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SINGULARITIES OF SOLUTIONS TO ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA II

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ABSTRACT. In this paper we shall study elastic mixed or initial-interface value problems and give an inner estimate of the location of singularities of reflected and refracted Riemann functions by making use of the localization method.

In this paper, we shall continue our study in the previous paper "Singularities of Solutions to Elastic Wave Propagation Problems in Stratified media" in RIMS Kokyuroku 994 (1997), Spectral and Scattering Theory and Its Related Topics pp.104-120.

1. Introduction

We consider elastic wave propagation problems in the following plane-stratified media $\mathbf{R}^3$ with the planar interface $x_3 = 0$:

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for} \quad x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2) & \text{for} \quad x_3 > 0. \end{cases}$$

Here the constants λ_1, λ_2, μ_1, μ_2 are called the Lamé constants and the constants ρ_1, ρ_2 are densities. We shall denote the lower half-space $\mathbf{R}^3_-$ by *Medium I* and the upper half-space $\mathbf{R}^3_+$ by *Medium II*, respectively, as in Figure 1.

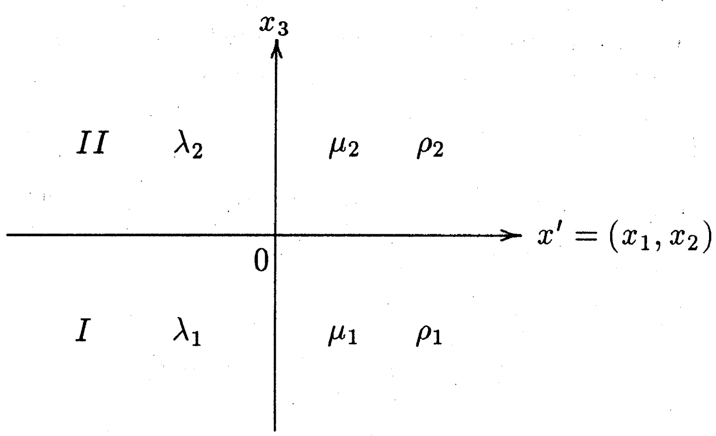

Figure 1　Stratified media I and II

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We assume that

$$\lambda_i + \mu_i > 0, \quad \mu_i > 0, \quad \rho_i > 0, \quad i = 1, 2. \tag{1.1}$$

(1.1) is the natural assumption in practical situation. From the roots of the characteristic equations of $P^I(D)$ and $P^{II}(D)$ which are defined below 3×3 matrix valued hyperbolic partial diferential operators in Medium I and Medium II, respectively, we obtain two speeds correspond to Pressuer or Primary wave (P wave) and Share or Secondary wave (S wave) on each medium. c_{p_1} denotes the speed of P wave in Medium I and c_{s_1} denotes the speed of S wave in Medium I. c_{p_2} and c_{s_2} denote the speed of P and S wave in Medium II, respectively. They are given by

$$c_{p_i}^2 = \frac{\lambda_i + 2\mu_i}{\rho_i}, \quad c_{s_i}^2 = \frac{\mu_i}{\rho_i}, \quad i = 1, 2.$$

By assumption (1.1), the speed of P wave is greater than that of S wave in each medium. On account of this, these are six cases of the order relation of the speeds of $\{c_{p_1}, c_{s_1}, c_{p_2}, c_{s_2}\}$. Here we assume that

$$c_{s_1} < c_{p_1} \leq c_{s_2} < c_{p_2}. \tag{1.2}$$

It is the standard case (cf. [Sh, Section 3]). The other cases can be treated in a similar manner.

Let $x = (x_0, x_1, x_2, x_3) = (x', x_3) = (x_0, x'') = (x_0, x''', x_3)$ in $\mathbf{R}^4$. The variable x_0 will play a role of time, and $x'' = (x_1, x_2, x_3)$ will play that of space. ξ is a real dual variable of x and is equal to $(\xi_0, \xi_1, \xi_2, \xi_3) = (\xi', \xi_3) = (\xi_0, \xi'') = (\xi_0, \xi''', \xi_3)$ in $\mathbf{R}^4_\xi$. We use the differential symble $D_j = i^{-1}\partial/\partial x_j$ $(j = 0, 1, 2, 3)$, where $i = \sqrt{-1}$. We shall denote by $\mathbf{R}^n_-$ the half-space $\{x = (x_1, \cdots, x_n) \in \mathbf{R}^n |\ x_n < 0\}$ and by $\mathbf{R}^n_+$ the half-space $\{x = (x_1, \cdots, x_n) \in \mathbf{R}^n |\ x_n > 0\}$, and also use the notation $|x| = \sqrt{x_1^2 + \cdots x_n^2}$.

Let $u(x) = {}^t(u_1(x), u_2(x), u_3(x)) \in \mathbf{R}^3$ be the displacement vector at time x_0 and position x''. The propagation problems of elastic waves in the stratified media is formulated as mixed (initial-interface value) problem:

$$\begin{cases} P^I(D)u(x) = f(x), & x_0 > 0,\ x'' = (x_1, x_2, x_3) \in \mathbf{R}^3_-, \\ P^{II}(D)u(x) = f(x), & x_0 > 0,\ x'' = (x_1, x_2, x_3) \in \mathbf{R}^3_+, \\ u(x)|_{x_3=-0} = u(x)|_{x_3=+0}, & x_0 > 0,\ x''' \in \mathbf{R}^2, \\ B^I(D)u(x)|_{x_3=-0} = B^{II}(D)u(x)|_{x_3=+0}, & x_0 > 0,\ x''' \in \mathbf{R}^2, \\ D_0^k u(x)|_{x_0=0} = g_k(x''), & k = 0, 1, \quad x'' \in \mathbf{R}^3. \end{cases} \tag{1.3}$$

Here

$$P^I(D)u = -D_0^2 Eu + \frac{\lambda_1 + \mu_1}{\rho_1}\nabla_{x''}(\nabla_{x''} \cdot u) + \frac{\mu_1}{\rho_1}\Delta_{x''}u,$$

is a 3×3 matrix valued second order hyperbolic differential operator with constant coefficients where E is a 3×3 identity matrix,

$$(B^I(D)u)_k = i\lambda_1(\nabla_{x''} \cdot u)\delta_{k3} + 2\mu_1\varepsilon_{k3}(u), \quad k = 1, 2, 3,$$

is the k-th component of symmetric stress tensors $B^I(D)u$ where

$$\varepsilon_{k3}(u) = i/2\,(D_3 u_k + D_k u_3)\,, \quad k = 1, 2, 3,$$

are strain tensors. The $P^{II}(D)u$ and $B^{II}(D)u$ are defined by replacing λ_1, μ_1, ρ_1 by λ_2, μ_2, ρ_2, respectively.

If we put unit impulse Dirac's delta $\delta(x-y)$ with $x_0 \geq y_0$ and $y_3 < 0$, that is, put it in Medium I, then the Riemann function of this elastic mixed problem is given by the following:

$$G(x,y) = \begin{cases} E^I(x-y) - F^I(x,y) & \text{for} \quad x_3 < 0, \\ F^{II}(x,y) & \text{for} \quad x_3 > 0, \end{cases}$$

where $E^I(x)$ is the fundamental solution in Medium I describing an incident wave, is defined by

$$E^I(x) = (2\pi)^{-4} \int_{\mathbf{R}^4_\xi} e^{ix\cdot(\xi+i\eta)} P^I(\xi+i\eta)^{-1} d\xi, \qquad \eta \in -s\vartheta - \Gamma,$$

with a positive real s large enough. Here ϑ and Γ are defiend below. Taking partial Fourier-Laplace transform with respect to x' for the mixed problem, we obtain a interface value problem for ordinary differential equation with parameters. Then taking partial inverse Fourier-Laplace transform for the solution, we obtain expicit expressions of reflected and refracted Riemann functions $F^I(x,y)$ and $F^{II}(x,y)$ which describe reflected and refracted waves, respectively.

In this paper, we give an inner estimate of the location of singularities of reflected and refracted Riemann functions $F^I(x,y)$ and $F^{II}(x,y)$ by making use of the localization method. This method is first studied by M. F. Atiyah, R. Bott, L. Gårding [A-B-G] for initial value problem, then studied by M. Matsumura [Ma 1], M. Tsuji [Ts], and S. Wakabayashi [Wa 1], [Wa 2] for half-space mixed problem. Matsumura studied the singularities of the ordinary wave propagation problems in the stratified media by applying above methods [Ma 2], [Ma 3]. They are useful references to our study.

We define a localization of polynomials according to Atiyah-Bott-Gårding (cf. [A-B-G]):

Definition 1. Let $P(\xi)$ be a polynomial of degree $m \geq 0$ and develop $\nu^m P(\nu^{-1}\xi + \eta)$ in ascendeing power of ν:

$$\nu^m P\left(\nu^{-1}\xi + \eta\right) = \nu^p P_\xi(\eta) + O(\nu^{p+1}) \qquad \text{as} \quad \nu \to 0, \tag{1.4}$$

where $P_\xi(\eta)$ is the first coefficient that does not vanish identically in η. The polynomial $P_\xi(\eta)$ is the localization of P at ξ, the number p is the multiplicity of ξ relative to P.

Moreover we introduce the following:

Definition 2. $\Gamma = \Gamma(P, \vartheta)$ is the component of $\mathbf{R}^n_\eta \setminus \{\eta \in \mathbf{R}^n_\eta,\ P(\eta) = 0\}$ which contains $\vartheta = (1, 0, \cdots, 0) \in \mathbf{R}^n$. Moreover $\Gamma' = \Gamma'(P, \vartheta) = \{x \in \mathbf{R}^n |\ x \cdot \eta \geq 0,\ \eta \in \Gamma\}$ is the dual cone of Γ and is called the propagation cone.

We obtain the following Main theorem I. This Main Theorem I is corresponding to Main Theorem in the previous paper. It means singular supports of reflected and refracted Riemann functions $F^I(x,y)$ and $F^{II}(x,y)$ are estimated innerly by localizations $F^I_{\xi^0}(x,y)$ and $F^{II}_{\xi^0}(x,y)$ of $F^I(x,y)$ and $F^{II}(x,y)$ at ξ^0, respectively.

Main Theorem I. *For $\xi^0 \in \mathbf{R}^4_\xi$ satisfying $(detP^I_j)(\xi^0) = 0$ $(j \in \{p_1, s_1\})$, that is*

$$(\det P^I_{p_1})(\xi^0) = \xi_0^{0^2} - c^2_{p_1} |\xi^{0''}|^2 = 0,$$

or

$$(\det P^I_{s_1})(\xi^0) = \xi_0^{0^2} - c^2_{s_1} |\xi^{0''}|^2 = 0,$$

we have the following:
(1) For the reflected Riemann function $F^I(x,y)$, we have

$$\lim_{\nu \to \infty} \nu e^{-i\nu\{(x'-y')\cdot\xi^{0'} + x_3\tau^-_k(\xi^{0'}) - y_3\xi^0_3\}} F^I(x,y) = F^I_{j\xi^0 k}(x,y), \tag{1.5}$$
$$j \in \{p_1, s_1\}, \quad k \in \{p_1, s_1\},$$

and if $\xi^{0'}$ are zeros of $\tau^+_m(\zeta')$, that is, $\xi^{0'}$ satisfy $|\xi^{0'''}| = \frac{\xi^0_0}{c_m}$ $(m \in \{p_1, p_2, s_2\})$, then we have

(1.6)
$$\lim_{\nu \to \infty} \left\{ \nu^{\frac{3}{2}} e^{-i\nu\{(x'-y')\cdot\xi^{0'} + x_3\tau^-_k(\xi^{0'}) - y_3\xi^0_3\}} F^I(x,y) - \nu^{\frac{1}{2}} F^I_{j\xi^0 k}(x,y) \right\}$$
$$= F^I_{j\xi^0 km}(x,y), \qquad j \in \{p_1, s_1\}, \quad k \in \{p_1, s_1\}, \quad m \in \{p_1, p_2, s_2\}$$

in the distribution sense with respect to $(x,y) \in \mathbf{R}^4_- \times \mathbf{R}^4_-$.
Moreover we have

$$\bigcup_{\xi^0 \neq 0} (\operatorname{supp} F^I_{j\xi^0 k}(x,y) \cup \operatorname{supp} F^I_{j\xi^0 km}(x,y)) \subset \operatorname{sing\,supp} F^I(x,y), \tag{1.7}$$

and

(1.8)
$$\operatorname{supp} F^I_{j\xi^0 k}(x,y) = (\Gamma_{j\xi^0})^I_k = \Big\{ (x,y) \in \mathbf{R}^4_- \times \mathbf{R}^4_- :$$
$$((x'-y') + x_3 \operatorname{grad}_\xi \tau^-_k(\xi^{0'})) \cdot \eta' - y_3\eta_3 \geq 0,\ \eta \in \Gamma_{j\xi^0} \Big\},$$
$$j \in \{p_1, s_1\}, \quad k \in \{p_1, s_1\}$$

for ξ^0 satisfying $F^I_{j\xi^0 k}(x,y) \neq 0$ $(j \in \{p_1, s_1\},\ k \in \{p_1, s_1\})$,

(1.9)
$$\operatorname{supp} F^I_{j\xi^0 km}(x,y) = (\Gamma_{j\xi^0 m})^I_k = \Big\{ (x,y) \in \mathbf{R}^4_- \times \mathbf{R}^4_- :$$

$$\left((x'-y')+x_3\mathrm{grad}_\xi\tau_k^-(\xi^{0'})\right)\cdot\eta'-y_3\,\eta_3\geq 0,\ \eta\in\Gamma_{j\xi^0 m}\Big\},$$
$$j\in\{p_1,s_1\},\quad k\in\{p_1,s_1\},\quad m\in\{p_1,p_2,s_2\}$$

for ξ^0 *satisfying* $F^I_{j\xi^0 km}(x,y)\neq 0$ *(*$j\in\{p_1,s_1\}$*,* $k\in\{p_1,s_1\}$*,* $m\in\{p_1,p_2,s_2\}$*).*
(2) For the refracted Riemann function $F^{II}(x,y)$*, we have*

(1.10)
$$\lim_{\nu\to\infty}\nu e^{-i\nu\{(x'-y')\cdot\xi^{0'}+x_3\tau_k^+(\xi^{0'})-y_3\xi_3^0\}}F^{II}(x,y)=F^{II}_{j\xi^0 k}(x,y),$$
$$j\in\{p_1,s_1\},\quad k\in\{p_2,s_2\},$$

and if $\xi^{0'}$ *are zeros of* $\tau_m^+(\zeta')$ *(*$m\in\{p_2\}$*), then we have*

