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Mourre theory for analytically fibered operators

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We present here some results obtained with C. Gérard about the Mourre theory for a class of operators which contains among other examples matrix valued differential operators with constant coefficients an periodic Schrödinger operators. We refer the reader to [5, 4] for further details

The framework is the one of a Hilbert space $\mathcal{H}$ equal to $L^2(M,\mu;\mathcal{H}') = \int^{\oplus}\mathcal{H}'d\mu(k)$ where $\mathcal{H}'$ is a separable Hilbert space and (M,μ) is a σ-finite measured space. In this framework, the fibered operators are the self-adjoint direct integrals of the form

$$H_0 = \int^{\oplus} H_0(k)d\mu(k).$$

The class of operators which we are interested in is characterized by the three conditions

i) the space M is a real-analytic manifold and μ is a $\mathcal{C}^\infty$-1-density.

ii) the resolvent $(H_0(k)+i)^{-1}$ is analytic with respect to k and $H_0(k)$ has only discrete spectrum for $k \in M$.

iii) the projection $p_{\mathbb{R}} : \Sigma \ni (\lambda, k) \mapsto \lambda \in \mathbb{R}$ is a proper map.

The set Σ mentionned above is the so called Bloch variety in the case of periodic Schrödinger operator. Under condition i) and ii) it is well defined in $\mathbb{R}\times M$ as a real analytic set (see [10]) by

$$\Sigma = \left\{(\lambda,k)\in\mathbb{R}\times M, \lambda\in\sigma(H_0(k))\right\}.$$

The main result states the existence of a conjugate operator for H_0 in any energy interval which avoids the discrete set of "thresholds" specified in the proof.

Theorem 1. *There exists a discrete set τ determined by H_0 so that for any interval $I \subset\subset \mathbb{R}\backslash\tau$ there exists an operator A_I essentially self-adjoint on $D(A_I) = \mathcal{C}^\infty_{comp}(M;$ $\mathcal{L}(\mathcal{H}'))$ satisfying the following properties:*

i) *For all $\chi \in \mathcal{C}^\infty_{comp}(I)$, there exists a constant $c_\chi > 0$ so that*

$$\chi(H_0)\,[H_0, iA_I]\,\chi(H_0) \geq c_\chi \chi(H_0)^2.$$

ii) *The multi-commutators $\mathrm{ad}^k_{A_I}(H_0)$ are bounded for all $k \in \mathbb{N}$.*

iii) *The operator A_I is a first order differential operator in k whose coefficients belong to $\mathcal{C}^\infty(M;$*
$\mathcal{L}(\mathcal{H}'))$ and there exists $\chi \in \mathcal{C}^\infty_{comp}(\mathbb{R}\setminus\tau)$ so that $A = \chi(H_0)A = A\chi(H_0)$.

Before giving the idea of the proof, let us recall some known consequences of this property. These are results of Mourre theory which hold for some natural class of perturbed Hamiltonians $H = H_0 + V$ (the result are even more precise for $V = 0$). For details about Mourre theory, we refer the details to [1], [7] and [8]).

Proposition 2. *Let A_I be a conjugate operator for H_0 associated with an arbitrary compact interval $I \subset \mathbb{R}\setminus\tau$. Let V be a symmetric operator on $\mathcal{H}$ so that:*

i) $V(H_0 + i)^{-1}$ *is compact,*

ii) $(H_0 + i)^{-1}[V, iA_I](H_0 + i)^{-1}$ *is compact,*

iii) $\int_0^1 \|(H_0 + i)^{-1}\left(e^{itA_I}[V, iA_I]e^{-itA_I} - [V, iA_I]\right)(H_0 + i)^{-1}\|\frac{dt}{t} < \infty.$

Then the following results hold:

i) *There exists a constant $c > 0$ and a compact operator K so that if $\chi \in \mathcal{C}^\infty_{comp}(I)$*

$$\chi(H)[H, iA_I]\chi(H) \geq c\chi^2(H) + K.$$

Consequently $\sigma_{\rm pp}(H)$ is of finite multiplicity in $\mathbb{R}\backslash\tau$ and has no accumulation points in $\mathbb{R}\backslash\tau$.

