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On the Spectrum of Dirac Operators with Potentials Diverging at Infinity

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1 Results.

In this report we consider the spectrum of the Dirac operator

$$L=\sum_{j=1}^{3}\alpha_{j}D_{j}+m(x)\beta+q(x)I_{4}\qquad\left(x\in\mathbf{R}^{3},\quad D_{j}=-i\frac{\partial}{\partial x_{j}}\right),$$

in the Hilbert space $\mathcal{H}=[L^{2}(\mathbf{R}^{3})]^{4}$, where

$$\alpha_{j}=\begin{pmatrix}\mathbf{0}&\sigma_{j}\\ \sigma_{j}&\mathbf{0}\end{pmatrix}\ (1\leq j\leq 3),\quad \beta=\begin{pmatrix}I_{2}&\mathbf{0}\\ \mathbf{0}&-I_{2}\end{pmatrix},\quad I_{4}=\begin{pmatrix}I_{2}&\mathbf{0}\\ \mathbf{0}&I_{2}\end{pmatrix},$$

$$\sigma_{1}=\begin{pmatrix}0&1\\ 1&0\end{pmatrix},\quad \sigma_{2}=\begin{pmatrix}0&-i\\ i&0\end{pmatrix},\quad \sigma_{3}=\begin{pmatrix}1&0\\ 0&-1\end{pmatrix},\quad I_{2}=\begin{pmatrix}1&0\\ 0&1\end{pmatrix},$$

and real valued functions $m(x)$ and $q(x)$ are assumed to be continuous in $\mathbf{R}^{3}$ and satisfy

$$m(x)\longrightarrow+\infty\quad(\text{or}\quad-\infty)\quad\text{as}\quad r=|x|\longrightarrow\infty.$$

or

$$q(x)\longrightarrow+\infty\quad(\text{or}\quad-\infty)\quad\text{as}\quad r\longrightarrow\infty.$$

Let us denote the unique self-adjoint realization of L by H. We will show below the the structure of the spectrum of the Dirac operator by dividing the problem into three cases ;

(a) $\limsup_{r\to\infty}\left|\frac{q(x)}{m(x)}\right|<1$,

(b) $\limsup_{r\to\infty}\left|\frac{m(x)}{q(x)}\right|<1$,

(c) $q(x)\equiv m(x)$ (or $q(x)\equiv-m(x)$).

Theorem A (Yamada [9], Theorem 1). Assume (a). Let $m, q \in C^1$ satisfy

(A.1) $m(x) \longrightarrow +\infty$ (or $-\infty$) as $r \to \infty$,

(A.2) $|\nabla m| = O(m(x))$, $|\nabla q| = O(m(x))$ as $r \to \infty$.

Then, we have $\sigma(H) = \sigma_d(H)$ (i.e., the set of discrete eigenvalues with finite multipicity), which is unbounded at $\pm\infty$ in $\mathbf{R}$.

Theorem B (Schmidt and Yamada [6], Theorem 1). Let $m \in C^1$ and $q \in C^0$ be spherically symmetric (i.e., $q = q(r)$, $m = m(r)$), and satisfy (b) and

(B.1) $q(r) \longrightarrow +\infty$ (or $-\infty$) as $r \to \infty$,

(B.2) $\liminf_{r\to\infty} |m(r)| > 0$,

(B.3) there exist a positive number R_0 and two distinct real values λ_1, λ_2 such that

$$\frac{m}{q-\lambda_j} \in BV[R_0,\infty) \quad (j=1,2),$$

that is, they are of bounded variation in the interval $[R_0, +\infty)$.

(B.4) there exist a positive number R_0 such that

$$\frac{m'}{r\,m\,q} \in L^1(R_0,\infty).$$

Then, we have $\sigma_{ac}(H) = \mathbf{R}$ and $\sigma_s(H) = \emptyset$ where $\sigma_{ac}(H)$ ($\sigma_s(H)$) is the absolutely continuous (singular) spectrum of H.

Remark 1. In Theorem B, if $m, q \in C^1$ satisfy (B.1), (B.2) and

$$\int_{R_0}^{\infty} \left(\left|\frac{m'}{q}\right| + \left|\frac{m\,q'}{q^2}\right| \right) dr < \infty$$

for some $R_0 > 0$, then (B.3) and (B.4) are satisfied.

Theorem C (Yamada [9], Theorem 2). Let $m, q \in C^0$ satisfy

(C.1) $m(x) \equiv q(x) \longrightarrow +\infty$ as $r \to \infty$,

Then, we have $\sigma(H) \cap (0, +\infty) \subset \sigma_d(H)$.