(1.11)
$$\lim_{\nu\to\infty}\left\{\nu^{\frac{3}{2}}e^{-i\nu\{(x'-y')\cdot\xi^{0'}+x_3\tau_k^+(\xi^{0'})-y_3\xi_3^0\}}F^{II}(x,y)-\nu^{\frac{1}{2}}F^{II}_{j\xi^0 k}(x,y)\right\}$$
$$=F^{II}_{j\xi^0 km}(x,y),\qquad j\in\{p_1,s_1\},\quad k\in\{p_1,s_1\},\quad m\in\{p_2\},$$

in the distribution sense with respect to $(x,y)\in\mathbf{R}^4_+\times\mathbf{R}^4_-$*.*
Moreover we have

$$\bigcup_{\xi^0\neq 0}\left(\mathrm{supp}\,F^{II}_{j\xi^0 k}(x,y)\cup\mathrm{supp}\,F^{II}_{j\xi^0 km}(x,y)\right)\subset\mathrm{sing\ supp}\,F^{II}(x,y),\tag{1.12}$$

and

(1.13)
$$\mathrm{supp}\,F^{II}_{j\xi^0 k}(x,y)=\left(\Gamma_{j\xi^0}\right)^{II}_k=\Big\{(x,y)\in\mathbf{R}^4_+\times\mathbf{R}^4_-:$$
$$\left((x'-y')+x_3\mathrm{grad}_\xi\tau_k^+(\xi^{0'})\right)\cdot\eta'-y_3\,\eta_3\geq 0,\ \eta\in\Gamma_{j\xi^0}\Big\},$$
$$j\in\{p_1,s_1\},\quad k\in\{p_2,s_2\}$$

for ξ^0 *satisfying* $F^{II}_{j\xi^0 k}(x,y)\neq 0$ *(*$j\in\{p_1,s_1\}$ $k\in\{p_2,s_2\}$*), and*

(1.14)
$$\mathrm{supp}\,F^{II}_{j\xi^0 km}(x,y)=\left(\Gamma_{j\xi^0 m}\right)^{II}_k=\Big\{(x,y)\in\mathbf{R}^4_-\times\mathbf{R}^4_-:$$
$$\left((x'-y')+x_3\mathrm{grad}_\xi\tau_k^+(\xi^{0'})\right)\cdot\eta'-y_3\,\eta_3\geq 0,\ \eta\in\Gamma_{j\xi^0 m}\Big\},$$
$$j\in\{p_1,s_1\},\quad k\in\{p_2,s_2\},\quad m\in\{p_2\}$$

for ξ^0 *satisfying* $F^{II}_{j\xi^0 km}(x,y)\neq 0$ *(*$j\in\{p_1,s_1\}$*,* $k\in\{p_2,s_2\}$*,* $m\in\{p_2\}$*).*
Here

(1.15)
$$\Gamma_{j\xi^0}=\Gamma\left((\det P^I_j)_{\xi^0}(\eta),\vartheta\right),\qquad\vartheta=(1,0,0,0),\quad j\in\{p_1,s_1\},$$

(1.16)

$$\Gamma_{j\xi^0 m} = \Gamma((\det P_j^I)_{\xi^0}(\eta), \vartheta) \cap \left\{ \Gamma\left(\frac{\xi_0^0}{c_m^2}\eta_0 - \xi_1^0\eta_1 - \xi_2^0\eta_2, \vartheta' \right) \times \mathbf{R}_\eta \right\},$$

$$\vartheta' = (1,0,0),\ j \in \{p_1, s_1\}, \quad m \in \{p_2\},$$

$$\tau_{p_1}^{\pm}(\xi') = \operatorname{sgn}(\mp\xi_0)\sqrt{\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0'''}|^2}, \quad \text{if} \quad \frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0'''}|^2 \geq 0,$$

and $\tau_{p_1}^{\pm}(\xi')$ *is taken a branch of* $\sqrt{\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0'''}|^2}$ *such that* $\pm\mathrm{Im}\tau_{p_1}^{\pm}(\xi') > 0$ *if* $\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0'''}|^2 < 0$. $\tau_{s_1}^{\pm}(\xi')$, $\tau_{p_2}^{\pm}(\xi')$, *and* $\tau_{s_2}^{\pm}(\xi')$ *are defined as the same as* $\tau_{p_1}^{\pm}(\xi')$ *substituting* c_{p_1} *for* c_{s_1}, c_{p_2}, *and* c_{s_2}, *respectively.*

Remark.1. The $(\Gamma_{j\xi^0})_k^I$ ($j \in \{p_1, s_1\}$, $k \in \{p_1, s_1\}$) represent k reflected wave for j incident wave. The $(\Gamma_{j\xi^0 m})_k^I$ ($j \in \{p_1, s_1\}$, $k \in \{p_1, s_1\}$ $m\{p_1, p_2, s_2\}$) represent m lateral wave of k reflected wave for j incident wave. The $(\Gamma_{j\xi^0})_k^{II}$ ($j \in \{p_1, s_1\}$, $k \in \{p_2, s_2\}$) represent k refracted wave for j incident wave. The $(\Gamma_{j\xi^0 m})_k^{II}$ ($j \in \{p_1, s_1\}$, $k \in \{p_2, s_2\}$ $m\{p_2\}$) represent m lateral wave of k refracted wave for j incident wave.

Remark.2. The $\tau_{p_1}^{\pm}(\xi')$, $\tau_{s_1}^{\pm}(\xi')$, $\tau_{p_2}^{\pm}(\xi')$, and $\tau_{s_2}^{\pm}(\xi')$ arise from

(1.17)

$$\begin{aligned}
\det P^I(\xi) &= \det P_1^I(\xi) \times \det P_2^I(\xi) \\
&= \{(-\xi_0^2 + c_{p_1}^2|\xi''|^2)(-\xi_0^2 + c_{s_1}^2|\xi''|^2)\} \times (-\xi_0^2 + c_{s_1}^2|\xi''|^2) \\
&= \det P_{p_1}^I(\xi) \times \{\det P_{s_1}^I(\xi)\}^2 \\
&= \{c_{p_1}^2 c_{s_1}^2(\xi_3 - \tau_{p_1}^+(\xi'))(\xi_3 - \tau_{p_1}^-(\xi'))(\xi_3 - \tau_{s_1}^+(\xi'))(\xi_3 - \tau_{s_1}^-(\xi'))\} \\
&\quad \times \{c_{s_1}^2(\xi_3 - \tau_{s_1}^+(\xi'))(\xi_3 - \tau_{s_1}^-(\xi'))\},
\end{aligned}$$

and the factor of $\det P^{II}(\xi)$ given with replaced p_1, s_1 by p_2, s_2, respectively.

Remark.3. If $(\det P^I)_j(\xi^0) \neq 0$, ($j \in \{p_1, s_1\}$) then $(\det P^I)_{j\xi^0}(\eta)$=$(\det P^I)_j(\xi^0)$ and is constant. So $\Gamma_{j\xi^0} = \Gamma_{j\xi^0 m} = \mathbf{R}^4$ and thus $(\Gamma_{j\xi^0})_k^I = (\Gamma_{j\xi^0 m})_k^I = \{0\} \subset \mathbf{R}_-^4 \times \mathbf{R}_-^4$ ($j \in \{p_1, s_1\}$, $k \in \{p_1, s_1\}$ $m \in \{p_1, p_2, s_2\}$) and $(\Gamma_{j\xi^0})_k^{II} = (\Gamma_{j\xi^0 m})_k^{II} = \{0\} \subset \mathbf{R}_+^4 \times \mathbf{R}_-^4$ ($j \in \{p_1, s_1\}$, $k \in \{p_1, s_1\}$ $m \in \{p_2\}$).

Remark.4. By the assumption (1.2), there are not any real ξ that are roots of $\zeta_0^2 - c_{p_1}^2|\zeta''|^2 = 0$ and zeros of $\tau_{s_1}^+(\xi')$. The sets of ξ^0 cause singularities are given in (3.1)-(3.19).

Remark.5. In (1.8), ξ^0 satisfying $F_{j\xi^0 k}^I(x,y) \neq 0$ is equivalent to $(Q_1(\xi^0), Q_2(\xi^0))$ $\neq 0$ in (3.18) below, or is equivalent to $Q_1(\xi^0) \neq 0$ in (3.20) below. In (1.9), ξ^0 satisfying $F_{j\xi^0 km}^I(x,y) \neq 0$ is equivalent to $T_1(\xi^0)R_1(\xi^{0'}) - Q_1(\xi^0)S(\xi^{0'}) \neq 0$ or $T_2(\xi^0)R_1(\xi^{0'}) - Q_2(\xi^0)S(\xi^{0'}) \neq 0$ in (3.21) below.

Lateral waves, in other words, glancing wave, arise from the presence of branch points of $\tau_{p_1}^{\pm}(\xi')$, $\tau_{s_1}^{\pm}(\xi')$, $\tau_{p_2}^{\pm}(\xi')$, and $\tau_{s_2}^{\pm}(\xi')$. In our problem, many lateral waves are appeared.

Concerning the Lopatinski determinant, there are two cases: One is the case that Lopatinski determinant has one real zero, and the other is the case that it has no zero. It depends on the Lamé constants and densities. We remark that the speed of the Stoneley wave c_{St} is less than or equal to c_{s_1} which is the minimum speed of $\{c_{s_1}, c_{p_1}, c_{s_2}, c_{p_2}\}$ (cf. [Sh Section 3]). If we put unit impulse δ on the interface $x_3 = 0$, then we have the Riemann functions

$$H_l(x) = \begin{cases} H_l^I(x) & \text{for} \quad x_3 < 0, \\ H_l^{II}(x) & \text{for} \quad x_3 > 0, \quad l = 1,2 \end{cases}$$

given (4.1)-(4.5) below. The singularity appears which corresponds to the interface wave of which name is the Stoneley wave, in the case that the Lopationski determinant has one real zero. For this case we have the following.

Main Theorem II. *For $\xi^{0\prime} \in \mathbf{R}^3_\xi$ satisfying $\xi_0^2 - c_{St}^2(\xi_1^2 + \xi_2^2) = 0$, we have the following:*
(1) For the Riemann function $H_l^I(x)$ $(l = 1,2)$, we have

$$\lim_{\nu\to\infty} \nu^{-1} e^{-i\nu x'\cdot\xi^{0\prime}} H_l^I(x) = H_{l\xi^0}^I(x)$$

in the distribution sense with respect to $x \in \mathbf{R}^4_-$.
Moreover we have

$$\bigcup_{\xi^{0\prime}\neq 0} \operatorname{supp} \lim_{x_3\to -0} H_{l\xi^0}^I(x) \subset \operatorname{sing\,supp} \lim_{x_3\to -0} H_l^I(x),$$

and

$$\operatorname{supp} \lim_{x_3\to -0} H_{l\xi^0}^I(x) = \left(\Gamma_{St\xi^{0\prime}}\right)' = \left\{ x' \in \mathbf{R}^3 : x'\cdot\eta' \geq 0,\ \eta' \in \Gamma_{St\xi^{0\prime}} \right\},$$

where

$$\Gamma_{St\xi^{0\prime}} = \Gamma\left(\eta_0 - c_{St}(\xi_1^0\eta_1 + \xi_2^0\eta_2)/\sqrt{\xi_1^{0\,2} + \xi_2^{0\,2}}, \vartheta'\right), \quad \vartheta' = (1,0,0). \tag{1.18}$$

(2) For the Riemann function $H_l^{II}(x)$ $(l = 1,2)$, we have

$$\lim_{\nu\to\infty} \nu^{-1} e^{-i\nu x'\cdot\xi^{0\prime}} H_l^{II}(x) = H_{l\xi^{0\prime}}^{II}(x)$$

in the distribution sense with respect to $x \in \mathbf{R}^4_+$.
Moreover we have

$$\bigcup_{\xi^{0\prime}\neq 0} \operatorname{supp} \lim_{x_3\to +0} H_{l\xi^{0\prime}}^{II}(x) \subset \operatorname{sing\,supp} \lim_{x_3\to +0} H_l^{II}(x),$$

and

$$\operatorname{supp} \lim_{x_3\to +0} H_{l\xi^{0\prime}}^{II}(x) = \left(\Gamma_{St\xi^{0\prime}}\right)' = \left\{ x' \in \mathbf{R}^3 : x'\cdot\eta' \geq 0,\ \eta' \in \Gamma_{St\xi^{0\prime}} \right\},$$

where $\Gamma_{\xi^{0\prime}}$ is the same as (1.18).

Remark. If the Lopatinski determinant has no zero, then a singularity corresponding to the Stoneley wave does not appear.

2. Proof of Main Theorem I

In this section, we give the precise proof of Main Theorem I. The procedure of getting explicit expressions of the reflected and refracted Riemann functions $F^I(x,y)$ and $F^{II}(x,y)$ are given in the previous paper.

We prove for the reflected Riemann function $F^I(x,y)$. A similar proof is given for the refracted Riemann function $F^{II}(x,y)$.

The first part of the theorem is derived by the localization method. First we prove the equation (1.5). We consider the case that $j = s_1$, $k = p_1$, that is, consider that P_1 reflected wave for S_1 incident wave, and consider that the point ξ^0 satisfying (3.2) below. We calculate

(2.1)

$$\nu e^{-i\nu\{(x'-y')\cdot\xi^{0'}+x_3\tau^-_{p_1}(\xi^{0'})-y_3\xi^0_3\}}F^I(x,y)$$
$$= (2\pi)^{-4}\int_{\mathbf{R}^3} e^{i(x'-y')\cdot(-\nu\xi^{0'}+\xi'+i\eta')}\int_{\mathbf{R}} e^{-iy_3(-\nu\xi^0_3+\xi_3+i\eta_3)}\mathrm{U}(\xi'''+i\eta''')\mathrm{C}\times$$

$$\left(\begin{matrix}
\dfrac{\begin{vmatrix} \{P^I_1(\zeta)^{-1}\}_1 & \cdots \\ & \cdots \\ B^I_1(\zeta)\{P^I_1(\zeta)^{-1}\}_1 & \cdots \\ & \cdots \end{vmatrix}}{\nu^{-1}R_1(\xi'+i\eta')}\begin{pmatrix} |\xi'''+i\eta'''| \\ -\tau^+_{p_1}(\xi'+i\eta') \end{pmatrix} e^{-i\{\tau^+_{p_1}(\xi'+i\eta')+\nu\tau^-_{p_1}(\xi^{0'})\}x_3} \\
+\dfrac{\begin{vmatrix} \cdot & \{P^I_1(\zeta)^{-1}\}_1 & \cdot\ \cdot \\ \cdot & & \cdot\ \cdot \\ \cdot & B^I_1(\zeta)\{P^I_1(\zeta)^{-1}\}_1 & \cdot\ \cdot \\ \cdot & & \cdot\ \cdot \end{vmatrix}}{\nu^{-1}R_1(\xi'+i\eta')}\begin{pmatrix} \tau^+_{s_1}(\xi'+i\eta') \\ |\xi'''+i\eta'''| \end{pmatrix} e^{-i\{\tau^+_{s_1}(\xi'+i\eta')+\nu\tau^-_{p_1}(\xi^{0'})\}x_3} \\
0 \\[2ex]
\dfrac{\begin{vmatrix} \{P^I_1(\zeta)^{-1}\}_2 & \cdots \\ & \cdots \\ B^I_1(\zeta)\{P^I_1(\zeta)^{-1}\}_2 & \cdots \\ & \cdots \end{vmatrix}}{\nu^{-1}R_1(\xi'+i\eta')}\begin{pmatrix} |\xi'''+i\eta'''| \\ -\tau^+_{p_1}(\xi'+i\eta') \end{pmatrix} e^{-i\{\tau^+_{p_1}(\xi'+i\eta')+\nu\tau^-_{p_1}(\xi^{0'})\}x_3} \\
+\dfrac{\begin{vmatrix} \cdot & \{P^I_1(\zeta)^{-1}\}_2 & \cdot\ \cdot \\ \cdot & & \cdot\ \cdot \\ \cdot & B^I_1(\zeta)\{P^I_1(\zeta)^{-1}\}_2 & \cdot\ \cdot \\ \cdot & & \cdot\ \cdot \end{vmatrix}}{\nu^{-1}R_1(\xi'+i\eta')}\begin{pmatrix} \tau^+_{s_1}(\xi'+i\eta') \\ |\xi'''+i\eta'''| \end{pmatrix} e^{-i\{\tau^+_{s_1}(\xi'+i\eta')+\nu\tau^-_{p_1}(\xi^{0'})\}x_3} \\
0 \\[2ex]
\begin{matrix} & 0 \\ & 0 \\ \dfrac{\begin{vmatrix} P^I_2(\zeta)^{-1} & \cdot \\ B^I_2(\zeta)P^I_2(\zeta)^{-1} & \cdot \end{vmatrix}}{\nu^{-1}R_2(\xi'+i\eta')} & e^{-i\{\tau^+_{s_1}(\xi'+i\eta')+\nu\tau^-_{p_1}(\xi^{0'})\}x_3} \end{matrix}
\end{matrix}\right)(\mathrm{U}(\xi'''+i\eta''')\mathrm{C})^{-1}\,d\xi_3 d\xi',$$

