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Single Point Singularity and Analyticity for the Korteweg - de Vries Equation

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1. INTRODUCTION

We study the smoothing effect for the following Korteweg-de Vries equation:

$$(1.1)\qquad \begin{cases} \partial_t v+\partial_x^3 v+\partial_x(v^2)=0, & t,x\in\mathbb{R},\\ v(0,x)=\phi(x).\end{cases}$$

Here the solution $u(t,x):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ denotes the surface displacement of the water wave.

There are plenty amount of literature for the study of KdV equation. Concerning the smoothing effect of the solution, Kato [11] firstly extract the smoothing effect from the linear part of the KdV equation:

$$(1.2)\qquad \begin{cases} \partial_t v+\partial_x^3 v=0, & t,x\in\mathbb{R},\\ v(0,x)=\phi(x).\end{cases}$$

Let $\chi(x)$ be a smooth non decreasing function with $\chi(x)=0$ if $x<-2R$ and $\chi(x)=1$ for $x>2R$ with $\partial_x\chi(x)=1$ on $-R<x<R$. Then a simple computation shows that

$$(1.3)\qquad \frac{d}{dt}\int\chi v^2dx+\int\partial_x\chi|\partial_x v|^2dx\le C(\|\partial_x^3\chi\|_\infty)\|v(t)\|_2^2.$$

This inequality combining with the L^2 conservation law immediately gives the local smoothing effect for the linear part of the KdV equation:

$$(1.4)\qquad \int_0^T\int_{-R}^R|\partial_x v|^2dxdt\le CR\|\phi\|_2^2+\int_0^T\|v(t)\|_2^2dt.$$

Later on as an extension of the Kato type smoothing estimate , Kenig-Ponce-Vega [13] obtained the L^p version of the homogeneous and inhomogeneous equation of the linear KdV equation:

$$(1.5)\qquad \|D_xe^{t\partial_x^3}\phi\|_{L_x^\infty(\mathbb{R};L_T^2)}\le C\|\phi\|_2$$

$$(1.6)\qquad \|D_x^2\int_0^{\cdot}e^{(\cdot-s)\partial_x^3}F(s)ds\|_{L_x^\infty(\mathbb{R};L_T^2)}\le C\|F\|_{L_x^1(\mathbb{R};L_T^2)}$$

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Using this estimate with some other extension, they showed that the KdV equation is well-posed in the Sobolev space $H^{3/4}$. The Uniqueness result is also obtained by Kurzkov-Faminski [18], Ginibre-Y. Tsutsumi [6] in the subspace of H^1. Since the KdV equation has infinitely many conserved quantities, if for example L^2 well-posedness is established and the dependence of the local existence time T is known by the term of $\|\phi\|_2$, it is shown that the global existence of the L^2 solution is obtained in the large data. Along the elegant method in the series of papers, Bourgain [2] obtained L^2 well-posedness of the KdV equation in the periodic boundary condition. His argument also works for the Cauchy problem 1.2 and the global well-posedness is established. Furthermore, by refining the method by Bourgain, Kenig-Ponce-Vega proved some bilinear estimate involving the negative exponent Sobolev space and established the local well-posedness for the Cauchy problem in the negative Sobolev space $H\ s(\mathbb{R})$ where $(-3/4 < s)$. This result is obtained by the method of Fourier restriction norm as well as the refining estimate for the quadratic nonlinear term in the KdV equation. In fact, the polynomial structure of the nonlinear term has a certain (very subtle) kind of smoothing effect.

On the other hand, very high regularity smoothing effect is also studied by several authors. Hayashi-K.Kato [7] obtained the analyticity for the nonlinear Schrödinger equation and de Bouard-Hayashi-Kato [5] established the analyticity for KdV equations from the Gevrey initial data.

Those results are basically obtained by using the commutation and almost commutation operators with the linear KdV equation.

In this paper, we discuss on the smoothing effect for the initial data has single point singularity at the origin. Since the solution we treat is in very weak space, we consider the equation as a corresponding integral equation. Let $V(t) = e^{-t\partial_x^3}$ be a free KdV evolution group. Then by the D'hamel principle, the solution of KdV equation 1.2 satisfies the following equation.

$$v(t) = V(t)\phi - \int_0^t V(t-t')\partial_x(v(t')^2 dt'.$$

Our result is the following:

Theorem 1.1. *Let $-3/4 < s$, $b \in (1/2, 7/12)$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty.$$

