5(\mathbb{R}^3_+)$ such that

(i) $\widetilde{f}^{\delta,\varepsilon} \longrightarrow \widetilde{f}^\delta$ in $H^5(\mathbb{R}^3_+)$ as $\varepsilon \longrightarrow 0$ and $\mathrm{supp}\widetilde{f}^{\delta,\varepsilon} \subset\subset \mathcal{B}(0)$.

(ii) $\widetilde{f}^{\delta,\varepsilon}$ satisfies the compatibility conditions of order 4 with respect to $(A.9)$.

By [11], the corresponding problem to (A.9) with initial data $\widetilde{f}^\varepsilon$ replaced by $\widetilde{f}^{\delta,\varepsilon}$ above has a unique solution $\widetilde{U}^{\delta,\varepsilon} \in C([0,T_1]; H^5(\mathbb{R}^3_+))$.

Here we remark that $\widetilde{U}^{\delta,\varepsilon}$ has a uniformly finite speed of propagation for any positive δ, ε provided δ, $\varepsilon << 1$ and $t < T_1 << 1$.

Moreover by Lemma A.5 we obtain the estimate (A.12) for the solution $\widetilde{U}^{\delta,\varepsilon}$ with $l = 5$.

Therefore $\{\widetilde{U}^{\delta,\varepsilon}\}$ is a bounded sequence in $\bigcap_{j=0}^5 C^j([0,T_1]; H^{5-j}_*(\mathbb{R}^3_+))$ for a fixed δ. Then for any t, $t' \in [0,T_1]$ and a positive constant C_δ independent of ε

$$||\partial_t^j \widetilde{U}^{\delta,\varepsilon}(t) - \partial_t^j \widetilde{U}^{\delta,\varepsilon}(t')||_{L^2(\mathbb{R}^3_+)} < C_\delta |t - t'|, \qquad 0 \leq j \leq 4.$$

Furthermore the adjoint space of $H^{5-j}_*(\mathbb{R}^3_+)$ contains $L^2(\mathbb{R}^3_+)$ densely, since the natural identity mapping: $H^{5-j}_*(\mathbb{R}^3_+) \longrightarrow L^2(\mathbb{R}^3_+)$ is injective and the image in $L^2(\mathbb{R}^3_+)$ is dense there. Hence by the Ascoli–Arzéla theorem we see that for a subsequence $\{\widetilde{U}^{\delta,\varepsilon'}\}$ and for some $\widetilde{U}^\delta$

$$\partial_t^j \widetilde{U}^{\delta,\varepsilon'} \longrightarrow \partial_t^j \widetilde{U}^\delta \quad \text{in } C_w([0,T_1]; H^{5-j}_*(\mathbb{R}^3_+)), \quad \text{as } \varepsilon' \longrightarrow 0,\ 0 \leq j \leq 4,$$

from which we obtain

$$\widetilde{U}^\delta \in C^1_w([0,T_1]; H^4_*(\mathbb{R}^3_+)) \subset C([0,T_1]; H^2(\mathbb{R}^3_+)).$$

Also the equations corresponding to (A.9) imply that $\widetilde{U}^\delta$ is a solution of $(3.1)_1$ with $\widetilde{M}\widetilde{U}^\delta = 0$ on $[0,T_1] \times \partial\mathbb{R}^3_+$ and $\widetilde{U}^\delta(0) = \widetilde{f}^\delta$. The proof of Lemma is now complete in view of Lemma 2.1 (ii). □

Here in order to give a simple proof of Lemma 3.2 we use $\overline{U}^\delta$ and $(\psi^\delta)^{-1}$ with regularities of higher order than we need.

In the following lemma we denote L^2–norm and L^2–inner product by $||\cdot||$ and $(\ , \ )$, respectively, if not stated otherwise. Furthermore in the remainder of this section, we write simply $\widetilde{A_i} = \widetilde{A}_i(\widetilde{\widetilde{U}})$ and $\widetilde{A_i}^\delta = \widetilde{A}_i(\widetilde{\widetilde{U}}^\delta)$, $i = 0,1,2,3$.

Lemma 3.3. *The solution $\widetilde{U}^\delta$ of the problem* (3.1) *satisfies the following two estimates:*

$$||\widetilde{U}^\delta(t)||_{H^1(\mathbb{R}^3_+)} \leq C||\widetilde{U}^\delta(0)||_{H^1(\mathbb{R}^3_+)} \qquad \textit{for } t \in [0, T_1], \tag{3.3}$$

$$\begin{aligned}&(\widetilde{A}_0^\delta \partial_1 \widetilde{U}^\delta, \partial_1 \widetilde{U}^\delta)(t) - (\widetilde{A}_0^\delta \partial_1 \widetilde{U}^\delta, \partial_1 \widetilde{U}^\delta)(t') \\ &\leq C \sum_{i,j} |(\widetilde{w}^{i,\delta}(t), \partial_1 \widetilde{w}^{j,\delta}(t)) - (\widetilde{w}^{i,\delta}(t'), \partial_1 \widetilde{w}^{j,\delta}(t'))| + C \int_{t'}^t ||\widetilde{U}^\delta||^2_{H^1} dt \\ &\qquad\qquad \textit{for } t, t' \in [0, T_1].\end{aligned} \tag{3.4}$$

Here the $w^{k,\delta}$'s, are certain linear combinations of the components of $\widetilde{U}^\delta$ whose coefficients are uniform bounded in $C^1([0, T_1] \times \overline{\mathbb{R}}^3_+)$ with respect to δ and $\sum_{i,j}$ is a certain finite sum (see the discussion following (3.7) *below). C is a positive constant independent of δ and unknown functions.*

Proof. We omit simply the indices δ and tilde in the proof.