Here $\zeta = \xi + i\eta$. The $\cdot$ means the same component of the Lopatinski matrices $\mathcal{R}_1(\zeta')$ $(\zeta' = \xi' + i\eta')$ and $\mathcal{R}_2(\zeta')$ given below, $\{P_1^I(\zeta)^{-1}\}_1$ and $\{P_1^I(\zeta)^{-1}\}_2$ are the 1 and 2 columns, respectively, of the inverse matrix of $P_1^I(\zeta)$ given by

(2.2)

$$P_1^I(\zeta)^{-1} = (\{P_1^I(\zeta)^{-1}\}_1, \{P_1^I(\zeta)^{-1}\}_2) = \frac{\mathrm{cof}P_1^I(\zeta)}{\det P_1^I(\zeta)} = \frac{1}{(\zeta_0^2 - c_{p_1}^2|\zeta''|^2)(\zeta_0^2 - c_{s_1}^2|\zeta''|^2)}$$
$$\times \begin{pmatrix} -\zeta_0^2 + \{c_{p_1}^2\zeta_3^2 + c_{s_1}^2(\zeta_1^2+\zeta_2^2)\} & -(c_{p_1}^2 - c_{s_1}^2)\sqrt{\zeta_1^2+\zeta_2^2}\,\zeta_3 \\ -(c_{p_1}^2 - c_{s_1}^2)\sqrt{\zeta_1^2+\zeta_2^2}\,\zeta_3 & -\zeta_0^2 + \{c_{s_1}^2\zeta_3^2 + c_{p_1}^2(\zeta_1^2+\zeta_2^2)\} \end{pmatrix},$$

$R_1(\zeta')$ and $R_2(\zeta')$ are the Lopatinski determinants of the systems $\{P_1^I(\zeta', D_3)$, $P_1^{II}(\zeta', D_3)$, $B_1^I(\zeta', D_3)$, $B_1^{II}(\zeta', D_3)\}$ and $\{P_2^I(\zeta', D_3)$, $P_2^{II}(\zeta', D_3)$, $B_2^I(\zeta', D_3)$, $B_2^{II}(\zeta', D_3)\}$, respectively, given by

$$R_1(\zeta') = \det \mathcal{R}_1(\zeta'), \tag{2.3}$$

(2.4)

$$\mathcal{R}_1(\zeta') = \begin{pmatrix} |\zeta'''| & \tau_{s_1}^+(\zeta') & |\zeta'''| & -\tau_{s_2}^+(\zeta') \\ -\tau_{p_1}^+(\zeta') & |\zeta'''| & \tau_{p_2}^+(\zeta') & |\zeta'''| \\ -2\rho_1 c_{s_1}^2\tau_{p_1}^+(\zeta')|\zeta'''| & -\rho_1 c_{s_1}^2(\tau_{s_1}^+(\zeta')^2 - |\zeta'''|^2) & 2\rho_2 c_{s_2}^2\tau_{p_2}^+(\zeta')|\zeta'''| & -\rho_2 c_{s_2}^2(\tau_{s_2}^+(\zeta')^2 - |\zeta'''|^2) \\ \rho_1 c_{s_1}^2(\tau_{s_1}^+(\zeta')^2 - |\zeta'''|^2) & -2\rho_1 c_{s_1}^2\tau_{s_1}^+(\zeta')|\zeta'''| & \rho_2 c_{s_2}^2(\tau_{s_2}^+(\zeta')^2 - |\zeta'''|^2) & 2\rho_2 c_{s_2}^2\tau_{s_2}^+(\zeta')|\zeta'''| \end{pmatrix},$$

$$R_2(\zeta') = \det \mathcal{R}_2(\zeta'), \tag{2.5}$$

$$\mathcal{R}_2(\zeta') = \begin{pmatrix} 1 & 1 \\ \rho_1 c_{s_1}^2\tau_{s_1}^+(\zeta') & -\rho_2 c_{s_2}^2\tau_{s_2}^+(\zeta') \end{pmatrix}. \tag{2.6}$$

We note that

$$\tau_{p_1}^+(\xi') = -\sqrt{\frac{\xi_0^2}{c_{p_1}^2} - (\xi_1^2 + \xi_2^2)} = -\tau_{p_1}^-(\xi')$$

since $\tau_{p_1}^+(\xi')$ is real. Similarly note that $\tau_j^+(\xi') = -\tau_j^-(\xi')$ $(j = \{s_1, p_2, s_2\})$. Making the change of variable $-\nu\xi^0 + \xi = \kappa$, then we have

(2.7)

$$= (2\pi)^{-4}\int_{\mathbf{R}^3} e^{i(x'-y')\cdot(\kappa'+i\eta')}\int_{\mathbf{R}} e^{-iy_3(\kappa_3+i\eta_3)}\mathrm{U}(\nu\xi^{0'''} + \kappa''' + i\eta''')\mathrm{C}\times$$

$$
\left(\begin{array}{l}
\dfrac{\begin{vmatrix} \{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdots \\ & \cdots \\ B_1^I(\nu\xi^0+\kappa+i\eta)\{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdots \\ & \cdots \end{vmatrix}}{\nu^{-1}R_1(\nu\xi^{0'}+\kappa'+i\eta')} \begin{pmatrix} |\nu\xi^{0'''}+\kappa'''+i\eta'''| \\ -\tau^+_{p_1}(\nu\xi^{0'}+\kappa'+i\eta') \end{pmatrix} \\
\qquad\qquad \times e^{-i\{\tau^+_{p_1}(\nu\xi^{0'}+\kappa'+i\eta')-\nu\tau^+_{p_1}(\xi^{0'})\}x_3} \\
+\dfrac{\begin{vmatrix} \cdot & \{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdot\ \cdot \\ \cdot & & \cdot\ \cdot \\ \cdot & B_1^I(\nu\xi^0+\kappa+i\eta)\{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdot\ \cdot \\ \cdot & & \cdot\ \cdot \end{vmatrix}}{\nu^{-1}R_1(\nu\xi^{0'}+\kappa'+i\eta')} \begin{pmatrix} |\nu\xi^{0'''}+\kappa'''+i\eta'''| \\ -\tau^+_{p_1}(\nu\xi^{0'}+\kappa'+i\eta') \end{pmatrix} \\
\qquad\qquad \times e^{-i\{\tau^+_{s_1}(\nu\xi^{0'}+\kappa'+i\eta')-\nu\tau^+_{p_1}(\xi^{0'})\}x_3} \\
\qquad\qquad\qquad 0 \\
\qquad\qquad \text{and so on}
\end{array}\right) \Big(\mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''')\mathrm{C}\Big)^{-1} d\kappa_3 d\kappa'.
$$

Note that

(2.8)

$$\tau_k^+(\nu\xi^{0'}+\kappa'+i\eta') = \nu\tau_k^+(\xi^{0'}) + \mathrm{grad}\tau_k^+(\xi^{0'})\cdot(\kappa'+i\eta') + O(\nu^{-1}), \quad k=\{p_1,s_1\},$$

and

(2.9)

$$\mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''') = \frac{1}{\sqrt{(\nu\xi_1^0+\kappa_1+i\eta_1)^2+(\nu\xi_2^0+\kappa_2+i\eta_2)^2}}$$

$$\times\begin{pmatrix} \nu\xi_1^0+\kappa_1+i\eta_1 & -\nu\xi_2^0-\kappa_2-i\eta_2 & 0 \\ \nu\xi_2^0+\kappa_2+i\eta_2 & \nu\xi_1^0+\kappa_+i\eta_1 & 0 \\ 0 & 0 & \sqrt{(\nu\xi_1^0+\kappa_1+i\eta_1)^2+(\nu\xi_2^0+\kappa_2+i\eta_2)^2} \end{pmatrix}$$

$$\to \frac{1}{\sqrt{{\xi_1^0}^2+{\xi_2^0}^2}}\begin{pmatrix} \xi_1^0 & -\xi_2^0 & 0 \\ \xi_2^0 & \xi_1^0 & 0 \\ 0 & 0 & \sqrt{{\xi_1^0}^2+{\xi_2^0}^2} \end{pmatrix} = U(\xi^{0'''}) \text{ as } \nu\to\infty \text{ for } \xi^{0'''}\neq 0,$$

(2.10)

$$\mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''')\to\mathrm{U}(\kappa'''+i\eta''') \quad \text{as} \quad \nu\to\infty \quad \text{for} \quad \xi^{0'''}=0,$$

(2.11)

$$\begin{aligned}\nu^{-1}|\nu\xi^{0'''}+\kappa'''+i\eta'''| &= \nu^{-1}\sqrt{(\nu\xi_1^0+\kappa_1+i\eta_1)^2+(\nu\xi_2^0+\kappa_2+i\eta_2)^2} \\ &= \sqrt{{\xi_1^0}^2+{\xi_2^0}^2}\sqrt{1+O(\nu^{-1})} \\ &\to \sqrt{{\xi_1^0}^2+{\xi_2^0}^2} = |\xi^{0'''}| \quad \text{as} \quad \nu\to\infty \quad \text{for} \quad \xi^{0'''}\neq 0,\end{aligned}$$

(2.12)

$$|\nu\xi^{0'''}+\kappa'''+i\eta'''|\to|\kappa'''+i\eta'''| \quad \text{as} \quad \nu\to\infty \quad \text{for} \quad \xi^{0'''}=0.$$

For 3×3 matirx valued function $\phi(x,y)\in C_0^\infty(\mathbf{R}_-^4\times\mathbf{R}_+^4)$, we have

$$\left\langle \nu e^{-i\nu\{(x'-y')\cdot\xi^{0'}+x_3\tau_{p_1}^-(\xi^{0'})-y_3\xi_3^0\}}F^I(x,y),\phi(x,y)\right\rangle_{x,y}$$

$$= (2\pi)^{-2}\Bigg\langle\Bigg\langle \mathrm{U}(\xi'''+i\eta''')\mathrm{C}\times$$

$$\left(\begin{matrix} \dfrac{\begin{vmatrix} \{P_1^I(\zeta)^{-1}\}_1 & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ B_1^I(\zeta)\{P_1^I(\zeta)^{-1}\}_1 & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{vmatrix}}{\nu^{-1}R_1(\xi'+i\eta')} \begin{pmatrix} |\xi'''+i\eta'''| \\ -\tau_{p_1}^+(\xi'+i\eta') \end{pmatrix} e^{-i\tau_{p_1}^+(\xi'+i\eta')x_3} \\ + \dfrac{\begin{vmatrix} \cdot & \{P_1^I(\zeta)^{-1}\}_1 & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & B_1^I(\zeta)\{P_1^I(\zeta)^{-1}\}_1 & \cdot & \cdot \\ \cdot & & \cdot & \cdot \end{vmatrix}}{\nu^{-1}R_1(\xi'+i\eta')} \begin{pmatrix} \tau_{s_1}^+(\xi'+i\eta') \\ |\xi'''+i\eta'''| \end{pmatrix} e^{-i\tau_{s_1}^+(\xi'+i\eta')x_3} \\ 0 \end{matrix}\right.$$

$$\left.\text{and so on}\right)(\mathrm{U}(\xi'''+i\eta''')\mathrm{C})^{-1},$$

$$e^{i\nu x_3\tau_{p_1}^+(\xi^{0'})}\tilde{\phi}(\nu\xi^{0'}-\xi'-i\eta',x_3,-\nu\xi^{0'}+\xi'+i\eta',-\nu\xi_3^0+\xi_3+i\eta_3)\Bigg\rangle_{\xi}\Bigg\rangle_{x_3},$$

making the change of variable $-\nu\xi^0+\xi=\kappa$, then we have

(2.13)

$$= (2\pi)^{-2}\Big\langle \mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''')\mathrm{C}\times$$

$$\left(\begin{matrix} \dfrac{\begin{vmatrix} \{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ B_1^I(\nu\xi^0+\kappa+i\eta)\{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{vmatrix}}{\nu^{-1}R_1(\nu\xi^{0'}+\kappa'+i\eta')} \begin{pmatrix} |\nu\xi^{0'''}+\kappa'''+i\eta'''| \\ -\tau_{p_1}^+(\nu\xi^{0'}+\kappa'+i\eta') \end{pmatrix} \\ \times e^{-i\tau_{p_1}^+(\nu\xi^{0'}+\kappa'+i\eta')x_3} \\ + \dfrac{\begin{vmatrix} \cdot & \{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & B_1^I(\nu\xi^0+\kappa+i\eta)\{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdot & \cdot \\ \cdot & & \cdot & \cdot \end{vmatrix}}{\nu^{-1}R_1(\nu\xi^{0'}+\kappa'+i\eta')} \begin{pmatrix} |\nu\xi^{0'''}+\kappa'''+i\eta'''| \\ -\tau_{p_1}^+(\nu\xi^{0'}+\kappa'+i\eta') \end{pmatrix} \\ \times e^{-i\{\tau_{s_1}^+(\nu\xi^{0'}+\kappa'+i\eta')x_3} \\ 0 \end{matrix}\right.$$

$$\left.\text{and so on}\right)\Big(\mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''')\mathrm{C}\Big)^{-1},$$

$$e^{i\nu x_3\tau^+_{p_1}(\xi^{0\prime})}\tilde{\phi}(-\kappa'-i\eta',x_3,\kappa'+i\eta',\kappa_3+i\kappa_3)\Big\rangle_{\kappa}\Big\rangle_{x_3}.$$

Here $<\ \ >$ denotes a sum of each component, and

$$\tilde{\phi}(\zeta',x_3,z)=\mathcal{FL}_{(x',y)}[\phi(x,y)](\zeta',z),$$

where $\mathcal{FL}$ denotes Fourier-Laplace transformation. If $\xi^{0'''}\neq 0$, then for the term including $e^{-i\tau^+_{p_1}(\nu\xi^{0\prime}+\kappa'+i\eta')x_3}$, we have by using (2.8)