Theorem C′ (Schmidt and Yamada [6], Theorem 2). Let $q \in C^1$ be a spherically symmetric function. Assume (C.1) and

(C.2) there exists a positive number R_0 such that

$$\frac{q'}{q^{3/2}} \in BV[R_0,\infty) \cap L^2(R_0,\infty).$$

Then, we have

$$\sigma(H) \cap (-\infty, 0) \subset \sigma_{ac}(H), \quad \text{and} \quad \sigma_s(H) \cap (-\infty, 0) = \emptyset.$$

Remark 2. In Theorem C$'$, if $q \in C^2$ satisfies

$$\int_{R_0}^{\infty} \left[\frac{|q''|}{q^{3/2}} + \frac{(q')^2}{q^{5/2}} \right] dr < \infty ,$$

then (C.2) is satisfied.

If $m \equiv -q \to -\infty$, then we have the similar result as in Theorem C$'$. On the other hand, if $m \equiv q \to -\infty$, or $m \equiv -q \to +\infty$, then see can see under the similar conditiions that the negative spectrum is discrete, and the positive spctrum is absolutely continuous.

Remark 3. For the sake of simplicity we assumed the continuity of $m(x)$ and $q(x)$ in $\mathbf{R}^3$. It turns out that, if real valued fumctions $m(x)$ and $q(x)$ belong to $L^2_{loc}(\mathbf{R}^3)$, a symmetric operator L defined on $\mathcal{C}_0^\infty = [C_0^\infty(\mathbf{R}^3)]^4$ has at least one self-adjoint extension. For, the symmetric operator L is real with respect to a conjugation J such that

$$J u = \alpha_1 \alpha_3 \overline{u} .$$

2 Outline of the Proofs.

We sketch the proof of Theorem A, C, B, C$'$, successively.

The Proof of Theorem A. It suffices to prove that $(H - i)^{-1}$ is a compact opeerator on $\mathcal{H}$. Let $\{f_n\}_{n=1,2,\cdots}$ be a bounded sequence in $\mathcal{H}$, and set $u_n = (H - i)^{-1} f_n$. Then, u_n satisfies

$$(\alpha \cdot D)\, u_n + m(x)\, \beta\, u_n = [i - q(x)]\, u_n + f_n , \tag{1}$$

and, by operating $(\alpha \cdot D)$ to (1),

$$(-\Delta + m^2 - q^2 + 1)\, u_n = [\beta\, (\alpha \cdot D\, m) - 2\, i\, q - (\alpha \cdot D\, q)]\, u_n$$
$$+ [(\alpha \cdot D) + (i - q + m\, \beta)]\, f_n$$

in the distribution sense, where $(\alpha \cdot D) = \sum_{j=1}^{3} \alpha_j D_j$. Using the assumptions we can find a positive constant C such that

$$\int_{\mathbf{R}^3} \left[|\nabla u_n|^2 + m^2(x)\, |u_n(x)|^2 \right] dx \le C\, \|f_n\|^2 ,$$

which and (A.1) imply the relative compactness of $\{u_n\}$.

Remark 4. We may adopt some local singularities of $q(x)$ and $m(x)$. If we assume that there exist positive constants C, R_0 and $\delta < 1$

$$\int_{|x| \le R_0} |\, [m(x)\, \beta + q(x)\, I_4]\, u(x)|^2\, dx \le \delta \int_{|x| \le R_0 + 1} |\nabla u|^2\, dx + C \int_{|x| \le R_0 + 1} |u(x)|^2\, dx \tag{2}$$

for any $u \in \mathcal{C}_0^\infty$, instead of the differentiability of $m(x)$ and $q(x)$ in $|x| \leq R_0$, then we obtain similarly from (1), (A.1) and (A.2) that

$$\int_{\mathbf{R}^3} |\nabla u_n|^2 \, dx + \int_{|x|\geq R_0} m^2(x) \, |u_n(x)|^2 \, dx \leq C \, \|f_n\|^2 ,$$

which implies also the relative compactness of $\{u_n\}$.

For example, if there exist positive constants R_0 and $\delta < 1$ such that

$$|m(x) \pm q(x)| \leq \frac{\delta}{2\,|x|} \qquad (|x| \leq R_0),$$

then (2) holds in view of the well-known inequality

$$\int_{\mathbf{R}^3} \frac{|u(x)|^2}{|x|^2} \, dx \leq 4 \int_{\mathbf{R}^3} |\nabla u|^2 \, dx$$

for any $u \in \mathcal{C}_0^\infty$.