First we prove (3.4). Since

$$(A_0 \partial_1 U, \partial_1 U)(t) - (A_0 \partial_1 U, \partial_1 U)(t') = \int_{t'}^t \int_{\mathbb{R}^3_+} \partial_\tau (A_0 \partial_1 U, \partial_1 U)(\tau, y) dy d\tau, \tag{3.5}$$

we have from (3.1) that for a constant $C > 0$ depending only on $\overline{U}$, $\mathbb{P}$ and their derivatives up to the second order

The right hand side of (3.5)

$$\leq - \int_{t'}^t \int_{\mathbb{R}^3_+} \sum_{j=1}^3 \partial_j (A_j \partial_1 U, \partial_1 U) dy d\tau + C \int_{t'}^t ||\partial_1 U|| \cdot ||U||_{H^1(\mathbb{R}^3_+)} d\tau. \tag{3.6}$$

Using $(3.1)_2$, $(3.1)_3$, and (2.5), we see from the corresponding form to (2.11) that

$$\begin{aligned}&\partial_1 q|_{y_1=0} = \widehat{l}_{21}|_{y_1=0}, \\ &\partial_1 u_1|_{y_1=0} = [-\overline{\alpha}\{\partial_t q + \overline{u}_2 \partial_2 q + \overline{u}_3 \partial_3 q - ({}^t\mathbb{P}\mathbb{P}\overline{H})_2 \partial_t H_2 - ({}^t\mathbb{P}\mathbb{P}\overline{H})_3 \partial_t H_3 \\ &\qquad -({}^t\mathbb{P}\mathbb{P}\overline{H})_2(\overline{u}_2\partial_2 + \overline{u}_3\partial_3)H_2 - ({}^t\mathbb{P}\mathbb{P}\overline{H})_3(\overline{u}_2\partial_2 + \overline{u}_3\partial_3)H_3\} \\ &\qquad -\partial_2 u_2 - \partial_3 u_3 + \widehat{l}_1]|_{y_1=0} \qquad\qquad \text{for } t \in [0, T_1].\end{aligned} \tag{3.7}$$

Note that $\overline{U} \in C^3$ and $\mathbb{P} \in C^3$ on a neighborhood of supp U. The first term on the right hand side of (3.6) can be estimated by

$$\begin{aligned}&\int_{t'}^t \int_{\partial\mathbb{R}^3_+} (A_1 \partial_1 U, \partial_1 U)(\tau, 0, y') dy' d\tau \\ &= 2 \int_{t'}^t \int_{\partial\mathbb{R}^3_+} (\partial_1 q, \partial_1 u_1)(\tau, 0, y') dy' d\tau \\ &\leq C \sum_{i\ j} \Big(\Big| \int_{t'}^t \int_{\mathbb{R}^3_+} \partial_1 (w^i, \partial_\tau w^j) dy d\tau \Big| + \Big| \int_{t'}^t \int_{\mathbb{R}^3_+} \partial_1 (w^i, \partial_2 w^j) dy d\tau \Big| \\ &\quad + \Big| \int_{t'}^t \int_{\mathbb{R}^3_+} \partial_1 (w^i, \partial_3 w^j) dy d\tau \Big| + \Big| \int_{t'}^t \int_{\mathbb{R}^3_+} \partial_1 (w^i, w^j) dy d\tau \Big| \Big).\end{aligned} \tag{3.8}$$

The first term in the parenthesis for fixed $i,\ j$ on the right hand side of (3.8) equals to

$$\left|\int_{t'}^{t}\int_{\mathbb{R}^3_+}\partial_\tau(w^i,\partial_1 w^j)dyd\tau+\int_{t'}^{t}\int_{\mathbb{R}^3_+}((\partial_1 w^i,\partial_\tau w^j)-(\partial_\tau w^i,\partial_1 w^j))dyd\tau\right|,$$

where $\partial_\tau w^k$, is written as a sum of the derivatives of components of U with respect to space variables. Therefore this term is bounded by

$$|(w^i(t),\partial_1 w^j(t))-(w^i(t'),\partial_1 w^j(t'))|+C\int_{t'}^{t}||U||^2_{H^1(\mathbb{R}^3_+)}d\tau.$$

The second term there equals to

$$\left|\int_{t'}^{t}\int_{\mathbb{R}^3_+}(\partial_1 w^i,\partial_2 w^j)dydt-\int_{t'}^{t}\int_{\mathbb{R}^3_+}(\partial_2 w^i,\partial_1 w^j))dydt\right|$$
$$\le C\int_{t'}^{t}||U||^2_{H^1(\mathbb{R}^3_+)}d\tau.$$

Therefore evaluating also the third and forth terms there in a similar way, we have the following: for a constant $C>0$ depending only on $\overline{U}$, $\mathbb{P}$ and their derivatives up to the second order

The right hand side of (3.8)

$$\le C\sum_{i,j}|(w^i(t),\partial_1 w^j(t))-(w^i(t'),\partial_1 w^j(t'))|+C\int_{t'}^{t}||U||^2_{H^1(\mathbb{R}^3_+)}dt. \qquad (3.9)$$

Thus applying the standard energy method to other terms of (3.6) and taking account of (3.2) we have our assertion (3.4).