ii) *For each* $\lambda \in I\backslash\sigma_{\rm pp}(H)$, *there exist* $\epsilon > 0$ *and* $c > 0$ *so that*

$$1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H, iA_I]1_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geq c1_{[\lambda-\epsilon,\lambda+\epsilon]}(H).$$

iii) *The limiting absorption principle holds on* $I\backslash\sigma_{\rm pp}(H)$:

$$\lim_{\epsilon\to\pm 0}(1+|A_I|)^{-s}(H-\lambda+i\epsilon)^{-1}(1+|A_I|)^{-s} \text{ exists and is bounded for all } s > \frac{1}{2}.$$

Consequently the singular continuous spectrum of H *is empty.*

iv) *If the operator* $(1+|A_I|)^s V(1+|A_I|)^s$ *is bounded for some* $s > \frac{1}{2}$, *then for any open interval* $\Delta \subset I$, *the wave operators*

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}1_\Delta(H_0) =: \Omega^\pm_\Delta$$

exist and are asymptotically complete,

$$1^{\rm c}_\Delta(H)\mathcal{H} = \Omega^\pm_\Delta\mathcal{H}.$$

Idea of the Proof : The proof is modeled on the case of dispersive hamiltonians $H_0 = p(D)$ on $L^2(\mathbb{R}^d)$ with $p \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ and $D = \frac{1}{i}\partial_x$. In such a case, the conjugate operator A is defined in the Fourier variable by

$$A = \frac{1}{2}(\partial_\xi p(\xi).D_\xi + \text{h. c.}).$$

Here the variable $k \in M$ plays the role of the Fourier variable while p has to be replaced by the eigenvalues of $H_0(k)$. Then one encounters the famous problem of the singularities of the Bloch variety. This can be bypassed with the next construction. First one makes a partition of Σ according to the multiplicity

$$\Sigma_i = \{(\lambda, k) \in \mathbb{R}\times M, \operatorname{rank}(1_\lambda(H_0(k))) = i\}$$

and one notices that the Σ_i are semi-analytic subsets of $\mathbb{R}\times M$ locally given by

$$\delta(\lambda,k) = \ldots = \partial_\lambda^{i-2}\delta(\lambda,k) = 0, \tag{1}$$

$$\partial_\lambda^{i-1}\delta(\lambda,k) = 0 \text{ and } \partial_\lambda^i\delta(\lambda,k) \neq 0, \tag{2}$$

where $\delta(\lambda,k)$ is some polynomial in λ analytically parametrized by k. Then the analyticity and properness assumptions ensure the existence of a stratification of the mapping $p_{\mathbb{R}} : \Sigma \to \mathbb{R}$ compatible with the partition $\Sigma = \cup_i \Sigma_i$ (see [2, 3, 6]). The set of thresholds is then given as the union of strata on $\mathbb{R}$ with dimension 0 (analogous to critical values in the dispersive case). Out of the thresholds, a stratum S_α of Σ with $S_\alpha \subset \Sigma_i$ can be written locally

$$S_\alpha \cap (I_0 \times V_0) = \left\{ \left(\tilde{\lambda}(k', 0), k', 0 \right), \ (k', 0) \in V_0 \right\},$$

where $\tilde{\lambda}$ is the solution to (2) given by implicit function theorem. This function is smooth with respect to k' on S_α and coincide with the eigenvalue which is noncritical. The conjugate operator is then given in a neighbourhood $I_0 \times V_0$ of $(\lambda_0, k_0) \in S_\alpha \in \Sigma$ by

$$A_{\lambda_0,k_0} = 1_{I_0}(H_0(k)) \circ \left[\frac{1}{2} \partial_{k'} \tilde{\lambda}(k) D_{k'} + \text{h. c.} \right] \circ 1_{I_0}(H_0(k)). \tag{3}$$

The operator coefficients depend analytically on $k \in V_0$ and the positive commutator estimate with H_0 come from the coincidence with the eigenvalue and the fact that $p_{\mathbb{R}}$ has no critical point on S_α. A compactness argument allowed by the properness assumption leads to the global definition of A_I. One then check the essential self-adjointness with the help of Nelson's commutator theorem [9]. $\square$

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