(2.14)

$$\longrightarrow(2\pi)^{-2}\Big\langle\Big\langle \mathrm{U}(\xi^{0'''})\mathrm{C}\frac{1}{(\det P^I_{s_1})_{\xi^0}(\kappa+i\eta)}$$
$$\left(\frac{Q_1(\xi^0)}{R_1(\xi^{0\prime})}\begin{pmatrix}|\xi^{0'''}|\\ -\tau^+_{p_1}(\xi^{0\prime})\end{pmatrix}e^{-i\mathrm{grad}\tau^+_{p_1}(\xi^{0\prime})x_3\cdot(\kappa'+i\eta')}\quad\text{and so on}\right)$$
$$\left(\mathrm{U}(\xi^{0'''})\mathrm{C}\right)^{-1},\ \tilde{\phi}(-\kappa'-i\eta',x_3,\kappa'+i\eta',\kappa_3+i\kappa_3)\Big\rangle_{\kappa}\Big\rangle_{x_3},$$

where $Q_1(\xi^0)$ is defined by (2.19) below. For the term including $e^{-i\tau^+_{s_1}(\nu\xi^{0\prime}+\kappa'+i\eta')x_3}$, the right-hand side of (2.13) is equal to

$$(2.15)\qquad (2\pi)^{-2}\Big\langle\Big\langle e^{-i\nu(\tau^+_{s_1}(\xi^{0\prime})-\tau^+_{p_1}(\xi^{0\prime}))x_3}f(\kappa+i\eta,\nu),$$
$$\tilde{\phi}(-\kappa'-i\eta',x_3,\kappa'+i\eta',\kappa_3+i\kappa_3)\Big\rangle_{\kappa}\Big\rangle_{x_3},$$

by using

$$\tau^+_{s_1}(\nu\xi^{0\prime}+\kappa'+i\eta')-\nu\tau^+_{p_1}(\xi^{0\prime})=\nu\{\tau^+_{s_1}(\xi^{0\prime})-\tau^+_{p_1}(\xi^{0\prime})\}$$
$$+\ \mathrm{grad}\tau^+_{s_1}(\xi^{0\prime})\cdot(\kappa'+i\eta')+O(\nu^{-1}).$$

We put

$$<\ \ >_{\kappa}=e^{-i\nu(\tau^+_{s_1}(\xi^{0\prime})-\tau^+_{p_1}(\xi^{0\prime}))x_3}g(x_3,\nu).$$

The $g(x_3,\nu)$ belongs to C_0^∞ with respect to x_3, and $\mathrm{supp}_{x_3}g(x_3,\nu)$ is included at a compact set independing on ν. Moreover

$$\left|\frac{\partial}{\partial x_3}g(x_3,\nu)\right|\le \exists C\quad\text{indep of }\nu.$$

So $<\ \ >_{x_3}$ in (2.15) is

$$(2.16)\qquad <\ \ >_{x_3}=\int e^{-i\nu(\tau^+_{s_1}(\xi^{0\prime})-\tau^+_{p_1}(\xi^{0\prime}))x_3}g(x_3,\nu)dx_3\to 0\quad\text{as}\quad \nu\to\infty$$

because of integral by parts or the Riemann-Lebesgue theorem. By (2.14) and (2.16), we obtain

(2.17)

$$\lim_{\nu\to\infty} < \nu e^{-i\nu\{(x'-y')\cdot\xi^{0\prime}+x_3\tau_k^-(\xi^{0\prime})-y_3\xi_3^0\}} F^I(x,y), \phi(x,y) >_{x,y}$$
$$=< F^I_{s_1\xi^0 p_1}(x,y), \phi(x,y) >_{x,y} \quad \text{for} \quad \phi(x,y) \in C_0^\infty(\mathbf{R}^4_- \times \mathbf{R}^4_-).$$

Here

(2.18)

$$F^I_{s_1\xi^0 p_1}(x,y)$$
$$= (2\pi)^{-4} \int_{\mathbf{R}^3} e^{i(x'-y'-\mathrm{grad}\,\tau^+_{p_1}(\xi^{0\prime})x_3)\cdot(\kappa'+i\eta')} \int_{\mathbf{R}} e^{-iy_3(\kappa_3+i\eta_3)}\, \mathrm{U}(\xi^{0\prime\prime\prime})\mathrm{C}$$
$$\times \frac{1}{(\det P^I_{s_1})_{\xi^0}(\kappa+i\eta)} \begin{pmatrix} \frac{Q_1(\xi^0)}{R_1(\xi^{0\prime})} \begin{pmatrix} |\xi^{0\prime\prime\prime}| \\ -\tau^+_{p_1}(\xi^{0\prime}) \\ 0 \end{pmatrix} & \frac{Q_2(\xi^0)}{R_1(\xi^{0\prime})} \begin{pmatrix} |\xi^{0\prime\prime\prime}| \\ -\tau^+_{p_1}(\xi^{0\prime}) \\ 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{pmatrix}$$
$$\left(\mathrm{U}(\xi^{0\prime\prime\prime})\mathrm{C}\right)^{-1} d\kappa_3 d\kappa',$$

where

$$(\det P^I_{s_1})_{\xi^0}(\kappa+i\eta) = 2\left(\xi_0^{0\,2} - c^2_{p_1}|\xi^{0\prime\prime}|^2\right)$$
$$\times \left\{\xi_0^0(\kappa_0+i\eta_0) - c^2_{s_1}(\xi_1^0(\kappa_1+i\eta_1)+\xi_2^0(\kappa_2+i\eta_2)+\xi_3^0(\kappa_3+i\eta_3))\right\},$$

and $Q_1(\xi^0)$ and $Q_2(\xi^0)$ are given by

(2.19)

$$Q_1(\xi^0) = (\det P^I_{s_1})_{\xi^0}(\xi^0) \begin{vmatrix} \{P^I_1(\xi^0)^{-1}\}_1 & \cdots \\ & \cdots \\ B^I_1(\xi^0)\{P^I_1(\xi^0)^{-1}\}_1 & \cdots \\ & \cdots \end{vmatrix}$$
$$= \begin{vmatrix} -\xi_0^{0\,2} + c^2_{s_1}|\xi^{0\prime\prime\prime}|^2 + c^2_{p_1}\xi_3^{0\,2} & \cdots \\ -(c^2_{p_1} - c^2_{s_1})|\xi^{0\prime\prime\prime}|\xi_3^0 & \cdots \\ i\rho_1 c^2_{s_1}\xi_3^0\{-\xi_0^{0\,2} + (2c^2_{s_1} - c^2_{p_1})|\xi^{0\prime\prime\prime}|^2 + c^2_{p_1}\xi_3^{0\,2}\} & \cdots \\ i\rho_1|\xi^{0\prime\prime\prime}|\{-(c^2_{p_1} - 2c^2_{s_1})\xi_0^{0\,2} + c^2_{s_1}(c^2_{p_1} - 2c^2_{s_1})|\xi^{0\prime\prime\prime}|^2 - c^2_{p_1}c^2_{s_1}\xi_3^{0\,2}\} & \cdots \end{vmatrix},$$

$$Q_2(\xi^0) = (\det P^I_{s_1})_{\xi^0}(\xi^0) \begin{vmatrix} \{P^I_1(\xi^0)^{-1}\}_2 & \cdots \\ & \cdots \\ B^I_1(\xi^0)\{P^I_1(\xi^0)^{-1}\}_2 & \cdots \\ & \cdots \end{vmatrix}$$
$$= \begin{vmatrix} -(c^2_{p_1} - c^2_{s_1})|\xi^{0\prime\prime\prime}|\xi_3^0 & \cdots \\ -\xi_0^{0\,2} + c^2_{p_1}|\xi^{0\prime\prime\prime}|^2 + c^2_{s_1}\xi_3^{0\,2} & \cdots \\ i\rho_1 c^2_{s_1}|\xi^{0\prime\prime\prime}|\{-\xi_0^{0\,2} + c^2_{p_1}|\xi^{0\prime\prime\prime}|^2 + (2c^2_{s_1} - c^2_{p_1})\xi_3^{0\,2}\} & \cdots \\ i\rho_1\xi_3^0\{-c^2_{p_1}\xi_0^{0\,2} + c^2_{s_1}(3c^2_{p_1} - 2c^2_{s_1})|\xi^{0\prime\prime\prime}|^2 + c^2_{p_1}c^2_{s_1}\xi_3^{0\,2}\} & \cdots \end{vmatrix},$$

where $\cdot$ means the same component of the Lopatinski matrix (2.4). If $\xi^{0'''}=0$, then using (2.10) and (2.12), the right-hand side of (2.13) is equal to

$$(2\pi)^{-2}\left\langle\left\langle \mathrm{U}(\kappa'''+i\eta''')\mathrm{C}\frac{1}{(\det P^I_{s_1})_{\xi^0}(\kappa+i\eta)}\right.\right.$$
$$\left(\frac{Q_1(\xi^0)}{R_1(\xi^{0'})}\begin{pmatrix}\frac{1}{\nu}|\kappa'''+i\eta'''|\\ -\tau^+_{p_1}(\xi^{0'})\end{pmatrix}e^{-i\,\mathrm{grad}\tau^+_{p_1}(\xi^{0'})x_3\cdot(\kappa'+i\eta')}\quad\text{and so on}\right)$$
$$\left.\left.(\mathrm{U}(\kappa'''+i\eta''')\mathrm{C})^{-1},\ \tilde\phi(-\kappa'-i\eta',x_3,\kappa'+i\eta',\kappa_3+i\kappa_3)\right\rangle_\kappa\right\rangle_{x_3}.$$

Since we could put ξ^0 satisfying (3.2) below to $(1,0,0,-\frac{1}{c_{s_1}})$, we obtain (2.17) with

(2.20)
$$F^I_{s_1\xi^0p_1}(x,y)=(2\pi)^{-4}\int_{\mathbf{R}^3}e^{i(x'-y'-(-\frac{1}{c_{p_1}},0,0)x_3)\cdot(\kappa'+i\eta')}\int_{\mathbf{R}}e^{-iy_3(\kappa_3+i\eta_3)}$$
$$\times\frac{1}{(\det P^I_{s_1})_{\xi^0}(\kappa+i\eta)}\frac{Q_1(\xi^0)}{R_1(\xi^{0'})}\begin{pmatrix}0&0&0\\0&0&0\\\frac{1}{c_{p_1}}&0&0\end{pmatrix}d\kappa_3d\kappa'.$$

Thus we prove the equation (1.5).

Secondly we prove the equation (1.6). We consider the case that $j=s_1$, $k=p_1$ and $m=s_2$, that is, consider that S_2 lateral wave of P_1 reflected wave for S_1 incident wave, and consider that the point ξ^0 satisfying (3.7) below.

We calculate

$$\nu^{\frac{3}{2}}e^{-i\nu\{(x'-y')\cdot\xi^{0'}+x_3\tau^-_{p_1}(\xi^{0'})+y_3\xi^0_3\}}F^I(x,y)-\nu^{\frac{1}{2}}F^I_{s_1\xi^0p_1}(x,y)$$
$$=(2\pi)^{-4}\nu^{\frac{1}{2}}\int_{\mathbf{R}^3}e^{i(x'-y')\cdot(\kappa'+i\eta')}\int_{\mathbf{R}}e^{-iy_3(\kappa_3+i\eta_3)}\mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''')\mathrm{C}\times$$

$$\left(\begin{array}{c}
\dfrac{\begin{vmatrix}\{P^I_1(\nu\xi^0+\kappa+i\eta)^{-1}\}_1&\cdot&\cdot&\cdot\\ &\cdot&\cdot&\cdot\\ B^I_1(\nu\xi^0+\kappa+i\eta)\{P^I_1(\nu\xi^0+\kappa+i\eta)^{-1}\}_1&\cdot&\cdot&\cdot\\ &\cdot&\cdot&\cdot\end{vmatrix}}{\nu^{-1}R_1(\nu\xi^{0'}+\kappa'+i\eta')}\begin{pmatrix}|\nu\xi^{0'''}+\kappa'''+i\eta'''|\\-\tau^+_{p_1}(\nu\xi^{0'}+\kappa'+i\eta')\end{pmatrix}\\
\times e^{-i\{\tau^+_{p_1}(\nu\xi^{0'}+\kappa'+i\eta')-\nu\tau^+_{p_1}(\xi^{0'})\}x_3}\\
+\dfrac{\begin{vmatrix}\cdot&\{P^I_1(\nu\xi^0+\kappa+i\eta)^{-1}\}_1&\cdot&\cdot\\ \cdot&&\cdot&\cdot\\ \cdot&B^I_1(\nu\xi^0+\kappa+i\eta)\{P^I_1(\nu\xi^0+\kappa+i\eta)^{-1}\}_1&\cdot&\cdot\\ \cdot&&\cdot&\cdot\end{vmatrix}}{\nu^{-1}R_1(\nu\xi^{0'}+\kappa'+i\eta')}\begin{pmatrix}|\nu\xi^{0'''}+\kappa'''+i\eta'''|\\-\tau^+_{p_1}(\nu\xi^{0'}+\kappa'+i\eta')\end{pmatrix}\\
\times e^{-i\{\tau^+_{s_1}(\nu\xi^{0'}+\kappa'+i\eta')-\nu\tau^+_{p_1}(\xi^{0'})\}x_3}\\
0\\
\text{and so on}
\end{array}\right)\left(\mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''')\mathrm{C}\right)^{-1}d\kappa_3d\kappa'$$

$$
\begin{aligned}
-(2\pi)^{-4}\nu^{\frac{1}{2}} \int_{\mathbf{R}^3} e^{i(x'-y'-\operatorname{grad}\tau^+_{p_1}(\xi^{0'})x_3)\cdot(\kappa'+i\eta')} \int_{\mathbf{R}} e^{-iy_3(\kappa_3+i\eta_3)}\, \mathrm{U}(\xi^{0'''})\mathrm{C} \\
\times \frac{1}{(\det P^I_{s_1})_{\xi^0}(\kappa+i\eta)} \begin{pmatrix} \frac{Q_1(\xi^0)}{R_1(\xi^{0'})} \begin{pmatrix} |\xi^{0'''}| \\ -\tau^+_{p_1}(\xi^{0'}) \\ 0 \end{pmatrix} & \frac{Q_2(\xi^0)}{R_1(\xi^{0'})} \begin{pmatrix} |\xi^{0'''}| \\ -\tau^+_{p_1}(\xi^{0'}) \\ 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{pmatrix} \\
\times \left(\mathrm{U}(\xi^{0'''})\mathrm{C}\right)^{-1} d\kappa_3 d\kappa'.
\end{aligned}
$$