Finally we show (3.3). Let $t'=0$ in (3.4). Then by the positive definiteness of A_0 we obtain

$$||\partial_1 U(t)||^2\le C(||U(t)||^2+||U(0)||||\partial_1 U(0)||+||\partial_1 U(0)||^2$$
$$+\int_0^t ||U(\tau)||_{H^1}||\partial_1 U(\tau)||d\tau).$$

We use L^2–estimates for $U(t)$ and the tangential derivatives $\partial_i U(t),\ i=2,3$, here. Then applying the Gronwall lemma, we get finally

$$||U(t)||^2_{H^1}\le C||U(0)||^2_{H^1}.$$

Thus the estimate (3.3) is established. □

Proof of Theorem I. From the remark in the proof of Lemma 3.2 we have

$$\text{supp}\widetilde{U}^\delta(t)\subset\subset\mathcal{B}(0)\quad\text{for } t<T_1<<1,$$

since supp$\widetilde{f}^\delta \subset\subset$ supp$\widetilde{\phi}_\alpha \subset\subset \mathcal{B}(0)$. For this reason we always assume in the following that $T_1 << 1$.

Then from (3.3) we have that for any $t,\ t' \in [0, T_1]$ and for a positive constant C independent of δ

$$||\widetilde{U}^\delta(t) - \widetilde{U}^\delta(t')|| \leq C|t - t'|,$$

since $\{\widetilde{U}^\delta\}$ is a bounded sequence in $C([0, T_1]; H^1(\mathbb{R}^3_+))$. Furthermore the facts that supp$\widetilde{U}^\delta(t) \subset$ a compact set for any $\delta,\ t \in [0, T_1]$ and that the adjoint space of H^1 contains $L^2(\mathbb{R}^3_+)$ densely imply also by the Ascoli–Arzéla theorem the following: there exist a subsequence $\{\widetilde{U}^{\delta'}\}$ and $\widetilde{U}$ such that

$$\widetilde{U}^{\delta'} \longrightarrow \widetilde{U} \quad \text{in} \quad C_w([0, T_1]; H^1(\mathbb{R}^3_+)) \bigcap C([0, T_1]; L^2(\mathbb{R}^3_+)) \text{ as } \delta' \longrightarrow 0. \tag{3.10}$$

Now, let

$$||\widetilde{U}(t)||^2_{\mathcal{H}^1(\mathbb{R}^3_+)} = (\widetilde{A}_0\widetilde{U}, \widetilde{U})(t) + \sum_{j=1}^{3}(\widetilde{A}_0\partial_j\widetilde{U}, \partial_j\widetilde{U})(t).$$

Then $||\cdot||_{H^1(\mathbb{R}^3_+)}$ is equivalent to $||\cdot||_{\mathcal{H}^1(\mathbb{R}^3_+)}$. To show $\widetilde{U} \in C([0, T_1]; H^1(\mathbb{R}^3_+))$ it suffices to prove

$$||\widetilde{U}(t)||_{\mathcal{H}^1(\mathbb{R}^3_+)} \longrightarrow ||\widetilde{U}(t')||_{\mathcal{H}^1(\mathbb{R}^3_+)} \quad \text{as } t \longrightarrow t'.$$

It follows form the energy inequalities that

$$\begin{aligned} &|(\widetilde{A}_0\widetilde{U}, \widetilde{U})(t) - (\widetilde{A}_0\widetilde{U}, \widetilde{U})(t')| \longrightarrow 0, \\ &|(\widetilde{A}_0\partial_j\widetilde{U}, \partial_j\widetilde{U})(t) - (\widetilde{A}_0\partial_j\widetilde{U}, \partial_j\widetilde{U})(t')| \longrightarrow 0, \quad j = 2, 3, \qquad \text{as } t \longrightarrow t'. \end{aligned}$$

To show that

$$|(\widetilde{A}_0\partial_1\widetilde{U}, \partial_1\widetilde{U})(t) - (\widetilde{A}_0\partial_1\widetilde{U}, \partial_1\widetilde{U})(t')| \longrightarrow 0 \quad \text{as } t \longrightarrow t', \tag{3.11}$$

first let $t > t'$. Regarding $\widetilde{U}(t')$ as initial data at t', as in Lemma 3.1 we approximate them by $\widetilde{U}^\delta(t')$ such that $\widetilde{U}^\delta(t') \longrightarrow \widetilde{U}(t')$ in $H^1(\mathbb{R}^3_+)$ as $\delta \longrightarrow 0$. Then by Lemma 3.2 we have the solution $\widetilde{U}^\delta(t)$ to the problem (3.1) with initial data $\widetilde{U}^\delta(t')$ at t'. Obviously we have

$$\liminf_{\delta \longrightarrow 0} (\widetilde{A}^\delta_0\partial_1\widetilde{U}^\delta, \partial_1\widetilde{U}^\delta)(t) \leq (\widetilde{A}_0\partial_1\widetilde{U}, \partial_1\widetilde{U})(t).$$

Therefore from (3.4), (3.10) we see

$$\begin{aligned} &(\widetilde{A}_0\partial_1\widetilde{U}, \partial_1\widetilde{U})(t) - (\widetilde{A}_0\partial_1\widetilde{U}, \partial_1\widetilde{U})(t') \\ &\leq \liminf_{\delta \longrightarrow 0} \{(\widetilde{A}^\delta_0\partial_1\widetilde{U}^\delta, \partial_1\widetilde{U}^\delta)(t) - (\widetilde{A}^\delta_0\partial_1\widetilde{U}^\delta, \partial_1\widetilde{U}^\delta)(t')\} \\ &\leq \liminf_{\delta \longrightarrow 0} \{C\sum_{i,j}|(\widetilde{w}^{i,\delta}(t), \partial_1\widetilde{w}^{j,\delta}(t)) - (\widetilde{w}^{i,\delta}(t'), \partial_1\widetilde{w}^{j,\delta}(t'))| + C\int_{t'}^{t} ||\widetilde{U}^\delta||^2_{H^1(\mathbb{R}^3_+)}dt\} \\ &= C\sum_{i,j}|(\widetilde{w}^i(t), \partial_1\widetilde{w}^j(t)) - (\widetilde{w}^i(t'), \partial_1\widetilde{w}^j(t'))| + C(t - t'). \end{aligned}$$