Then there exist $T > 0$ and a unique solution $v \in C((-T,T), H^s) \cap X_b^s$ of the KdV equation (1.2) and the solution is time locally well-posed, i.e. the solution continuously depends on the initial data. Moreover the solution v is analytic at $(t,x) \in (-T,0) \cup (0,T) \times \mathbb{R}$ where *we define*

$$\|f\|_{X_b^s} = \left(\iint \langle\tau - \xi^3\rangle^{2b}\langle\xi\rangle^{2s}|\hat{f}(\tau,\xi)|^2 d\tau d\xi\right)^{1/2} = \|V(-\cdot)f(\cdot)\|_{H_t^b(\mathbb{R};H_x^s(\mathbb{R}))}$$

and $V(t) = e^{-t\partial_x^3}$ *is the unitary group of the free KdV evolution.*

Remark 1. A typical example of the initial data satisfying the assumption of the above theorem is the Dirac delta measure, since $(x\partial_x)^k\delta(x) = (-1)^k\delta(x)$. The other example of the data is $p.v.\frac{1}{x}$, where $p.v.$ denotes Cauchy's principal value. Any possible linear combination of those functions with an analytic, H^s data satisfying the assumption can be also the initial data. In this sense, Dirac's delta measure adding the soliton initial data can also be taken as a initial data.

Remark 2. For a non-smooth initial data, it is known that the global in time solution has been obtained (see [4], [8]) by the inverse scattering method. Also recently the analyticity for the inverse scattering solution with a weighted initial data was obtained by Tarama [20]. However, since our method is based on the fact that the solution is in H^s, we don't know if our result is true globally in time.

By a almost similar argument of Theorem 1.1, one can also show the following corollary.

Corollary 1.2. *Let* $-3/4 < s$, $b \in (1/2, 7/12)$. *Suppose that for some* $A_0 > 0$, *the initial data* $\phi \in H^s(\mathbb{R})$ *and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{(k!)^3}\|(x\partial_x)^k\phi\|_{H^s} < \infty,$$

then there exist $T > 0$ *and a unique solution* $v \in C((-T,T), H^s) \cap X_b^s$ *of the KdV equation (*1.2*) and for any* $t \in (-T,0) \cup (0,T)$ $v(t,\cdot)$ *is analytic function in space variable and for* $x \in \mathbb{R}$, $v(\cdot,x)$ *is of Gevery* 3 *as a time variable function.*

Remark 3. Both in Theorem and Corollary, the assumption on the initial data implies the analyticity and Gevrey 3 regularity except the origin respectively. In this sense, those results are stating that the singularity at the origin immediately disappear after $t > 0$ or $t < 0$ up to analyticity.

Remark 4. Recently, some related results are obtained for the linear and nonlinear Schr see Kajitani-Wakabayashi [9] and for nonlinear case, Chihara [3]. They are giving a global weighted uniform estimates of the solution with arbitrary order derivative in space variable. In our case, it is still unknown if the weighted uniform bounds are possible or not.

2. Method

Our method is based on the following observation. Firstly, we introduce the generator of the dilation $P = 3t\partial_t + x\partial_x$ for the linear part of the KdV equation. Noting the commutation relation with the linear KdV operator $L = \partial_t + \partial_x^3$:

$$[L, P] = 3L,$$

it follows

$$LP^k = (P+3)^k L, \tag{2.1}$$

$$(P+3)^k \partial_x = \partial_x (P+2)^k \tag{2.2}$$

for any $k = 1, 2, \cdots$. Applying $P = 3t\partial_x + x\partial_x$ to the KdV equation, we have

$$\begin{aligned} \partial_t(P^k v) + \partial_x^3(P^k v) &= (P+3)^k Lv = (P+3)^k(-\partial_x(v^2)) \\ &= -\partial_x(P+2)^k v^2. \end{aligned} \tag{2.3}$$

We set $v_k = P^k v$ and $B_k(v,v) = \partial_x(P+2)^k v^2$. Then noting that

$$\begin{aligned} (P+2)^l v &= (P+2)^{l-1}Pv + 2(P+2)^{l-1}v = \cdots \\ &= \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} 2^{l-j} P^j v \end{aligned} \tag{2.4}$$

we see

$$\begin{aligned} B_k(v,v) =& \partial_x(P+2)^k(v^2) = \partial_x \sum_{l=0}^{k} \binom{k}{l} (P+2)^l v P^{k-l} v \\ =& \partial_x \sum_{l=0}^{k} \sum_{m=0}^{l} \binom{k}{l}\binom{l}{m} 2^{l-m} P^m v P^{k-l} v \\ =& \sum_{k=k_1+k_2+k_3} \frac{k!}{k_1! k_2! k_3!} 2^{k_1} \partial_x(v_{k_2} v_{k_3}) \end{aligned}$$

We remark that the above nonlinear term keeps the bilinear structure like the original KdV equation. This is because the Leibniz law can be applicable for a operation of P. Now each v_k satisfies the following system of equations;

$$\begin{cases} \partial_t v_k + \partial_x^3 v_k + B_k(v,v) = 0, & t, x \in \mathbb{R}, \\ v_k(0,x) = (x\partial_x)^k \phi(x). \end{cases} \tag{2.5}$$