We have for τ_{s_1}, τ_{p_1}, τ_{p_2},

$$
\nu^{-1}\tau^+_l(\nu\xi^{0'}+\kappa'+i\eta') \longrightarrow \tau^+_l(\xi^{0'}) \quad \text{as} \quad \nu\to\infty \quad \text{for} \quad l=\{s_1,p_1,p_2\},
$$

and for τ_{s_2}

$$
\begin{aligned}
&\nu^{-\frac{1}{2}}\tau^+_{s_2}(\nu\xi^{0'}+\kappa'+i\eta') \\
&\longrightarrow \sqrt{2\left\{\frac{\xi^0_0}{c^2_{s_2}}(\kappa_0+i\eta_0)-\xi^{0'''}\cdot(\kappa'''+i\eta''')\right\}} \quad \text{as} \quad \nu\to\infty,
\end{aligned}
$$

where $\sqrt{\cdot}$ satisfies $\mathrm{Im}\sqrt{\cdot}>0$. We have

$$
\begin{aligned}
&R_1(\nu\xi^{0'}+\kappa'+i\eta') \\
&= \nu^6 \left\{ R_1(\xi^{0'}) + \nu^{-\frac{1}{2}} \sqrt{2\left\{\frac{\xi^0_0}{c^2_{s_2}}(\kappa_0+i\eta_0)-\xi^{0'''}\cdot(\kappa'''+i\eta''')\right\}} \times \right. \\
&\left(\begin{vmatrix} -\tau^+_{p_1}(\zeta') & |\zeta'''| & |\zeta'''| \\ -2\rho_1 c^2_{s_1}\tau^+_{p_1}(\zeta')|\zeta'''| & -\rho_1 c^2_{s_1}(\tau^+_{s_1}(\zeta')^2-|\zeta'''|^2) & -\rho_2 c^2_{s_2}(\tau^+_{s_2}(\zeta')^2-|\zeta'''|^2) \\ \rho_1 c^2_{s_1}(\tau^+_{s_1}(\zeta')^2-|\zeta'''|^2) & -2\rho_1 c^2_{s_1}\tau^+_{s_1}(\zeta')|\zeta'''| & 2\rho_2 c^2_{s_2}\tau^+_{s_2}(\zeta')|\zeta'''| \end{vmatrix} \right. \\
&+ \rho_2 c^2_{s_2} \begin{vmatrix} |\zeta'''| & \tau^+_{s_1}(\zeta') & -\tau^+_{s_2}(\zeta') \\ -\tau^+_{p_1}(\zeta') & |\zeta'''| & |\zeta'''| \\ -2\rho_1 c^2_{s_1}\tau^+_{p_1}(\zeta')|\zeta'''| & -\rho_1 c^2_{s_1}(\tau^+_{s_1}(\zeta')^2-|\zeta'''|^2) & -\rho_2 c^2_{s_2}(\tau^+_{s_2}(\zeta')^2-|\zeta'''|^2) \end{vmatrix} \left. \right) \left. + O(\nu^{-1}) \right\} \\
&= \nu^6 \left\{ R_1(\xi^{0'}) + \nu^{-\frac{1}{2}} \sqrt{2\left\{\frac{\xi^0_0}{c^2_{s_2}}(\kappa_0+i\eta_0)-\xi^{0'''}\cdot(\kappa'''+i\eta''')\right\}} S(\xi^{0'}) + O(\nu^{-1}) \right\}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\nu \begin{vmatrix} \{P^I_1(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdots \\ & \cdots \\ B^I_1(\nu\xi^0+\kappa+i\eta)\{P^I_1(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdots \\ & \cdots \end{vmatrix} = \frac{\nu^6}{(\det P^I_{s_1})_{\xi^0}(\kappa+i\eta)} \\
&\times \left\{ Q_1(\xi^0) + \nu^{-\frac{1}{2}} \sqrt{2\left\{\frac{\xi^0_0}{c^2_{s_2}}(\kappa_0+i\eta_0)-\xi^{0'''}\cdot(\kappa'''+i\eta''')\right\}} T_1(\xi^0) + O(\nu^{-1}) \right\},
\end{aligned}
$$

$$\nu \begin{vmatrix} \cdot & \{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & B_1^I(\nu\xi^0+\kappa+i\eta)\{P_1^I(\nu\xi^0+\kappa+i\eta)^{-1}\}_1 & \cdot & \cdot \\ \cdot & & \cdot & \cdot \end{vmatrix} = \frac{\nu^6}{(\det P_{s_1}^I)_{\xi^0}(\kappa+i\eta)}$$

$$\times \left\{ Q_2(\xi^0) + \nu^{-\frac{1}{2}} \sqrt{2\left\{ \frac{\xi_0^0}{c_{s_2}^2}(\kappa_0+i\eta_0) - \xi^{0'''}\cdot(\kappa'''+i\eta''') \right\}} T_2(\xi^0) + O(\nu^{-1}) \right\}.$$

In a similar manner as the proof of the equation (1.5), we obtain

$$\left\langle \nu^{\frac{3}{2}} e^{-i\nu\{(x'-y')\cdot\xi^{0'}+x_3\tau_{p_1}^-(\xi^{0'})+y_3\xi_3^0\}} F^I(x,y) - \nu^{\frac{1}{2}} F^I_{s_1\xi^0 p_1}(x,y), \phi(x,y) \right\rangle_{x,y}$$
$$=< F_{s_1\xi^0 p_1 s_2}(x,y), \phi(x,y) >_{x,y} \quad \text{for} \quad \phi(x,y) \in C_0^\infty(\mathbf{R}_-^4 \times \mathbf{R}_-^4),$$

where

(2.21)

$$F^I_{s_1\xi^0 p_1 s_2}(x,y)$$
$$= (2\pi)^{-4} \int_{\mathbf{R}^3} e^{i(x'-y'-\mathrm{grad}\,\tau_{p_1}^+(\xi^{0'})x_3)\cdot(\kappa'+i\eta')} \int_{\mathbf{R}} e^{-iy_3(\kappa_3+i\eta_3)}\, \mathrm{U}(\xi^{0'''})\mathrm{C}$$
$$\times \frac{1}{(\det P^I_{s_1})_{\xi^0}(\kappa+i\eta)} \sqrt{2\left\{ \frac{\xi_0^0}{c_{s_2}^2}(\kappa_0+i\eta_0) - \xi^{0'''}\cdot(\kappa'''+i\eta''') \right\}}$$
$$\times \begin{pmatrix} \frac{T_1(\xi^0)R_1(\xi^{0'})-Q_1(\xi^0)S(\xi^{0'})}{R_1(\xi^{0'})^2} \begin{pmatrix} |\xi^{0'''}| \\ -\tau_{p_1}^+(\xi^{0'}) \end{pmatrix} & \frac{T_2(\xi^0)R_1(\xi^{0'})-Q_2(\xi^0)S(\xi^{0'})}{R_1(\xi^{0'})^2} \begin{pmatrix} |\xi^{0'''}| \\ -\tau_{p_1}^+(\xi^{0'}) \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 0 \end{pmatrix} \left(\mathrm{U}(\xi^{0'''})\mathrm{C}\right)^{-1} d\kappa_3 d\kappa.$$

Here we remark that

$$\left(T_1(\xi^0)R_1(\xi^{0'}) - Q_1(\xi^0)S(\xi^{0'}),\ T_2(\xi^0)R_1(\xi^{0'}) - Q_2(\xi^0)S(\xi^{0'})\right) \not\equiv (0,\ 0)$$

since there is at least one no zero point.

If we localize at the point ξ^0 satisfying (3.5) below, that is $j = s_1$, $k = s_1$, $m = p_1$, then the proof of (1.6) is as follows. For the term including $e^{-i\tau_{s_1}^+(\nu\xi^{0'}+\kappa'+i\eta')}$ we prove as same as (2.14). For the term including $e^{-i\tau_{p_1}^+(\nu\xi^{0'}+\kappa'+i\eta')}$, we have

$$\left\langle \nu^{\frac{3}{2}} e^{-i\nu\{(x'-y')\cdot\xi^{0'}+x_3\tau_{p_1}^-(\xi^{0'})+y_3\xi_3^0\}} F^I(x,y) - \nu^{\frac{1}{2}} F^I_{s_1\xi^0 s_1}(x,y), \phi(x,y) \right\rangle_{x,y}$$
$$= \Big\langle \nu^{\frac{3}{2}} e^{-i\{\tau_{p_1}^+(\nu\xi^{0'}+\kappa'+i\eta')-\nu\tau_{s_1}^+(\xi^{0'})\}x_3} f(\kappa+i\eta,\nu),$$
$$\tilde{\phi}(-\kappa'-i\eta', x_3, \kappa'+i\eta', \kappa_3+i\eta_3) \Big\rangle_{\kappa,x_3}$$
$$= \int \nu^{\frac{1}{2}} e^{i\left\{\nu\tau_{s_1}^+(\xi^{0'}) - \nu^{\frac{1}{2}}\sqrt{2\{\frac{\xi_0^0}{c_{p_1}^2}(\kappa_0+i\eta_0)-\xi^{0'''}\cdot(\kappa'''+i\eta''')\}}\right\}x_3}$$

$$\times \bar{f}(\kappa + i\eta, x_3, \nu)\tilde{\phi}(-\kappa' - i\eta', x_3, \kappa' + i\eta', \kappa_3 + i\eta_3)\, d\kappa dx_3$$
$$\equiv I(\nu)$$

for $\phi(x, y) \in C_0^\infty(\mathbf{R}^4_- \times \mathbf{R}^4_-)$, where

$$\tilde{\phi}(\zeta', x_3, z) = \mathcal{FL}_{(x',y)}[\phi(x,y)](\zeta', z),$$
$$\bar{f}(\kappa + i\eta, x_3, \nu) = e^{\{\mathrm{grad}\tau^+_{p_1}(\xi^{0\prime})\cdot(\kappa'+i\eta')+o(\nu^{-1})\}x_3} f(\kappa + i\eta, \nu).$$

If we put

$${}^tL = -\frac{\sqrt{2\{\frac{\xi_0^0}{c_{p_1}^2}(\kappa_0 + i\eta_0) - \xi^{0\prime\prime\prime}\cdot(\kappa''' + i\eta''')\}}}{\nu^{\frac{1}{2}} x_3 \frac{\xi_0^0}{c_{p_1}^2}} \frac{1}{i}\frac{\partial}{\partial \kappa_0},$$

then we obtain

$$I(\nu) = \int e^{i\left\{\nu\tau^+_{s_1}(\xi^{0\prime}) - \nu^{\frac{1}{2}}\sqrt{2\{\frac{\xi_0^0}{c_{p_1}^2}(\kappa_0+i\eta_0)-\xi^{0\prime\prime\prime}\cdot(\kappa'''+i\eta''')\}}\right\}x_3}$$
$$\times \nu^{\frac{1}{2}} L^2 \left[\bar{f}(\kappa + i\eta, x_3, \nu)\tilde{\phi}(-\kappa' - i\eta', x_3, \kappa' + i\eta', \kappa_3 + i\eta_3)\right] d\kappa dx_3$$
$$\longrightarrow 0 \quad \text{as} \quad \nu \to \infty,$$

since

$$\left|\left(\frac{\partial}{\partial \kappa_0}\right)^j \left[\bar{f}(\kappa + i\eta, x_3, \nu)\tilde{\phi}(-\kappa' - i\eta', x_3, \kappa' + i\eta', \kappa_3 + i\eta_3)\right]\right|$$
$$\leq C(\eta)(|\kappa| + |\eta|)^{-6} \quad \text{if} \quad \nu \geq 1, \quad j \leq 2,$$

and

$$-\exists C_1 \leq x_3 \leq -\exists C_2 < 0 \quad \text{on} \quad \mathrm{supp}\,\tilde{\phi}.$$

Thus we prove the equation (1.6).

The inclusion relation (1.7) is proved in the previous paper, so we omit the proof of (1.7).

Finally we prove the formula (1.8) and (1.9). If $(Q_1(\xi^0), Q_2(\xi^0)) \neq (0,0)$ in (2.18) or if $Q_1(\xi^0) \neq 0$ in (2.20), then we could put

$$F^I_{s_1\xi^0 p_1}(x, y) = \mathrm{Const.}(2\pi)^{-4} \int_{\mathbf{R}^4} \frac{e^{i(x'-y'-\mathrm{grad}\tau^+_{p_1}(\xi^{0\prime})x_3)\cdot(\kappa'+i\eta')-y_3(\kappa_3+i\eta_3)}}{(\kappa_0 + i\eta_0) - \frac{c_{s_1}^2}{\xi_0^0}\xi^{0\prime\prime}\cdot(\kappa'' + i\eta'')} d\kappa,$$

and would like to obtain $\mathrm{supp} F^I_{s_1\xi^0 p_1}(x, y)$. If we put

$$G_1(x) = (2\pi)^{-4} \int_{\mathbf{R}^4} \frac{e^{ix\cdot(\kappa+i\eta)}}{(\kappa_0 + i\eta_0) - \frac{c_{s_1}^2}{\xi_0^0}\xi^{0\prime\prime}\cdot(\kappa'' + i\eta'')} d\kappa,$$

then

$$F^I_{s_1\xi^0 p_1}(x,y) = G_1(x' - y' - \mathrm{grad}\tau^+_{p_1}(\xi^{0\prime})x_3, -y_3).$$

So it is sufficient that we consider $\mathrm{supp}G_1$. From the Paley-Wiener-Schwartz theorem,

$$ch[\mathrm{supp}G_1] = \{x \in \mathbf{R}^4 |\ x \cdot \eta \geq 0 \quad \text{for} \quad \forall \eta \in \Gamma_{s_1\xi^0}\},$$

where ch denotes a convex hull, and

$$\Gamma_{s_1\xi^0} = \left\{ \eta \in \mathbf{R}^4_\eta \,\middle|\, \eta_0 - \frac{c^2_{s_1}}{\xi^0_0}\xi^{0\prime\prime} \cdot \eta'' > 0 \right\}. \tag{2.22}$$

By (2.22), we have

$$ch[\mathrm{supp}G_1] = \left\{ x \in \mathbf{R}^4 \,\middle|\, x = \lambda\left(1, -\frac{c^2_{s_1}}{\xi^0_0}\xi^{0\prime\prime}\right), \quad \lambda \geq 0 \right\}$$

and it is half-line. So we obtain

$$\mathrm{supp}G_1 = ch[\mathrm{supp}G_1] = \left\{ x \in \mathbf{R}^4 \,\middle|\, x = \lambda\left(1, -\frac{c^2_{s_1}}{\xi^0_0}\xi^{0\prime\prime}\right), \quad \lambda \geq 0 \right\},$$

since G_1 is a homogeneous distribution. Thus we prove the formula (1.8). Next we prove the formula (1.9). If

$$\Big(T_1(\xi^0)R_1(\xi^{0\prime}) - Q_1(\xi^0)S(\xi^{0\prime}),\ T_2(\xi^0)R_1(\xi^{0\prime}) - Q_2(\xi^0)S(\xi^{0\prime})\Big) \neq (0,\ 0)$$

in (2.21), then we could put

$$F^I_{s_1\xi^0 p_1 s_2}(x,y) = \mathrm{Const.}(2\pi)^{-4} \int_{\mathbf{R}^4} \frac{e^{i(x'-y'-\mathrm{grad}\tau^+_{p_1}(\xi^{0\prime})x_3)\cdot(\kappa'+i\eta') - y_3(\kappa_3+i\eta_3)}}{(\kappa_0 + i\eta_0) - \frac{c^2_{s_1}}{\xi^0_0}\xi^{0\prime\prime} \cdot (\kappa'' + i\eta'')}$$
$$\times \sqrt{\frac{\xi^0_0}{c^2_{s_2}}(\kappa_0 + i\eta_0) - \xi^{0\prime\prime\prime} \cdot (\kappa''' + i\eta''')}\, d\kappa,$$

and would like to obtain $\mathrm{supp}F^I_{s_1\xi^0 p_1 s_2}(x,y)$. If we put

$$G_2(x) = (2\pi)^{-4} \int_{\mathbf{R}^4} \frac{e^{ix\cdot(\kappa+i\eta)}}{(\kappa_0 + i\eta_0) - \frac{c^2_{s_1}}{\xi^0_0}\xi^{0\prime\prime} \cdot (\kappa'' + i\eta'')}$$
$$\times \sqrt{\frac{\xi^0_0}{c^2_{s_2}}(\kappa_0 + i\eta_0) - \xi^{0\prime\prime\prime} \cdot (\kappa''' + i\eta''')}\, d\kappa,$$

then

$$F^I_{s_1\xi^0 p_1 s_2}(x,y) = G_2(x' - y' - \mathrm{grad}\tau^+_{p_1}(\xi^{0\prime})x_3, -y_3).$$