Using the reversibility in time of our problem (2.6) we may regard $\widetilde{U}(t)$ as initial data at t and solve the problem (3.1) for $t' < t$ and for approximate initial data at t. Using the same argument as above we obtain the analogous estimate with respect to the absolute value of the left hand side of the above inequality. Therefore we have (3.11) since $\widetilde{U} \in C_w([0,T_1];H^1(\mathbb{R}^3_+)) \bigcap C([0,T_1];L^2(\mathbb{R}^3_+))$. Thus we see that $\widetilde{U} \in C([0,T_1];H^1(\mathbb{R}^3_+))$. Finally using (3.1) and (3.10), by certain limit processes we obtain that $\widetilde{U}$ is the uniqueness solution of (2.6). The proof of Theorem I is complete. □

Appendix

Here we summarize Lemmas referred in previous sections and give outlines of those simple proofs for reader's convenience and for completion of our paper.

Throughout Appendix we assume that for some $l \geq 1$ $\widetilde{\widetilde{U}}$, $\widetilde{\widetilde{U}}^{\varepsilon} \in C^{l+1}([0,T_1]\times\overline{\mathbb{R}^3_+})$ and $\mathbb{P}$, $\mathbb{P}^{\varepsilon} \in C^{l+1}(\overline{\mathbb{R}^3_+})$ such that $\widetilde{\widetilde{U}}^{\varepsilon} \longrightarrow \widetilde{\widetilde{U}}$ in $C^{l+1}([0,T_1]\times\overline{\mathbb{R}^3_+})$, $\mathbb{P}^{\varepsilon} \longrightarrow \mathbb{P}$ in $C^{l+1}(\overline{\mathbb{R}^3_+})$, if not stated otherwise. Moreover we assume previously that $\widetilde{M\widetilde{\widetilde{U}}} = \widetilde{M\widetilde{\widetilde{U}}^{\varepsilon}} = 0$ and $\widetilde{N}\widetilde{\widetilde{U}} = \widetilde{N}\widetilde{\widetilde{U}}^{\varepsilon} = 0$ on $[0,T]\times\partial\mathbb{R}^3_+$. In the proof of following lemmas we drop the tilde over letters and denote simply $\widetilde{A}_i(\widetilde{\widetilde{U}})$, $\widetilde{B}(\widetilde{\widetilde{U}})$ and $\widetilde{A}_i(\widetilde{\widetilde{U}}^{\varepsilon})$, $\widetilde{B}(\widetilde{\widetilde{U}}^{\varepsilon})$ by A_i, B and A_i^{ε}, B^{ε} which involve smoothly also entries of $\mathbb{P}$ and $\mathbb{P}^{\delta}$ with their derivatives of the first order, respectively.

A.1 Compatibility condition

Recall the compatibility conditions of order $l-1$ defined as follows: given the system (2.4), boundary condition $MU = 0$ on $[0,T]\times\partial\Omega$ and initial condition $U(0,x) = f(x)$ for $x \in \Omega$, we define $f^{(p)}$, $p \geq 1$ successively by formally taking derivatives of order up to $p-1$ of the system with respect to the time variable, solving for $\partial_t^p U$ and evaluating at $t=0$. Thus $f^{(p)}$ is written as a sum of the derivatives (with respect to the space variable) of f of order at most p. We set $f^{(0)} = f$. Then the compatibility conditions of order $l-1$ are that $Mf^{(p)} = 0$ on $\partial\Omega$, $0 \leq p \leq l-1$.

Then the initial data f are said to satisfy the compatibility conditions of order $l-1$ for the equations $(2.4)_1$ and $(2.4)_2$.

Lemma A.1. *Let $\widetilde{f}$ belong to $H^l(\mathbb{R}^3_+)$ which satisfies the following conditions* (i) *and* (ii)*:*

(i) *$\widetilde{f}$ enjoys the compatibility conditions of order $l-1$ for $(A.1)$ corresponding to* (2.6)*:*

$$\begin{aligned} &\widetilde{A}_0(\widetilde{\widetilde{U}})\partial_t\widetilde{U} + \sum_{j=1}^{3}\widetilde{A}_j(\widetilde{\widetilde{U}})\partial_j\widetilde{U} + \widetilde{B}(\widetilde{\widetilde{U}})\widetilde{U} = 0 \quad \text{in } [0,T]\times\mathbb{R}^3_+, \\ &\widetilde{M}\widetilde{U} = 0 \quad \text{on } [0,T]\times\partial\mathbb{R}^3_+. \end{aligned} \tag{A.1}$$

(ii) $\widetilde{N}\widetilde{f} = 0$ *on* $\partial\mathbb{R}^3_+$.

Then there exist $\tilde{f}^\varepsilon \in H^l(\mathbb{R}^3_+)$ *such that*

(i) $\tilde{f}^\varepsilon \longrightarrow \tilde{f}$ *in* $H^l(\mathbb{R}^3_+)$ *as* $\varepsilon \longrightarrow 0$, *supp* $\tilde{f}^\varepsilon \subset\subset$ *a neighborhood of supp* $\tilde{f}$ *and supp* $\tilde{f}^\varepsilon$ *are contained in a compact set for* $\varepsilon \leq 1$.