Therefore we firstly establish the local well-posedness of the solution to the following infinitely coupled system of KdV equation in a suitable weak space:

$$\begin{cases} \partial_t v_k + \partial_x^3 v_k + B_k(v,v) = 0, & t, x \in \mathbb{R}, \\ v_k(0,x) = \phi_k(x). \end{cases} \tag{2.6}$$

Then taking $\phi_k = (x\partial_x)^k\phi(x)$, the uniqueness and local well-posedness allow us to say $v_k = P^k v$ for all $k = 0, 1, \cdots$.

According to Bourgain [2], we introduce the Fourier restriction space as

$$X_b^s = \{f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{X_b^s} < \infty\},$$

where

$$\|f\|_{X_b^s}^2 = c\iint \langle\tau - \xi^3\rangle^{2b}\langle\xi\rangle^{2s}|\hat{f}(\tau,\xi)|^2 d\tau d\xi = \|V(-t)f\|_{H_t^b(\mathbb{R};H_x^s)}^2.$$

The KdV equation is proven to be well-posed in the above space X_b^s up to $s > -3/4$ with $b > 1/2$. The space where we solve the system is infinitely sum of this space. Let $f = (f_0, f_1, \cdots, f_k, \cdots)$ denotes the infinity series of distributions and define

$$\mathcal{A}_{A_0}(X_b^s) = \{f = (f_0, f_1, \cdots, f_k, \cdots), f_i \in X_b^s \quad (i = 0, 1, 2, \cdots) \text{ such that} \|f\|_{\mathcal{A}_{A_0}} < \infty\},$$

where

$$\|f\|_{\mathcal{A}_{A_0}} \equiv \sum_{k=0}^{\infty} \frac{A_0^k}{k!}\|f_k\|_{X_b^s}.$$

The system will be shown to be well-posed in the above space if $s > -3/4$ and $b > 1/2$.

The well-posedness is derived by utilizing the contraction principle argument to the corresponding system of integral equations:

$$\psi(t)v_k(t) = \psi(t)V(t)\phi_k - \psi(t)\int_0^t V(t-t')\psi_T(t')B_k(v,v)(t')dt' \tag{2.7}$$

The following estimates of linear and nonlinear part due to Bourgain [2] and refined by Kenig-Ponce-Vega [?] are our essential tools.

Lemma 2.1. *Let $s \in \mathbb{R}$, $a, a' \in (0, 1/2)$, $b \in (1/2, 1)$and $\delta < 1$. Then for any $k = 0, 1, 2, \cdots$, we have*

$$\|\psi_\delta\phi_k\|_{X_{-a}^s} \le C\delta^{(a-a')/4(1-a')}\|\phi_k\|_{X_{-a'}^s}, \tag{2.8}$$

$$\|\psi_\delta V(t)\phi_k\|_{X_b^s} \le C\delta^{1/2-b}\|\phi_k\|_{H^s} \tag{2.9}$$

$$\|\psi_\delta\int_0^t V(t-t')F_k(t')dt'\|_{X_b^s} \le C\delta^{1/2-b}\|F_k\|_{X_b^s} \tag{2.10}$$

Lemma 2.2. *Let $s > -3/4$, $b, b' \in (1/2, 7/12)$ with $b < b'$ and $\delta < 1$. Then for any $k, l = 0, 1, 2, \cdots$, we have*

$$\|\partial_x(u_k v_l)\|_{X_{b'-1}^s} \le C\delta^{1/2-b}\|v_k\|_{X_b^s}\|v_l\|_{X_b^s} \tag{2.11}$$

Proof of Lemma 2.1 and 2.2. See [13]. ❑

¿From Lemma 2.2, it is immediately obtained the bilinear estimate for the nonlinearity for the system.

Corollary 2.3. ?? *Let $s > -3/4$, $b, b' \in (1/2, 7/12)$ with $b < b'$ and $\delta < 1$. Then, we have*

$$\|B_k(v,v)\|_{X^s_{b'-1}} \le C\delta^{1/2-b} \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{k!}{k_1!k_2!k_3!} \|v_{k_2}\|_{X^s_b} \|v_{k_3}\|_{X^s_b} \tag{2.12}$$