The Proof of Theorem C. Let us show that any positive number λ does not belong to the essential spectrum $\sigma_{ess}(H)$. If otherwise, there exist a positive number λ and an orhtonormal sequence $\{u_n\}_{n=1,2,\cdots}$ in $\mathcal{H}$ such that

$$(H - \lambda) \, u_n \longrightarrow 0$$

in $\mathcal{H}$. Then, write

$$u_n = \begin{pmatrix} v_n \\ w_n \end{pmatrix}, \qquad (H - \lambda) \, u_n = \begin{pmatrix} f_n \\ g_n \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then, we have

$$\begin{cases} (\sigma \cdot D) \, w_n + 2 \, q \, v_n - \lambda \, v_n = f_n \\ (\sigma \cdot D) \, v_n - \lambda \, w_n = g_n , \end{cases} \qquad (3)$$

where $(\sigma \cdot D) = \sum_{j=1}^{3} \sigma_j \, D$. Combining these identities we obtain

$$-\Delta \, v_n + 2 \, \lambda \, q \, v_n = (\sigma \cdot D) \, g_n + \lambda \, f_n + \lambda^2 \, v_n . \qquad (4)$$

which implies

$$\int_{\mathbf{R}^3} \left[|\nabla v_n|^2 + 2 \, \lambda \, |q(x)| \, |v_n(x)|^2 \right] \, dx \leq C \, \left(\|f_n\|^2 + \|g_n\|^2 + \|v_n\|^2 \right) . \qquad (5)$$

where C is a positive constant independent of $n = 1, 2, \cdots$. Thus, we can select a strongly convergent subsequence $\left\{ v_{n_j} \right\}_{j=1,2,\cdots}$ in $\mathbf{h} = [L^2(\mathbf{R}^3)]^2$. Since u_n tends weakly to 0 in $\mathcal{H}$, we see

$$v_{n_j} \longrightarrow 0 \ \text{ in } \ \mathbf{h} \ \text{ as } \ j \to \infty,$$

which, (3) and (5) imply

$$w_{n_j} \longrightarrow 0 \text{ in } \mathbf{h} \text{ as } j \to \infty ,$$

This fact contradicts to $\|u_n\| = 1$ $(n = 1, 2, \cdots)$.

Remark 5. In the proof of Theorem C, we may adopt a local singularity of $q(x)$ in a ball, that is,

$$\int_{|x| \le R_0} |q(x)|^2 \, dx < \infty ,$$

instead of the continuity in the ball. For, if $q(x)$ satisfies the above condition, then we have

$$\int_{|x| \le R_0} |q(x)| \, |f(x)|^2 \, dx \le \delta \int_{|x| \le R_0+1} |\nabla f|^2 \, dx + C_\delta \int_{|x| \le R_0+1} |f(x)|^2 \, dx ,$$

$$\int_{|x| \le R_0} |q(x) \, f(x)|^2 \, dx \le \delta \int_{|x| \le R_0+1} |\Delta f|^2 \, dx + C_\delta \int_{|x| \le R_0+1} |f(x)|^2 \, dx \qquad (6)$$

for any $f \in C_0^\infty$, any positive number $\delta < 1$, and a positive constant C_δ. Therefore, we can show (5) by means of (3) and (4). Thus, we can include Coulomb potentials in Theorem C without the restriction of the size of the constant.

Remark 6. If $m(x) \equiv q(x)$ is an $L^2_{loc}(\mathbf{R}^3)$ function, then the symmetric operator H_0 with the domain $D(H_0) = C_0^\infty$ such that $H_0 \, u = L \, u$ $(u \in D(H_0)$ is essentially self-adjoint. Indeed, we can see that the ranges of $(H_0 \pm i)$ are dense in $\mathcal{H}$. If otherwise, we could take non-zero vectors v and $w \in \mathbf{h}$ such that

$$\begin{cases} (\sigma \cdot D) \, w + 2 \, q \, v = \eta \, v \\ (\sigma \cdot D) \, v = \eta \, w , \end{cases} \qquad (7)$$

where $\eta = i$ or $-i$, and

$$-\Delta \, v + 2 \, \eta \, q \, v = -v \qquad (8)$$

in the distribution sense. Then, we have $|\nabla v| \in L^2(\mathbf{R}^3)$ by means of the latter of (7) and $\Delta \, v \in \mathbf{h}$ in view of the assumption and (6), (8), which give

$$\int_{|x| \le R} \left[|\nabla v|^2 + |v|^2 \right] \, dx = \int_{|x| = R} \mathrm{Re} < \frac{\partial v}{\partial r}, \bar{v} > \, dS . \qquad (9)$$

Since the right hand side of (9) tends to 0 by a sequence $\{R_n\}_{n=1,2,\cdots}$ with $R_n \to \infty$, we have $v = 0$, which and the second identity of (7), gives $w = 0$. This is a contradiction.