(ii) *Each* $\tilde{f}^\varepsilon$ *satisfies the compatibility conditions of order* $l-1$ *for* $(A.2)$:

$$\begin{aligned} &\tilde{A}_0(\tilde{\tilde{U}}^\varepsilon)\partial_t\tilde{U}^\varepsilon + \sum_{j=1}^{3}\tilde{A}_j(\tilde{\tilde{U}}^\varepsilon)\partial_j\tilde{U}^\varepsilon + \tilde{B}(\tilde{\tilde{U}}^\varepsilon)\tilde{U}^\varepsilon = 0 \quad \text{in } [0,T]\times\mathbb{R}^3_+, \\ &\widetilde{M}\tilde{U}^\varepsilon = 0 \qquad\qquad \text{on } [0,T]\times\partial\mathbb{R}^3_+. \end{aligned} \tag{A.2}$$

(iii) $\tilde{N}\tilde{f}^\varepsilon = 0$ *on* $\partial\mathbb{R}^3_+$ *for any* ε.

Proof. First we find $g^\varepsilon \in H^l(\mathbb{R}^3_+)$ such that $g^\varepsilon \longrightarrow f$ in $H^l(\mathbb{R}^3_+)$ as $\varepsilon \longrightarrow 0$, supp g^ε is compact and $Ng^\varepsilon = 0$ on $\partial\mathbb{R}^3_+$. Using the same notation as in [11] we shall prove the existence of vector valued functions h^ε which satisfy the following relations:

$$\begin{aligned} &h^\varepsilon \in H^l(\mathbb{R}^3_+), \\ &h^\varepsilon \longrightarrow 0 \quad \text{in } H^l(\mathbb{R}^3_+), \\ &MB^\varepsilon_p h^\varepsilon = MB^\varepsilon_p g^\varepsilon \quad \text{on } \partial\mathbb{R}^3_+,\ 1\leq p\leq l-1, \\ &\text{where } B^\varepsilon_0 = I, \\ &\quad B^\varepsilon_p g^\varepsilon = ((A^\varepsilon_0)^{-1}A^\varepsilon_1)^p\partial^p_1 g^\varepsilon + \sum_{i=0}^{p-1} C^\varepsilon_{p,p-i}\partial^i_1 g^\varepsilon (= (g^\varepsilon)^{(p)}), \\ &\quad \text{here } C^\varepsilon_{p,p-i} \text{ is a differential operater of order at most } p-i \\ &\quad \text{involving only the differentiation } \partial_{y_2} \text{ and } \partial_{y_3}, \\ &Nh^\varepsilon = 0 \quad \text{on } \partial\mathbb{R}^3_+. \end{aligned} \tag{A.3}$$

Then setting $f^\varepsilon = g^\varepsilon - h^\varepsilon$, we have the desired f^ε. To construct such h^ε, we rewrite $(A.3)_3$ as follows:

$$\begin{aligned} &Mh^\varepsilon = Mg^\varepsilon &&\text{on } \partial\mathbb{R}^3_+, \\ &M(\hat{A}^\varepsilon_1)^p\partial^p_1 h^\varepsilon = MB^\varepsilon_p g^\varepsilon - MK^\varepsilon_p &&\text{on } \partial\mathbb{R}^3_+,\quad 1\leq p\leq l-1, \end{aligned} \tag{A.4}$$

where

$$\hat{A}^\varepsilon_1 = (A^\varepsilon_0)^{-1}A^\varepsilon_1,\ K^\varepsilon_p = \sum_{i=0}^{p-1} C^\varepsilon_{p,p-i}\partial^i_1 h^\varepsilon.$$

Here we notice from (2.7), (2.9) and (2.10) that

$$\begin{aligned} &M = M^2,\ MA^\varepsilon_0 = A^\varepsilon_0 M,\ \text{Ker } \hat{A}^\varepsilon_1 \cap \text{Range } \hat{A}^\varepsilon_1 = \{0\}, \\ &\text{Ker } \hat{A}^\varepsilon_1 = \text{Ker } A^\varepsilon_1 \subset \text{Ker } M \quad \text{on } \partial\mathbb{R}^3_+. \end{aligned} \tag{A.5}$$

Now let x' be an arbitrary point in $\partial\mathbb{R}^3_+$. Let $C(x')$ be a sum of circles each of which contains only non–zero eigenvalue of $\hat{A}^\varepsilon_1$. Define $T^\varepsilon(x')$ by

$$T^\varepsilon_p = T^\varepsilon_p(x') = \frac{1}{2\pi i}\int_{C(x')}\frac{1}{\lambda^p}(\lambda - \hat{A}^\varepsilon_1)^{-1}d\lambda \quad \text{on } \partial\mathbb{R}^3_+, \tag{A.6}$$

which is a real matrix-valued function on $\partial\mathbb{R}^3_+$ since A^ε_i, $i=0,1$, and the eigenvalues of $\widehat{A}^\varepsilon_1$ are all real. Then by the definition it follows that

$$T^\varepsilon_0 = P_{\text{Range } \widehat{A}^\varepsilon_1}, \quad T^\varepsilon_0 = (\widehat{A}^\varepsilon_1)^p T^\varepsilon_p = T^\varepsilon_p(\widehat{A}^\varepsilon_1)^p \quad \text{on } \partial\mathbb{R}^3_+. \tag{A.7}$$

Finally we define the boundary values b^ε_p, of h^ε to be found, inductively as follows:

$$\begin{aligned} &b^\varepsilon_0 = Mg^\varepsilon, \\ &b^\varepsilon_p = T^\varepsilon_p(MB^\varepsilon_p g^\varepsilon - P_{\text{Range } \widehat{A}^\varepsilon_1}\sum_{i=0}^{p-1} C^\varepsilon_{p,p-i} b^\varepsilon_i) \quad \text{on } \partial\mathbb{R}^3_+,\ 1 \le p \le l-1. \end{aligned} \tag{A.8}$$