Set a map $\Phi : \{v_k\}_{k=0}^{\infty} \to \{v_k(t)\}_{k=0}^{\infty}$ such that $\Phi = (\Phi_0, \Phi_1, \cdots)$ and

$$\Phi_k(\phi_k) \equiv \psi V(t)\phi_k - \psi \int_0^t V(t-t')B_k(v,v)(t')dt'$$

Then it is shown that $\Phi_k : \mathcal{A}_{A_0}(H^s) \to \mathcal{A}_{A_1}(X^s_b)$ is a contraction. In fact, by using Lemma 2.1 and Lemma 2.2, we easily see that

$$\begin{aligned}
\|\Phi\|_{\mathcal{A}_{A_1}(X^s_b)} =& \sum_{k=0}^{\infty} \frac{A_1^k}{k!} \|v_k\|_{X^s_b} \\
=& C_0 \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|\phi_k\|_{H^s} + C_1 T^\mu \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{k!}{k_1!k_2!k_3!} \|v_{k_2}\|_{X^s_b} \|v_{k_3}\|_{X^s_b} \\
=& C_0 \|v\|_{\mathcal{A}_{A_0}(H^s)} + C_1 T^\mu \sum_{k=0}^{\infty} \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{A_0^{k_1}}{k_1!} \frac{A_0^{k_2}}{k_2!} \|v_{k_2}\|_{X^s_b} \frac{A_0^{k_3}}{k_3!} \|v_{k_3}\|_{X^s_b} \\
\le& C_0 \|v\|_{\mathcal{A}_{A_0}(H^s)} + C_1 T^\mu \sum_{k_1=0}^{\infty} 2^{k_1} \frac{A_0^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} \frac{A_0^{k_2}}{k_2!} \|v_{k_2}\|_{X^s_b} \sum_{k_3=0}^{\infty} \frac{A_0^{k_3}}{k_3!} \|v_{k_3}\|_{X^s_b}.
\end{aligned}$$

It follows

$$\|\Phi(v)\|_{\mathcal{A}_{A_1}(X^s_b)} \le C_0 \|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1 e^{2A_0} T^\mu \|v\|^2_{\mathcal{A}_{A_1}(X^s_b)}$$

and also we have the estimate for the difference

$$\|\Phi(v^{(1)}) - \Phi(v^{(2)})\|_{\mathcal{A}_{A_1}(X^s_b)} \le C_1 e^{2A_0} T^\mu (\|v^{(1)}\|_{\mathcal{A}_{A_1}(X^s_b)} + \|v^{(2)}\|_{\mathcal{A}_{A_1}(X^s_b)}) \|v^{(1)} - v^{(2)}\|_{\mathcal{A}_{A_1}(X^s_b)}.$$

Choosing T small enough, the map Φ is contraction from

$$X_T = \{f = (f_0, f_1, \cdots); f_i \in X^s_b, \sum_0^{\infty} \frac{A_0^k}{k!} \|f_k\|_{X^s_b} \le 2C_0 M_0\}$$

to itself, where $M_0 = \|v\|_{\mathcal{A}_{A_0}(H^s)}$. This shows the well-posedness.

Then for the original equation with the assumption for the initial data

$$\|(x\partial_x)^k \phi\|_{H^s} \le C A_1^k k! \quad k = 0, 1, \cdots,$$

we show the corresponding solution to (KdV) is obtained with the following estimate

$$\|P^k v\|_{X^s_b} \le C A_0^k k! \quad k = 0, 1, \cdots.$$

Now by the localization argument, the operator P plays the role of the vector field $P_0 = 3t_0\partial_t + x_0\partial_x$ where $(t_0, x_0) \in \{(-T, 0) \cup (0, T)\} \times \mathbb{R}$ is any fixed point. Since the Fourier restriction norm has originally contains the regularity with the characteristic derivative $L^b = \langle \partial_t + \partial_x^3 \rangle^b$, we combine the both derivative L^b and p_0^k (and by the localization argument) to derive the regularity. If we set a smooth cut-off $a(t,x)$ whose support are around the point (t_0, x_0) with supp $a \subset B_{2\varepsilon}$. Then we firstly derive

$$\|aP^k v\|_{L^2_{t,x}(\mathbb{R}^2)} \le CA_2^k k! \quad k = 0,1,2,\cdots .$$

Based on this estimate, we forward the step into

$$\|aP^k v\|_{H^{7/2}_{t,x}(\mathbb{R}^2)} \le CA_3^k k! \quad k = 0,1,2,\cdots .$$

Then by the bootstrap argument, we have in step by step that

$$\sup_t \|a\partial_x^l P^k v\|_{H^1_{t,x}(\mathbb{R}^2)} \le CA_4^{k+l}(k+l)! \quad k,l = 0,1,2,\cdots ,$$

and

$$\sup_t \|a\partial_t^m \partial_x^l v\|_{H^1_{t,x}(\mathbb{R}^2)} \le CA_5^{l+m}(l+m)! \quad l,m = 0,1,2,\cdots .$$

This gives the regularity for the solution.

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