So it is sufficient that we consider suppG_2. From the Paley-Wiener-Schwartz theorem,

$$ch[\mathrm{supp}G_2] = \{x \in \mathbf{R}^4 |\ x \cdot \eta \geq 0 \quad \text{for} \quad \forall \eta \in \Gamma_{s_1\xi^0 s_2}\},$$

where ch denotes a convex hull, and

$$\Gamma_{s_1\xi^0 s_2} = \left\{ \eta \in \mathbf{R}^4_\eta \,\middle|\, \eta_0 - \frac{c_{s_1}^2}{\xi_0^0}\xi^{0\prime\prime} \cdot \eta^{\prime\prime} > 0, \quad \eta_0 - \frac{c_{s_2}^2}{\xi_0^0}\xi^{0\prime\prime\prime} \cdot \eta^{\prime\prime\prime} > 0 \right\}. \tag{2.23}$$

By (2.19), we have

$$\begin{aligned} ch[\mathrm{supp}G_2] = \Bigg\{ x \in \mathbf{R}^4 \,\Bigg|\, x = k_1 \left(1, -\frac{c_{s_1}^2}{\xi_0^0}\xi^{0\prime\prime}\right) + k_2 \left(1, -\frac{c_{s_2}^2}{\xi_0^0}\xi^{0\prime\prime\prime}, 0\right), \\ k_1,\ k_2 \geq 0 \Bigg\}. \end{aligned}$$

We would like to verify ch[suppG_2]=suppG_2. We take the change of coordinates such as

$$p = A\kappa, \qquad A = \begin{pmatrix} 1 & -\frac{c_{s_1}^2}{\xi_0^0}\xi_1^0 & -\frac{c_{s_1}^2}{\xi_0^0}\xi_2^0 & -\frac{c_{s_1}^2}{\xi_0^0}\xi_3^0 \\ 1 & -\frac{c_{s_2}^2}{\xi_0^0}\xi_1^0 & -\frac{c_{s_2}^2}{\xi_0^0}\xi_2^0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we note that A is holomorphic matrix by the assumption (1.2). Then $G_2(x)$ is given by

$$G_2(x) = (2\pi)^{-4} \frac{1}{|\det A|} \int_{\mathbf{R}^4} \frac{e^{i\,{}^tA^{-1}x\cdot(p - iA\vartheta)}}{c_{s_2}\xi_0^0} \times \frac{\sqrt{\xi_0^0(p_0 - i)}}{p_1 - i}\, dp,$$

where $\vartheta = {}^t(1,0,0,0)$ and $\sqrt{\xi_0^0(p_0 - i)}$ is taken the branch such that $\mathrm{Im}\sqrt{\xi_0^0(p_0 - i)} > 0$. By $A\vartheta = {}^t(1,1,0,0)$ and the Cauchy integral theorem, we obtain

$$\begin{aligned} G_2(x) &= (2\pi)^{-4} \frac{1}{|\det A|} \int_{\mathbf{R}^4} e^{i\,{}^tA^{-1}x\cdot(p - i\,{}^t(1,1,0,0))} \frac{1}{c_{s_2}\xi_0^0} \frac{\sqrt{\xi_0^0(p_0 - i)}}{p_1 - i}\, dp \\ &= (2\pi)^{-4} \frac{1}{|\det A|} \int_{\mathbf{R}^4} e^{i\,{}^tA^{-1}x\cdot p} \frac{1}{c_{s_2}\xi_0^0} \frac{\sqrt{\xi_0^0(p_0 - i0)}}{p_1 - i0}\, dp. \end{aligned} \tag{2.24}$$

By the Fourier transform formula (cf. [Hö, Example 7.1.17]), we deduce

$$\mathcal{F}^{-1}\left[\frac{1}{\xi - i0}\right](x) = iH(x), \tag{2.25}$$

$$\mathcal{F}^{-1}\left[(\xi - i0)^{\frac{1}{2}}\right](x) = -\frac{e^{-\frac{i}{4}\pi}}{2\sqrt{\pi}}\, x_+^{-\frac{3}{2}}, \tag{2.26}$$

where $H(x)$ denotes the Heaviside function and

$$x_+^a = \begin{cases} x^a & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \leq 0 \end{cases}$$

for $a \in \mathbf{C}$. By (2.25) and (2.26), the right-hand side of (2.24) is equal to

$$\begin{cases} \dfrac{-1}{|\det A| c_{s_2}\sqrt{\xi_0^0}} \dfrac{e^{-\frac{i}{4}\pi}}{2\sqrt{\pi}}(z_0)_+^{-\frac{3}{2}} \otimes iH(z_1)\otimes\delta(z_2,z_3) & \text{for} \quad \xi_0^0>0, \\ \dfrac{-1}{|\det A| c_{s_2}\sqrt{|\xi_0^0|}} \dfrac{e^{\frac{i}{4}\pi}}{2\sqrt{\pi}}(z_0)_+^{-\frac{3}{2}} \otimes iH(z_1)\otimes\delta(z_2,z_3) & \text{for} \quad \xi_0^0<0 \end{cases}$$
$$= \frac{-ie^{-i\,\mathrm{sgn}\xi_0^0\frac{\pi}{4}}}{2\sqrt{\pi}|\det A| c_{s_2}\sqrt{|\xi_0^0|}}(z_0)_+^{-\frac{3}{2}} \otimes H(z_1)\otimes\delta(z_2,z_3),$$

where $z = {}^tA^{-1}x$. Thus we get

$$\begin{aligned} \operatorname{supp} G_2 &= \{x\in\mathbf{R}^4 |\ z_0\ge 0,\ z_1\ge 0,\ z_2=z_3=0\} \\ &= \left\{ x\in\mathbf{R}^4 \;\middle|\; x={}^tA\begin{pmatrix}k_1\\k_2\\0\\0\end{pmatrix},\quad k_1,\ k_2\ge 0\right\} \\ &= \Bigg\{ x\in\mathbf{R}^4 \;\Bigg|\; x=k_1\left(1,-\frac{c_{s_1}^2}{\xi_0^0}\xi^{0''}\right)+k_2\left(1,-\frac{c_{s_2}^2}{\xi_0^0}\xi^{0'''},0\right), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k_1,\ k_2\ge 0\Bigg\} \\ &= ch[\operatorname{supp} G_2], \end{aligned}$$

thereby we prove the formula (1.9).

This completes the proof of Main Theorem I.

3. The Location of Singularities

By using Main Theorem I, we find an inner estimate of the location of singularities of reflected and refracted Riemann functions $F^I(x,y)$ and $F^{II}(x,y)$.

In the expressions $F^I(x,y)$ and $F^{II}(x,y)$, the parts put between $U(\xi'''+i\eta''')C$ and $(U(\xi'''+i\eta''')C)^{-1}$ are decomposed into 2×2 and 1×1 matrices valued Riemann functions $F^I_{2\times2}(x,y)$ and $F^I_{1\times1}(x,y)$ for $F^I(x,y)$, and $F^{II}_{2\times2}(x,y)$ and $F^{II}_{1\times1}(x,y)$ for $F^{II}(x,y)$. The displacement vector of $F^\iota_{2\times2}(x,y)$ ($\iota=\{I,II\}$) lies in $(x_1-y_1)(x_3-y_3)$-plane and that of $F^\iota_{1\times1}(x,y)$ ($\iota=\{I,II\}$) lies in (x_2-y_2)-axis. Thus we can treat $F^\iota_{2\times2}(x,y)$ and $F^\iota_{1\times1}(x,y)$ ($\iota=\{I,II\}$) independently.

For $F^I_{2\times2}(x,y)$, we have the following 4 sets of ξ^0 that are roots of $\det P^I_1(\xi^0)=0$ and are not zeros of $\tau^+_m(\xi^0)$ ($m\in\{p_1,p_2,s_2\}$); roots of $\det P^I_{s_1}(\xi^0)={\xi_0^0}^2-c_{s_1}^2|\xi^{0''}|^2=0$ are

(3.1) $$\xi^0=(1,\xi_1^0,\xi_2^0,\tau^+_{s_1}(\xi^{0'}))\text{ for }\tau^-_{s_1}(\xi^{0'})\text{ in (1.5) with }|\xi^{0'''}|<\frac{1}{c_{s_1}},$$

(3.2) $$\xi^0=(1,\xi_1^0,\xi_2^0,\tau^+_{s_1}(\xi^{0'}))\text{ for }\tau^-_{p_1}(\xi^{0'})\text{ in (1.5) with }|\xi^{0'''}|<\frac{1}{c_{p_1}},$$

and roots of $\det P^I_{p_1}(\xi^0)={\xi_0^0}^2-c_{p_1}^2|\xi^{0''}|^2=0$ are

(3.3) $$\xi^0=(1,\xi_1^0,\xi_2^0,\tau^+_{p_1}(\xi^{0'}))\text{ for }\tau^-_{s_1}(\xi^{0'})\text{ in (1.5) with }|\xi^{0'''}|<\frac{1}{c_{p_1}},$$

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^{0'})) \text{ for } \tau_{p_1}^-(\xi^{0'}) \text{ in (1.5) with } |\xi^{0'''}| < \frac{1}{c_{p_1}}. \tag{3.4}$$

(3.1) and (3.2) (resp. (3.3) and (3.4)) correspond to P_1 and S_1 reflected waves for S_1 (resp. P_1) incident wave, respectively.

We have the following 9 sets of ξ^0 that are roots of $\det P_1^I(\xi^0) = 0$ and are not zeros of $\tau_m^+(\xi^0)$ ($m \in \{p_1, s_2, p_2\}$); (3.5)-(3.9) are roots of $\det P_{s_1}^I(\xi^0) = 0$ and

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^{0'})) \text{ for } \tau_{s_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{p_1}} \tag{3.5}$$

are zeros of $\tau_{p_1}^+(\xi^{0'})$,

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^{0'})) \text{ for } \tau_{s_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{s_2}}, \tag{3.6}$$

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^{0'})) \text{ for } \tau_{p_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{s_2}} \tag{3.7}$$

are zeros of $\tau_{s_2}^+(\xi^{0'})$,

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^{0'})) \text{ for } \tau_{s_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{p_2}}, \tag{3.8}$$

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^{0'})) \text{ for } \tau_{p_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{p_2}} \tag{3.9}$$

are zeros of $\tau_{p_2}^+(\xi^{0'})$, moreover (3.10)-(3.13) are roots of $\det P_{s_1}^I(\xi^0) = 0$ and

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^{0'})) \text{ for } \tau_{s_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{s_2}}, \tag{3.10}$$

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^{0'})) \text{ for } \tau_{p_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{s_2}} \tag{3.11}$$

are zeros of $\tau_{s_2}^+(\xi^{0'})$,

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^{0'})) \text{ for } \tau_{s_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{p_2}}, \tag{3.12}$$

$$\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^{0'})) \text{ for } \tau_{p_1}^-(\xi^{0'}) \text{ in (1.6) with } |\xi^{0'''}| = \frac{1}{c_{p_2}} \tag{3.13}$$

are zeros of $\tau_{p_2}^+(\xi^{0'})$. (3.5) corresponds to S_1 lateral or glancing wave for S_1 incident wave with P_1 influence. (3.6) and (3.7) (resp. (3.8) and (3.9)) correspond to P_1

and S_1 lateral waves for S_1 incident wave with S_2 (resp. P_2) influence, respectively. (3.10) and (3.11) (resp. (3.12) and (3.13)) correspond to P_1 and S_1 lateral waves for P_1 incident wave with S_2 (resp. P_2) influence, respectively.

For $F^{II}_{2\times 2}(x,y)$, we have the following the 4 sets of ξ^0 that are roots of $\det P^I_1(\xi^0) = 0$ and are not zeros of $\tau^+_{p_2}(\xi^0)$; roots of $\det P^I_{s_1}(\xi^0) = 0$ are

(3.14)
$$\xi^0 = (1, \xi^0_1, \xi^0_2, \tau^+_{s_1}(\xi^{0\prime})) \text{ for } \tau^+_{s_2}(\xi^{0\prime}) \text{ in (1.10) with } |\xi^{0\prime\prime\prime}| < \frac{1}{c_{s_2}},$$

(3.15)
$$\xi^0 = (1, \xi^0_1, \xi^0_2, \tau^+_{s_1}(\xi^{0\prime})) \text{ for } \tau^+_{p_2}(\xi^{0\prime}) \text{ in (1.10) with } |\xi^{0\prime\prime\prime}| < \frac{1}{c_{p_2}},$$

and roots of $\det P^I_{p_1}(\xi^0) = 0$ are

(3.16)
$$\xi^0 = (1, \xi^0_1, \xi^0_2, \tau^+_{p_1}(\xi^{0\prime})) \text{ for } \tau^+_{s_2}(\xi^{0\prime}) \text{ in (1.10) with } |\xi^{0\prime\prime\prime}| < \frac{1}{c_{s_2}},$$

(3.17)
$$\xi^0 = (1, \xi^0_1, \xi^0_2, \tau^+_{p_1}(\xi^{0\prime})) \text{ for } \tau^+_{p_2}(\xi^{0\prime}) \text{ in (1.10) with } |\xi^{0\prime\prime\prime}| < \frac{1}{c_{p_2}}.$$

(3.14) and (3.15) (resp. (3.16) and (3.17)) correspond to S_2 and P_2 reflected waves for S_1 (resp. P_1) incident wave, respectively.