Then by the same way as in the proof of Lemma 3.3 in [11] we see conversely that there exists h^ε such that $b^\varepsilon_p = \partial^p_1 h^\varepsilon$ on $\partial\mathbb{R}^3_+$, $0 \le p \le l-1$. Thus we have that the resulting h^ε satisfies (A.3). Because from (A.5) it is seen that $K^\varepsilon_p = P_{\text{Ker } \widehat{A}^\varepsilon_1} K^\varepsilon_p + P_{\text{Range } \widehat{A}^\varepsilon_1} K^\varepsilon_p$, Range $\widehat{A}^\varepsilon_1 \supset$ Range M on $\partial\mathbb{R}^3_+$, from which (A.7) yields that $b^\varepsilon_p \in$ Range $\widehat{A}^\varepsilon_1$ and that $(A.3)_3$ is valid. Next by the fact that $NM = 0$ we see that $(A.3)_4$ is valid. Furthermore by $(A.6)$, the smoothness of $\overline{U}$, $\overline{U}^\varepsilon$, the constancy of rank A_1 and the compatibility conditions with respect to U_0 we obtain that $b^\varepsilon_p \longrightarrow 0$ in $H^{l-1-\frac{1}{2}}(\partial\mathbb{R}^3_+)$ as $\varepsilon \longrightarrow 0$, $0 \le p \le l-1$, from which it follows $(A.3)_1$ and $(A.3)_2$. This completes the proof of Lemma A.1. □

Corollary A.1. *Let $\widetilde{f}$ satisfy* (i) *and $\widetilde{f}_H = 0$ on $\mathbb{R}^3_+$ instead of* (ii) *in the assumption of* Lemma A.1. *Then there are $\widetilde{f}^\varepsilon \in H^l(\mathbb{R}^3_+)$ satisfy*

(iv) $(\widetilde{f}^\varepsilon)_H = 0$ *in* $\mathbb{R}^3_+$

with both (i) *and* (ii) *in conclusion of* Lemma A.1.

Proof. As in the proof of Lemma A.1, but setting $(g^\varepsilon)_H = 0$ in $\mathbb{R}^3_+$ instead of that $Ng^\varepsilon = 0$ on $\partial\mathbb{R}^3_+$, we define b^ε_0 and b^ε_p as follows:

$$\begin{aligned} &b^\varepsilon_0 = Mg^\varepsilon, \\ &b^\varepsilon_p = P_{\text{Range} A^\varepsilon_1} T^\varepsilon_p(MB^\varepsilon_p g^\varepsilon - P_{\text{Range } \widehat{A}^\varepsilon_1} K^\varepsilon_p) \quad \text{on } \partial\mathbb{R}^3_+,\ 1 \le p \le l-1. \end{aligned}$$

Since

$$P_{\text{Range} A^\varepsilon_1} = \begin{array}{l}\text{all elements are equal to zero except} \\ \text{that the (1,1) and (2,2) entries are equal to 1}\end{array} \quad \text{on } \partial\mathbb{R}^3_+,$$

we have

$$(A^\varepsilon_0)^{-1} A^\varepsilon_1 P_{\text{Range} A^\varepsilon_1} = (A^\varepsilon_0)^{-1} A^\varepsilon_1,\ MP_{\text{Range} A^\varepsilon_1} = M \quad \text{on } \partial\mathbb{R}^3_+.$$

Therefore the b^ε_p defined above has the same properties as in Lemma A.1, except that $b^\varepsilon_p \in \text{Range} A^\varepsilon_1$ on $\partial\mathbb{R}^3_+$, $0 \le p \le l-1$, from which it follows that $(b^\varepsilon_p)_H = 0$ on $\partial\mathbb{R}^3_+$, $0 \le p \le l-1$. Accordingly from the proof of Lemma 3.3 in [11], we see that $(h^\varepsilon)_H = 0$ on $\mathbb{R}^3_+$. □

Lemma A.2. *Let $m \geq 1$ be integer. Let initial data $\widetilde{f} \in H^l(\mathbb{R}^3_+)$ satisfy the compatibility conditions of order $l-1$ for (A.1). Here we assume that $\overline{\widetilde{U}}^\varepsilon \in C^{l+2m+1}([0,T]\times \overline{\mathbb{R}^3_+})$. Then there exist $\widetilde{f}^\varepsilon$ having the following properties:*

(i) $\widetilde{f}^\varepsilon \in H^{l+m}(\mathbb{R}^3_+)$.

(ii) $\widetilde{f}^\varepsilon \longrightarrow \widetilde{f}$ *in* $H^l(\mathbb{R}^3_+)$ *as* $\varepsilon \longrightarrow 0$ *and supp* $\widetilde{f}^\varepsilon \subset\subset$ *a neighborhood of supp* $\widetilde{f}$.

(iii) *Each* $\widetilde{f}^\varepsilon$ *satisfies the compatibility conditions of order* $(l-1)+m$ *with respect to* (A.2).

(iv) $\widetilde{N}\widetilde{f}^\varepsilon = 0$ *on* $\{y_1 = 0\}$, *if* $\widetilde{N}\widetilde{f} = 0$ *on* $\{y_1 = 0\}$.

(v) $(\widetilde{f}^\varepsilon)_H = 0$ *in* $\mathbb{R}^3_+$, *if* $\widetilde{f}_H = 0$ *there.*

Proof. Here we may consider only the case where $f_H = 0$ in $\mathbb{R}^3_+$.