We have the following 2 sets of ξ^0 that are roots of det $P^I_1(\xi^0) = 0$ and are not zeros of $\tau^+_{p_2}(\xi^0)$;

$$\xi^0 = (1, \xi^0_1, \xi^0_2, \tau^+_{s_1}(\xi^{0\prime})) \text{ for } \tau^+_{s_2}(\xi^{0\prime}) \text{ in (1.11) with } |\xi^{0\prime\prime\prime}| = \frac{1}{c_{p_2}} \tag{3.18}$$

are roots of $\det P^I_{s_1}(\xi^0) = 0$ and zeros of $\tau^+_{p_2}(\xi^0)$,

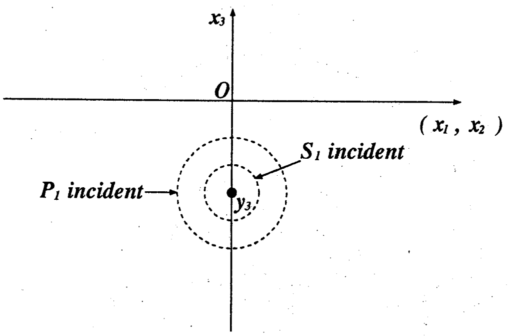
$$\xi^0 = (1, \xi^0_1, \xi^0_2, \tau^+_{p_1}(\xi^{0\prime})) \text{ for } \tau^+_{s_2}(\xi^{0\prime}) \text{ in (1.11) with } |\xi^{0\prime\prime\prime}| = \frac{1}{c_{p_2}} \tag{3.19}$$

are roots of $\det P^I_{p_1}(\xi^0) = 0$ and zeros of $\tau^+_{p_2}(\xi^0)$. (3.18) (resp. (3.19)) corresponds to S_2 lateral wave for S_1 (resp. P_1) incident wave with P_2 influence.

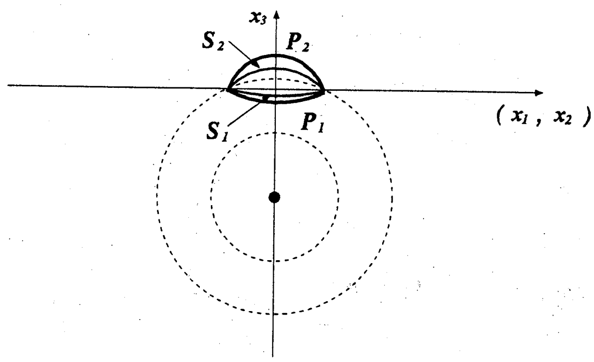
Remark. It is sufficient to consider only the case $\xi^0_0 = 1$ since $(\Gamma_{j\xi^0})^I_k = (\Gamma_{j(t\xi^0)})^I_k$, $(\Gamma_{j\xi^0 m})^I_k = (\Gamma_{j(t\xi^0)m})^I_k$ ($j = \{p_1, s_1\}$, $k = \{p_1, s_1\}$, $m = \{p_1, p_2, s_2\}$), and $(\Gamma_{j\xi^0})^{II}_k = (\Gamma_{j(t\xi^0)})^{II}_k$, $(\Gamma_{j\xi^0 m})^{II}_k = (\Gamma_{j(t\xi^0)m})^{II}_k$ ($j = \{p_1, s_1\}$, $k = \{p_1, s_1\}$, $m = \{p_2,\}$) for $t \in \mathbf{R} \setminus \{0\}$.

The figures of inner estimate of the location of singularities of the reflected and refracted Riemann functions $F^I_{2\times 2}(x,y)$ and $F^{II}_{2\times 2}(x,y)$ are given as in Figure 2 and Figure 3 in the previous paper. However they are wrong. Here we show the correct figure of the inner estimate of the location of singularities of $F^I_{2\times 2}(x,y)$ and $F^{II}_{2\times 2}(x,y)$ with pass of time as in Figure 2.

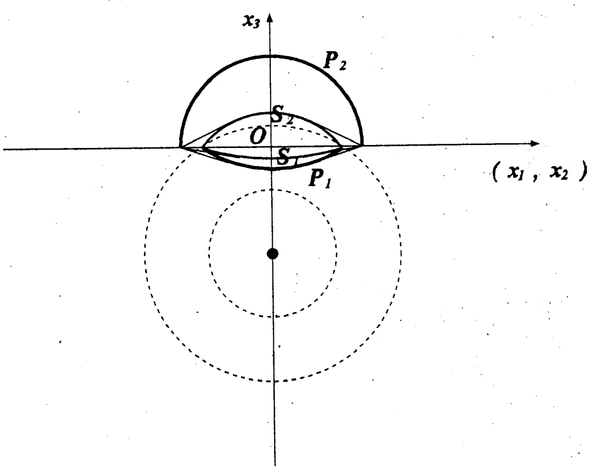
$$x_0 < -\frac{y_3}{c_{p_1}}$$



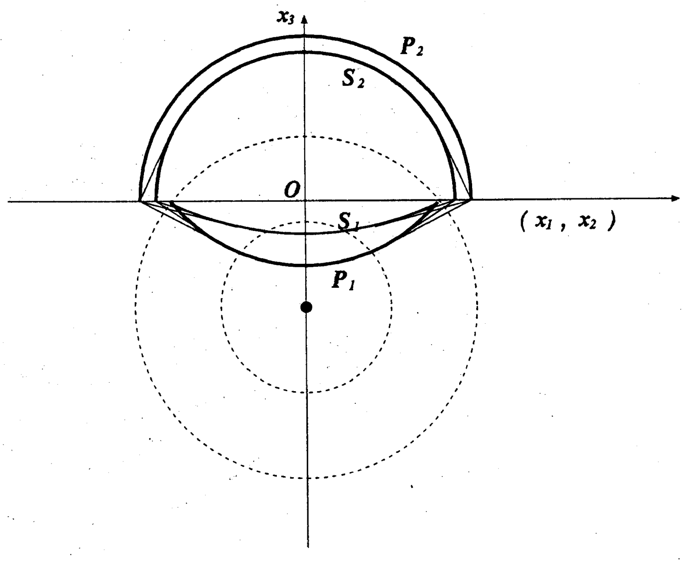
$$-\frac{y_3}{c_{p_1}} \leq x_0 < \frac{c_{p_2}(-y_3)}{c_{p_1}\sqrt{c_{p_2}^2 - c_{p_1}^2}}$$



$$\frac{c_{p_2}(-y_3)}{c_{p_1}\sqrt{c_{p_2}^2 - c_{p_1}^2}} \leq x_0 < \frac{c_{s_2}(-y_3)}{c_{p_1}\sqrt{c_{s_2}^2 - c_{p_1}^2}}$$



$$\frac{c_{s_2}(-y_3)}{c_{p_1}\sqrt{c_{s_2}^2 - c_{p_1}^2}} \leq x_0 < \frac{c_{p_2}(-y_3)}{c_{s_1}\sqrt{c_{p_2}^2 - c_{s_1}^2}}$$



$$\frac{c_{p_2}(-y_3)}{c_{s_1}\sqrt{c_{p_2}^2 - c_{s_1}^2}} \le x_0 < \frac{c_{s_2}(-y_3)}{c_{s_1}\sqrt{c_{s_2}^2 - c_{s_1}^2}} \qquad \frac{c_{s_2}(-y_3)}{c_{s_1}\sqrt{c_{s_2}^2 - c_{s_1}^2}} \le x_0 < \frac{c_{p_1}(-y_3)}{c_{s_1}\sqrt{c_{p_1}^2 - c_{s_1}^2}}$$

$$\frac{c_{p_1}(-y_3)}{c_{s_1}\sqrt{c_{p_1}^2 - c_{s_1}^2}} \le x_0$$

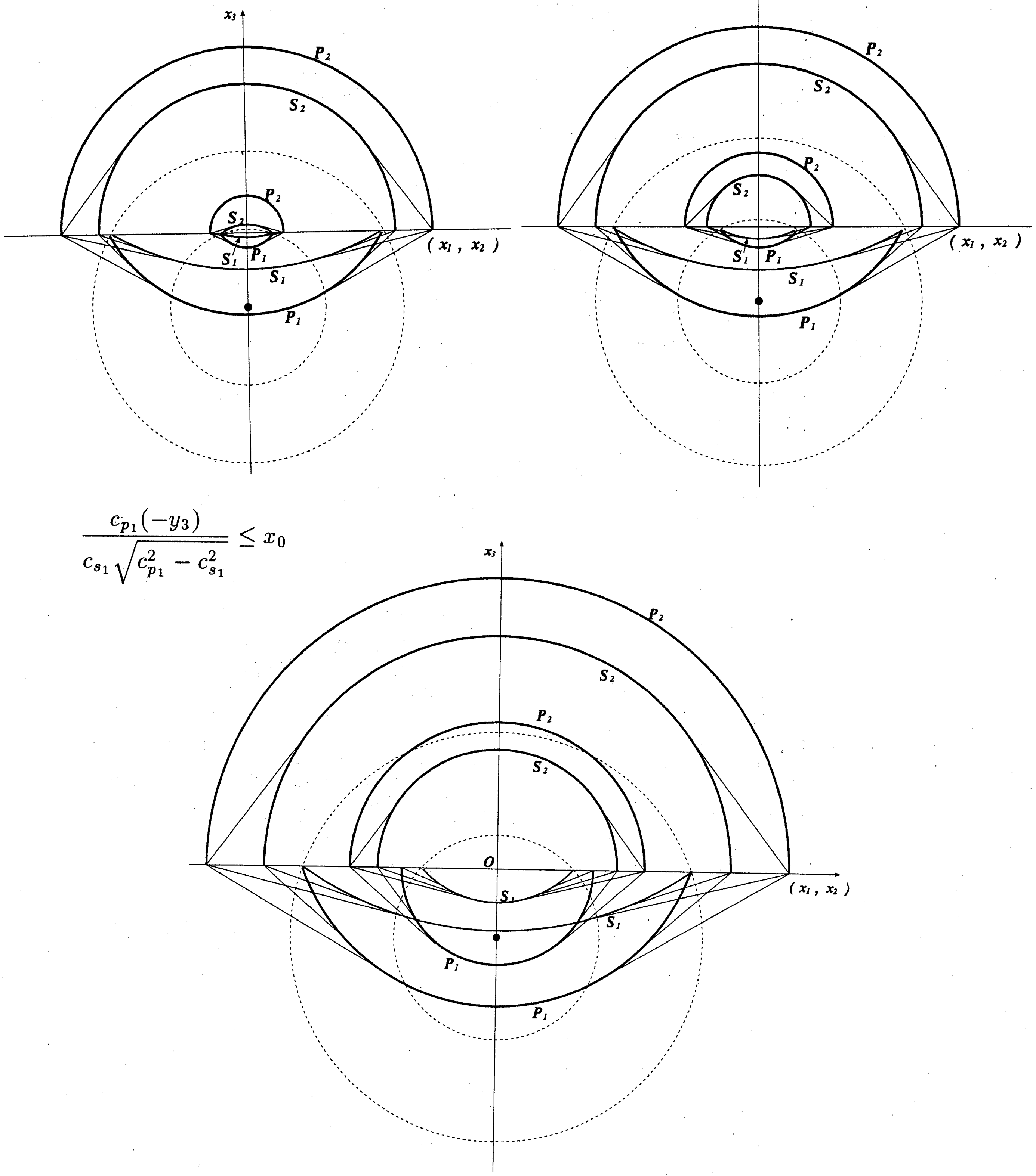


Figure 2 Inner estimate of the location of singularities of $F^{I}_{2\times2}(x,y)$ and $F^{II}_{2\times2}(x,y)$

4. The Stoneley Wave

In this section, we consider the elastic wave propagation if unit impulse Dirac's delta is put on the interface $x_3 = 0$.

The Riemann functions of the elastic mixed problem (1.3) is given by

$$H_l(x) = \begin{cases} H_l^I(x) & \text{for} \quad x_3 < 0, \\ H_l^{II}(x) & \text{for} \quad x_3 > 0, \quad l = 1, 2, \end{cases} \tag{4.1}$$

that are obtained to solve the following interface problems:

$$\begin{cases} P^I(D_x)H_1^I(x) = 0, \quad x \in \mathbf{R}_-^4, \\ P^{II}(D_x)H_1^{II}(x) = 0, \quad x \in \mathbf{R}_+^4, \\ H_1^I(x)|_{x_3=-0} - H_1^{II}(x)|_{x_3=+0} = \delta(x', 0)E, \quad x' \in \mathbf{R}^3, \\ B^I(D_x)H_1^I(x)|_{x_3=-0} = B^{II}(D_x)H_1^{II}(x)|_{x_3=+0}, \end{cases} \tag{4.2}$$

$$\begin{cases} P^I(D_x)H_2^I(x) = 0, \quad x \in \mathbf{R}_-^4, \\ P^{II}(D_x)H_2^{II}(x) = 0, \quad x \in \mathbf{R}_+^4, \\ H_2^I(x)|_{x_3=-0} = H_2^{II}(x)|_{x_3=+0}, \\ B^I(D_x)H_1^I(x)|_{x_3=-0} - B^{II}(D_x)H_1^{II}(x)|_{x_3=+0} = \delta(x', 0)E, \quad x' \in \mathbf{R}^3, \end{cases} \tag{4.3}$$

where E is a 3×3 identity matrix.

As the same procedure to get the explicit expression of the reflected and refracted Riemann finctions $F^I(x, y)$ and $F^{II}(x, y)$, we obtain the expression of Riemann functions $H_1^I(x)$ and $H_1^{II}(x)$. $H_2^I(x)$ and $H_2^{II}(x)$ are given by similar expression.

$$H_1^I(x) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{ix' \cdot (\xi' + i\eta')} \mathrm{U}(\xi''' + i\eta''')\mathrm{C} \tag{4.4}$$

$$\times \left(\begin{array}{c} \dfrac{\begin{vmatrix} 1 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{vmatrix}}{R_1(\xi' + i\eta')} \begin{pmatrix} |\xi''' + i\eta'''| \\ -\tau_{p_1}^+(\xi' + i\eta') \end{pmatrix} e^{-i\tau_{p_1}^+(\xi' + i\eta')x_3} \\ + \dfrac{\begin{vmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \end{vmatrix}}{R_1(\xi' + i\eta')} \begin{pmatrix} \tau_{s_1}^+(\xi' + i\eta') \\ |\xi''' + i\eta'''| \end{pmatrix} e^{-i\tau_{s_1}^+(\xi' + i\eta')x_3} \\ 0 \end{array} \right.$$

$$
\frac{\begin{vmatrix} 0 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{vmatrix}}{R_1(\xi'+i\eta')} \begin{pmatrix} |\xi'''+i\eta'''| \\ -\tau_{p_1}^{+}(\xi'+i\eta') \end{pmatrix} e^{-i\tau_{p_1}^{+}(\xi'+i\eta')x_3}
+ \frac{\begin{vmatrix} \cdot & 0 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \end{vmatrix}}{R_1(\xi'+i\eta')} \begin{pmatrix} \tau_{s_1}^{+}(\xi'+i\eta') \\ |\xi'''+i\eta'''| \end{pmatrix} e^{-i\tau_{s_1}^{+}(\xi'+i\eta')x_3}
$$

$$0$$

$$
\left.\begin{matrix} 0 \\ 0 \\ \dfrac{\begin{vmatrix} 1 & \cdot \\ 0 & \cdot \end{vmatrix}}{R_2(\xi'+i\eta')} e^{-i\tau_{s_1}^{+}(\xi'+i\eta')x_3} \end{matrix}\right) \times (\mathrm{U}(\xi'''+i\eta''')\mathrm{C})^{-1} d\xi', \quad x_3 < 0,
$$

$$
(4.5) \qquad H_1^{II}(x) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{ix'\cdot(\xi'+i\eta')} \mathrm{U}(\xi'''+i\eta''')\mathrm{C}
$$

$$
\times \left(\begin{matrix}
\dfrac{\begin{vmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot \end{vmatrix}}{R_1(\xi'+i\eta')} \begin{pmatrix} |\xi'''+i\eta'''| \\ \tau_{p_2}^{+}(\xi'+i\eta') \end{pmatrix} e^{i\tau_{p_2}^{+}(\xi'+i\eta')x_3}
+ \dfrac{\begin{vmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \end{vmatrix}}{R_1(\xi'+i\eta')} \begin{pmatrix} -\tau_{s_2}^{+}(\xi'+i\eta') \\ |\xi'''+i\eta'''| \end{pmatrix} e^{i\tau_{s_2}^{+}(\xi'+i\eta')x_3} \\
0
\end{matrix} \right.
$$

$$
\frac{\begin{vmatrix} \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot \end{vmatrix}}{R_1(\xi'+i\eta')} \begin{pmatrix} |\xi'''+i\eta'''| \\ \tau_{p_2}^{+}(\xi'+i\eta') \end{pmatrix} e^{i\tau_{p_2}^{+}(\xi'+i\eta')x_3}
+ \frac{\begin{vmatrix} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \end{vmatrix}}{R_1(\xi'+i\eta')} \begin{pmatrix} -\tau_{s_2}^{+}(\xi'+i\eta') \\ |\xi'''+i\eta'''| \end{pmatrix} e^{i\tau_{s_2}^{+}(\xi'+i\eta')x_3}
$$

$$0$$

$$\left.\begin{matrix} 0 \\ 0 \\ \dfrac{\begin{vmatrix} \cdot & 1 \\ \cdot & 0 \end{vmatrix}}{R_2(\xi'+i\eta')} e^{i\tau^+_{s_2}(\xi'+i\eta')x_3} \end{matrix}\right) \times (\mathrm{U}(\xi'''+i\eta''')\mathrm{C})^{-1}d\xi', \quad x_3>0.$$