Let $g^\varepsilon \in H^{l+2m}(\mathbb{R}^3_+)$ such that $g^\varepsilon \longrightarrow f$ in $H^l(\mathbb{R}^3_+)$ as $\varepsilon \longrightarrow 0$ and $(g^\varepsilon)_H = 0$ in $\mathbb{R}^3_+$. Then we shall show the existence of h^ε such that

$$\begin{aligned} &h^\varepsilon \in H^{l+m}(\mathbb{R}^3_+), \quad h^\varepsilon \longrightarrow 0 \quad \text{in } H^l(\mathbb{R}^3_+), \\ &MB_p h^\varepsilon = MB_p g^\varepsilon \quad \text{on } \partial\mathbb{R}^3_+,\ 1 \leq p \leq (l-1)+m, \\ &(h^\varepsilon)_H = 0 \quad \text{in } \mathbb{R}^3_+. \end{aligned}$$

Using regularity of higher order of $\overline{U}^\varepsilon$ and g^ε than that in Lemma A.1, by the same way as in this lemma and in Corollary A.1, we obtain b^ε_p such that

$$\begin{aligned} &b^\varepsilon_p \in H^{l+2m-p-\frac{1}{2}}(\partial\mathbb{R}^3_+),\ 0 \leq p \leq (l-1)+m, \\ &(b^\varepsilon_p)_H = 0 \quad \text{on } \partial\mathbb{R}^3_+, \\ &b^\varepsilon_p \longrightarrow 0 \quad \text{in } H^{l-p-\frac{1}{2}}(\partial\mathbb{R}^3_+),\ 0 \leq p \leq l-1. \end{aligned}$$

Then by a certain refinement of the proof of Lemma 3.3 in [11] and by using the b^ε_p above mentioned we construct h^ε directly as follows:

$$h^\varepsilon \in H^{l+m}(\mathbb{R}^3_+), \quad (h^\varepsilon)_H = 0 \quad \text{in } \mathbb{R}^3_+,\ h^\varepsilon \longrightarrow 0 \quad \text{in } H^l(\mathbb{R}^3_+)$$

$$\text{and } \partial_1^p h^\varepsilon = b^\varepsilon_p \quad \text{on } \partial\mathbb{R}^3_+, \quad 0 \leq p \leq (l-1)+m.$$

Therefore setting $f^\varepsilon = g^\varepsilon - h^\varepsilon$ we see the assertion of Lemma A.2. □

Now we consider, as in [10], the non–characteristic initial boundary value problem for ε, $0 < \varepsilon << 1$, whose boundary condition is maximal nonnegative:

$$\begin{aligned} &\widetilde{A}_0(\overline{\widetilde{U}})\partial_t\widetilde{U}^\varepsilon + \sum_{j=1}^{3}\widetilde{A}_j(\overline{\widetilde{U}})\partial_j\widetilde{U}^\varepsilon - \varepsilon\widetilde{A}_0(\overline{\widetilde{U}})\partial_1\widetilde{U}^\varepsilon + \widetilde{B}(\overline{\widetilde{U}})\widetilde{U}^\varepsilon = 0 \quad \text{in } [0,T_1]\times\mathbb{R}^3_+, \\ &\widetilde{M}\widetilde{U}^\varepsilon = 0 \qquad \text{on } [0,T_1]\times\partial\mathbb{R}^3_+, \qquad\qquad (A.9) \\ &\widetilde{U}^\varepsilon(0,y) = \widetilde{f}^\varepsilon \qquad \text{with compact support in } \overline{\mathbb{R}^3_+}. \end{aligned}$$

Then we have

Lemma A.3. *Let $\widetilde{f} \in H^l(\mathbb{R}^3_+)$ satisfy compatibility conditions for $(A.9)$ with $\varepsilon = 0$. Then there are $\widetilde{f}^\varepsilon$ such that*

(i) $\widetilde{f}^\varepsilon \in H^l(\mathbb{R}^3_+)$.

(ii) $\widetilde{f}^\varepsilon \longrightarrow \widetilde{f}$ *in* $H^l(\mathbb{R}^3_+)$ *as* $\varepsilon \longrightarrow 0$ *and* $supp\widetilde{f}^\varepsilon \subset\subset$ *a neighborhood of* $supp\widetilde{f}$.

(iii) *Each* $\widetilde{f}^\varepsilon$ *satisfies the compatibility conditions of order* $l-1$ *for* $(A.9)$.

Proof. Here we shall construct h^ε in the analogous way in the proof of Lemma A.1. Since the boundary matrix of (A.9) is $A_1 - \varepsilon A_0$, we must solve the following equations:

$$\begin{aligned} &h^\varepsilon = Mf, \\ &(\widehat{A}_1 - \varepsilon I)^p \partial_1^p h^\varepsilon = (MB_p^\varepsilon g^\varepsilon + P_{\mathrm{Ker}\ \widehat{A}_1} K_p^\varepsilon) - K_p^\varepsilon \quad \text{on } \partial\mathbb{R}^3_+, \quad 1 \le p \le (l-1). \end{aligned}$$

Here we set

$$\widehat{A}_1 - \varepsilon I = (\widehat{A}_1 - \varepsilon P_{\mathrm{Range}\widehat{A}_1}) - \varepsilon P_{\mathrm{Ker}\widehat{A}_1} \equiv \overline{\overline{A}}_1 + \overline{\overline{A}}_2$$

and

$$(\overline{\overline{A}}_1 + \overline{\overline{A}}_2)^p \equiv \overline{\overline{A}}_1^p + B_p.$$

Then form $\overline{\overline{A}}_1 \cdot \overline{\overline{A}}_2 = \overline{\overline{A}}_2 \cdot \overline{\overline{A}}_1$ it follows that $MB_p = 0$ on $\partial\mathbb{R}^3_+$. Thus we reduce our equations to the following:

$$\begin{aligned} &h^\varepsilon = Mf, \\ &\overline{\overline{A}}_1^p \partial_1^p h^\varepsilon = (MB_p^\varepsilon g^\varepsilon + P_{\mathrm{Ker}\ \widehat{A}_1} K_p^\varepsilon) - K_p^\varepsilon \quad \text{on } \partial\mathbb{R}^3_+, \quad 1 \le p \le (l-1), \end{aligned}$$

which we can solve as the same way in the proof of Lemma A.1. □

A.2. H_*-space

We recall the definition of H_*–space and outline of the proof of the estimate (A.10) described below, which is an extension of that with respect to H_{tan}–space (see Theorem 10 in [10]). Here we may restrict only to the case where $\Omega = \mathbb{R}^3_+$. Given integer $l \ge 1$ the function space $H^l_*(\mathbb{R}^3_+)$ defined as the set of functions $u \in L^2(\mathbb{R}^3_+)$ with the following property: $\partial_*^\alpha \partial_1^k u \in L^2(\mathbb{R}^3_+)$ if $|\alpha| + 2k \le l$, where $\partial_*^\alpha \equiv (\sigma(x_1)\partial_1)^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$. Here $\sigma(x_1)$ is the monotone increasing function such that $\sigma(x_1) \in C^\infty([0,\infty))$, and $\sigma(x_1) = x_1$ for $0 < x_1 < \frac{1}{2}$, $= 1$ for $x_1 > 1$. Then the H^l_*–norm is

$$||U||^2_{H^l_*(\mathbb{R}^3_+)} \equiv \sum_{|\alpha|+2k \le l} ||\partial_*^\alpha \partial_1^k U||^2.$$

Note that ∂_*^α can be replaced by $\sigma(x_1)^{\alpha_1} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ because the corresponding norms are equivalent to each other.

Now from (A.9) we have the following a-priori estimate

Lemma A.4.

(i) *For regular solution* $\widetilde{U}^\varepsilon \in C([0,T_1];H^l(\mathbb{R}^3_+))$ *to* $(A.9)$

$$\sum_{j=0}^{l} \|\partial_t^j \widetilde{U}^\varepsilon(t)\|_{H_*^{l-j}(\mathbb{R}^3_+)} \le C \sum_{j=0}^{l} \|\partial_t^j \widetilde{U}^\varepsilon(0)\|_{H_*^{l-j}(\mathbb{R}^3_+)} \qquad \text{for } t \in [0,T_1], \tag{A.10}$$

where C *is a positive constant depending* T_1, *but independent of* ε *and* $\widetilde{U}^\varepsilon(0)$. *Here we assume that supp* $\widetilde{U}^\varepsilon \subset [0,T_1] \times S(0,r_0)$, $0 < T_1 << 1$, $0 < r_0 << 1$.

(ii) *Let* $\overline{\overline{U}}$ *be constant vector and set* $\mathbb{P} = I$. *Then for regular solution* $\widetilde{U}^\varepsilon \in C([0,\infty);H^l(\mathbb{R}^3_+))$ *to* $(A.9)$

$$\sum_{j=0}^{l} \|\partial_t^j \widetilde{U}^\varepsilon(t)\|_{H_*^{l-j}(\mathbb{R}^3_+)} \le Ce^{\gamma t} \sum_{j=0}^{l} \|\partial_t^j \widetilde{U}^\varepsilon(0)\|_{H_*^{l-j}(\mathbb{R}^3_+)}, \tag{A.11}$$

for all $t > 0$, *where* C, γ *are sufficiently large positive number, but independent of* ε *and* $\widetilde{U}^\varepsilon(0)$.

Outline of the proof. Using an certain orthogonal matrix-valued function T smoothly depending on $\overline{U}$ and $\mathbb{P}$ we can reduce our equations to the form such that

$$\partial_t V^\varepsilon + \sum_{j=1}^{3} \overline{A}_j \partial_j V^\varepsilon - \varepsilon \partial_1 V^\varepsilon + \overline{B} V^\varepsilon = 0. \tag{A.12}$$

Here

$$\overline{A}_j = {}^tTA_0^{-\frac{1}{2}} A_j A_0^{-\frac{1}{2}} T, \ j = 0,1,2,3, \ V^\varepsilon = {}^tTA_0^{\frac{1}{2}} U^\varepsilon$$

and if we set

$$\begin{pmatrix} \overline{A}_1^{II} & \overline{A}_1^{III} \\ \overline{A}_1^{III} & \overline{A}_1^{IIII} \end{pmatrix} = \overline{A}_1,$$

then $\overline{A}_1^{II}$ is nonsingular and $\overline{A}_1^{III} = \overline{A}_1^{III} = \overline{A}_1^{IIII} = 0$ over $[0,T_1] \times (\partial\mathbb{R}^3_+ \bigcap S(0,r_0))$. Furthermore we may choose the above T and a constant real matrix $\overline{M}$ as follows: $\overline{M}V = 0$ if and only if $MA_0^{-\frac{1}{2}}TV = 0$ there for any vector V.

Thus we may regard $\partial\mathbb{R}^3_+$ as characteristic of constant multiplicity with respect to $\overline{A}_1$ and $\overline{M}$. In such a situation we have the following a-priori estimate:

$$\sum_{j=0}^{l} \|\partial_t^j V^\varepsilon(t)\|_{H_*^{l-j}(\mathbb{R}^3_+)} \le K \sum_{j=0}^{l} \|\partial_t^j V^\varepsilon(0)\|_{H_*^{l-j}(\mathbb{R}^3_+)} \quad \text{for } t \in [0,T_1], \tag{A.13}$$

where $V^\varepsilon(t)$ is the regular solution to (A.12) with boundary condition: $\overline{M}V^\varepsilon = 0$ on $\partial\mathbb{R}^3_+$, and K is a positive constant independent of ε. (For the proof of (A.13) see, e.g., [2] and [10] or [9].)

Finally, under assumption (ii), from (A.13) without regard to supp V^ε we obtain the desired estimate. □

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