Here the Lopatinski determinants $R_1(\zeta')$ ($\zeta'=\xi'+i\eta'$) and $R_2(\zeta')$ are defined (2.3)-(2.4) and (2.5)-(2.6), respectively. $\cdot$ means the same component of the Lopatinski matrices $\mathcal{R}_1(\zeta')$ and $\mathcal{R}_2(\zeta')$ defined by (2.4) and (2.6).

We obtained the following conclusion for the Lopatinski determinant $R_1(\xi')$ (cf.[Sh, Section 3]). Here c_{St} is the speed of the Stoneley wave corresponding to the interface wave and is less than or equal to most minimum speed c_{s_1}. The zero does not necessarily exist. For the discreminant

$$\begin{aligned}\mathrm{Dis}(c_{s_1}^2) = \left(\mu_1 - 2\mu_2 + \frac{c_{s_1}^2}{c_{s_2}^2}\mu_2\right)^2 + \sqrt{\frac{c_{s_1}^2}{c_{p_2}^2}-1}\sqrt{\frac{c_{s_1}^2}{c_{s_2}^2}-1}(\mu_1-2\mu_2)^2 \\ + \sqrt{\frac{c_{s_1}^2}{c_{p_1}^2}-1}\sqrt{\frac{c_{s_1}^2}{c_{s_2}^2}-1}\mu_1\mu_2\frac{c_{s_1}^2}{c_{s_2}^2},\end{aligned}$$

we get

(i)

$$\mathrm{Dis}(c_{s_1}^2) > 0 \implies \text{The zero } \xi_0 = c_{St}|\xi'''| \text{ of } R_1(\xi') \text{ exists in } [0,\ c_{s_1}|\xi'''|) \text{ with order 1.}$$

(ii)

$$\mathrm{Dis}(c_{s_1}^2) = 0 \implies c_{St} = c_{s_1} \text{ and we shall consider this case under some restricted conditions (cf. [Sh, Lemma 6.4]).}$$

(iii)

$$\mathrm{Dis}(c_{s_1}^2) < 0 \implies R_1(\xi') \text{ has no zero.}$$

If $\mathrm{Dis}(c_{s_1}^2) > 0$, then we find an inner estimate of the location of singularities of the Riemann functions $H^I(x)$ and $H^{II}(x)$ by using Main Theorem II. And if $\mathrm{Dis}(c_{s_1}^2) < 0$, then a singularity corresponding to the Stoneley wave does not appear. For the Lopatinski determinant $R_2(\xi')$, it has no zeros.

The Lopatinski determinant $R_1(\xi')$ has the expression

$$\begin{aligned}(4.6) \qquad R_1(\xi') = &\left\{\frac{1}{c_{p_1}c_{s_1}}\frac{\mu_2^2}{c_{s_2}^4} + \frac{1}{c_{p_2}c_{s_2}}\frac{\mu_1^2}{c_{s_1}^4} + \frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}\left(\frac{1}{c_{p_1}c_{s_2}} + \frac{1}{c_{p_2}c_{s_1}}\right)\right\} \\ &(\xi_0 - c_{St}|\xi'''|)\left(\xi_0^5 + \text{lower term of } \xi_0\right).\end{aligned}$$

The first factor of the right-hand side of (4.6) is constant, and the third factor of that does not have the factor of zeros of $R_1(\xi')$. A point causes a singularity corresponding to the Stoneley wave is

$$(4.7) \qquad \xi^{0'} = (1, \xi_1^0, \xi_2^0), \qquad |\xi^{0'''}| = \frac{1}{c_{St}}.$$

We note that there are not any real ξ' that are zeros of $\xi_0 - c_{St}|\xi'''|$ and some of $\tau^+_{s_1}(\xi')$, $\tau^+_{p_1}(\xi')$, $\tau^+_{s_2}(\xi')$, and $\tau^+_{p_2}(\xi')$ are real because of $0 < c_{St} \leq c_{s_1}$ and the assumption (1.2).

We consider that

$$
\begin{aligned}
&\nu^{-1}e^{-i\nu x'\cdot\xi^{0'}}H^I_1(x)\\
&= (2\pi)^{-3}\int_{\mathbf{R}^3} e^{ix'\cdot(-\nu\xi^{0'}+\xi'+i\eta')}\mathrm{U}(\xi'''+i\eta''')\mathrm{C}\times\nu^{-1}\\
&\times\left(\begin{array}{c}\dfrac{\begin{vmatrix}1&\cdot&\cdot&\cdot\\0&\cdot&\cdot&\cdot\\0&\cdot&\cdot&\cdot\\0&\cdot&\cdot&\cdot\end{vmatrix}}{R_1(\xi'+i\eta')}\begin{pmatrix}|\xi'''+i\eta'''|\\-\tau^+_{p_1}(\xi'+i\eta')\end{pmatrix}e^{-i\tau^+_{p_1}(\xi'+i\eta')x_3}\\ \text{and so on}\end{array}\right)\times(\mathrm{U}(\xi'''+i\eta''')\mathrm{C})^{-1}d\xi'.
\end{aligned}
$$

Making the change of variable $-\nu\xi^{0'}+\xi'=\kappa'$, then we have

$$
\begin{aligned}
&= (2\pi)^{-3}\int_{\mathbf{R}^3} e^{ix'\cdot(\kappa'+i\eta')}\mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''')\mathrm{C}\\
&\times\nu^{-1}\left(\begin{array}{c}\dfrac{\begin{vmatrix}1&\cdot&\cdot&\cdot\\0&\cdot&\cdot&\cdot\\0&\cdot&\cdot&\cdot\\0&\cdot&\cdot&\cdot\end{vmatrix}}{R_1(\nu\xi^{0'}+\kappa'+i\eta')}\begin{pmatrix}|\nu\xi^{0'''}+\kappa'''+i\eta'''|\\-\tau^+_{p_1}(\nu\xi^{0'}+\kappa'+i\eta')\end{pmatrix}e^{-i\tau^+_{p_1}(\nu\xi^{0'}+\kappa'+i\eta')x_3}\\ \\ \text{and so on}\end{array}\right)\times(\mathrm{U}(\nu\xi^{0'''}+\kappa'''+i\eta''')\mathrm{C})^{-1}d\kappa'.
\end{aligned}
$$

By (2.8), we have for the point (4.7)

$$
\nu^{-1}\tau^+_k(\nu\xi^{0'}+\kappa'+i\eta')=\tau^+_k(\xi^{0'})+O(\nu^{-1}),\quad k=\{p_1,s_1\}, \tag{4.8}
$$

where

$$
\tau^+_{p_1}(\xi^{0'})=i\frac{\sqrt{c^2_{p_1}-c^2_{St}}}{c_{p_1}c_{St}},\quad \tau^+_{s_1}(\xi^{0'})=i\frac{\sqrt{c^2_{s_1}-c^2_{St}}}{c_{s_1}c_{St}}. \tag{4.9}
$$

By (4.6), we have for the point (4.7)

(4.10)

$$
\nu^{-5}R_1(\nu\xi^{0'}+\kappa'+i\eta')=\mathrm{Const.}(\xi^{0'})\left(\eta_0-c_{St}\frac{\xi^0_1\eta_1+\xi^0_2\eta_2}{|\xi^{0'''}|}\right)+O(\nu^{-1})
$$

$$
\begin{aligned}
&= \mathrm{Const.}(\xi^{0'})\left((\kappa_0+i\eta_0)-c_{St}\big((\kappa_1+i\eta_1)+(\kappa_2+i\eta_2)\big)\right)+O(\nu^{-1})\\
&= R_{1\xi^{0'}}(\kappa'+i\eta')+O(\nu^{-1}).
\end{aligned}
$$

Moreover if we put

$$
\begin{vmatrix}
1 & \tau_{s_1}^{+}(\xi^{0'}) & |\xi^{0'''}| & -\tau_{s_2}^{+}(\xi^{0'}) \\
0 & |\xi^{0'''}| & \tau_{p_2}^{+}(\xi^{0'}) & |\xi^{0'''}| \\
0 & -\rho_1 c_{s_1}^2\left(\tau_{s_1}^{+}(\xi^{0'})^2-|\xi^{0'''}|^2\right) & 2\rho_2 c_{s_2}^2\tau_{p_2}^{+}(\xi^{0'})|\xi^{0'''}| & -\rho_2 c_{s_2}^2\left(\tau_{s_2}^{+}(\xi^{0'})^2-|\xi^{0'''}|^2\right) \\
0 & -2\rho_1 c_{s_1}^2\tau_{s_1}^{+}(\xi^{0'})|\xi^{0'''}| & \rho_2 c_{s_2}^2\left(\tau_{s_2}^{+}(\xi^{0'})^2-|\xi^{0'''}|^2\right) & 2\rho_2 c_{s_2}^2\tau_{s_2}^{+}(\xi^{0'})|\xi^{0'''}|
\end{vmatrix}
$$

is equal to $V_1(\xi^{0'})$ (resp.

$$
\begin{vmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \end{vmatrix} = V_2(\xi^{0'}),\quad
\begin{vmatrix} 0 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{vmatrix} = V_3(\xi^{0'}),\quad
\begin{vmatrix} \cdot & 0 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \end{vmatrix} = V_4(\xi^{0'})),
$$

then

$$
\nu^{-5}V_n(\nu\xi^{0'}+\kappa'+i\eta') = V_n(\xi^{0'})+O(\nu^{-1}),\quad n=1,2,3,4. \tag{4.11}
$$

Thus we get from (2.9), (2.10), (4.8), (4.10), and (4.11),

$$
\begin{aligned}
&\lim_{\nu\to\infty}\nu^{-1}e^{-i\nu x'\cdot\xi^{0'}}H_1^I(x) = (2\pi)^{-3}\int_{\mathbf{R}^3} e^{ix'\cdot(\xi'+i\eta')}\mathrm{U}(\xi^{0'''})\mathrm{C}\times\\
&\begin{pmatrix}
\frac{V_1(\xi^{0'})}{R_{1\xi^{0'}}(\kappa'+i\eta')}\begin{pmatrix}|\xi^{0'''}|\\ -\tau_{p_1}^{+}(\xi^{0'})\end{pmatrix} e^{\frac{\sqrt{c_{p_1}^2-c_{St}^2}}{c_{p_1}c_{St}}x_3} & \frac{V_3(\xi^{0'})}{R_{1\xi^{0'}}(\kappa'+i\eta')}\begin{pmatrix}|\xi^{0'''}|\\ -\tau_{p_1}^{+}(\xi^{0'})\end{pmatrix} e^{\frac{\sqrt{c_{p_1}^2-c_{St}^2}}{c_{p_1}c_{St}}x_3} & \\
+\frac{V_2(\xi^{0'})}{R_{1\xi^{0'}}(\kappa'+i\eta')}\begin{pmatrix}\tau_{s_1}^{+}(\xi^{0'})\\ |\xi^{0'''}|\end{pmatrix} e^{\frac{\sqrt{c_{s_1}^2-c_{St}^2}}{c_{s_1}c_{St}}x_3} & +\frac{V_4(\xi^{0'})}{R_{1\xi^{0'}}(\kappa'+i\eta')}\begin{pmatrix}\tau_{s_1}^{+}(\xi^{0'})\\ |\xi^{0'''}|\end{pmatrix} e^{\frac{\sqrt{c_{s_1}^2-c_{St}^2}}{c_{s_1}c_{St}}x_3} & \\
0 & 0 & \\
 & & \begin{matrix}0\\0\\0\end{matrix}
\end{pmatrix}\times(\mathrm{U}(\xi^{0'''})\mathrm{C})^{-1}d\kappa' = H_{1\xi^{0'}}^I(x).
\end{aligned}
$$

Therefore we can prove Main Theorem II in the same way as Main Theorem I.

Finally, we calculate the singularity of $\lim_{x_3\to -0}H_1^I(x)$ caused by the point (4.7) as the Stoneley wave.

$$
\Gamma_{St\xi^{0'}} = \{\eta\in\mathbf{R}^3 |\quad \eta_0-c_{St}^2(\xi_1^0\eta_1+\xi_2^0\eta_2)>0\}\times\mathbf{R}_\eta,
$$

and

$$\operatorname{supp} \lim_{x_3 \to -0} H^I_{1\xi^0}(x) = \left(\Gamma_{St\xi^{0\prime}}\right)' = \left\{x' \in \mathbf{R}^3 : x' \cdot \eta' \geq 0, \quad \eta' \in \Gamma_{St\xi^{0\prime}}\right\}$$
$$= \left\{x' \in \mathbf{R}^3 : x' = u(1, -c_{St}^2\xi_1^0, -c_{St}^2\xi_1^0), \quad u \geq 0\right\}.$$

Thus an inner estimate of the location of singularities of $\lim_{x_3 \to -0} H_1^I(x)$, corresponding to the Stoneley wave on the interface $x_3 = 0$ is given by

$$(4.12) \qquad \bigcup_{|\xi^{0\prime\prime\prime}| = \frac{1}{c_{St}}} \left(\Gamma_{St\xi^{0\prime}}\right)' = \{x' \in \mathbf{R}^3 : \ x_1^2 + x_2^2 = c_{St}^2 x_0^2, \quad x_0 \geq 0\}.$$

Thus an inner estimate of the location of singularities of $\lim_{x_3 \to +0} H_1^{II}(x)$ is also given by (4.12) in a similar manner.

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