The Proof Theorem B. If $m = m(r)$ and $q = q(r)$ are spherical symmetric functions, the spectral problem of H is reduces to the one of one-dimensional Dirac operators

$$l_k = -i \, \sigma_2 \, \frac{d}{dr} + m(r) \, \sigma_3 + q(r) \, I_2 + \frac{k}{r} \, \sigma_1 \qquad (k = \pm 1, \pm 2, \cdots)$$

in $[L^2(0, \infty)]^2$ (see, e.g., Arai [1]).

The proof of the Theorem B is given on the line of the follwoing Lemma 1 given by Behncke [3], Theorem 1.

Lemma 1. Let m, p and q be real valued functions and belong to $L^1_{loc}(0,\infty)$, and

$$l = -i\,\sigma_2 \frac{d}{dr} + m\,(r)\,\sigma_3 + q\,(r)\,I_2 + p\,(r)\,\sigma_1\,.$$

Let h_0 be the minimal operator such that

$$h_0\,u = l\,u\,, \qquad u \in D(h_0) = \{u \in C_0^\infty(0,\infty) \;\mid\; l\,u \in \mathbf{h}\}\,,$$

which is a densely defined symmetric operator in $\mathbf{h}$ (see, e.g., Weidmann [8], Theorem 3.7). and let h be a self-adjoint extension of h_0 (note that h_0 is a real operator). Assume that I is an interval such that every solution of

$$l\,v = \lambda\,v \qquad (\lambda \in I) \tag{10}$$

is bounded at infinity. Then, we have

$$I \subset \sigma_{ac}(h) \;\text{ and }\; \sigma_s(h) \cap I = \emptyset\,.$$

The above lemma is closely related to Gilbert–Pearson [4] and Weidmann [7]. There is also a direct proof by Schmidt [5], Appendix.

In order to obtain the boundedness of v of (10) at infinity, we prepare the following Lemma 2.

Lemma 2. Let M, P and Q be real valued functions and belong to $L^1_{loc}(0,\infty)$ such that

$$\lim_{r\to\infty} Q\,(r) = \infty \tag{11}$$

$$\limsup_{r\to\infty} \frac{\sqrt{M\,(r)^2 + P\,(r)^2}}{Q\,(r)} < 1\,, \tag{12}$$

and

$$\frac{\sqrt{M^2+P^2}}{Q-\sqrt{M^2+P^2}}\,, \quad \frac{M}{Q-\sqrt{M^2+P^2}}\,, \quad \frac{P}{Q-\sqrt{M^2+P^2}} \in BV[R_0,\infty) \tag{13}$$

for some $R_0 > 0$. Then, every solution of

$$-i\,\sigma_2 \frac{d}{dr} v + M\,\sigma_3\,v + P\,\sigma_1\,v + Q\,v = 0$$

is bounded at infinity. If $P \equiv 0$, the condition (13) may be read as ;

$$\frac{M}{Q-M} \in BV[R_0,\infty)\,.. \tag{14}$$

Theorem B is shown by seeing the boundedness of any solution v of

$$l_k\, v = -i\,\sigma_2 \frac{d}{dr}\, v + m\,(r)\,\sigma_3 + q\,(r)\, v + \frac{k}{r}\,\sigma_1\, v = \lambda\, v \quad (\lambda < 0\,, \quad k = \pm 1, \pm 2, \cdots)$$

at infinity. If we set

$$Q = q - \lambda\,, \qquad M = m\,, \qquad P = \frac{k}{r}$$

in Lemma 2, we can guarantee (11), (12) and (13) in Lemma 2 by means of the asuumptions (B.1), (B.2), (B.3) and (B.4). Therefore, we get the boundedness of v at infinity, and the absolute continuity of the spectrum of H in view of Lemma 1.

Remark 7. If $q\,(r)$ and $m\,(r)$ are locally bounded in $[0, \infty)$, every l_k $(k = \pm 1, \pm 2, \cdots)$ is of limit point type at 0 and ∞. If $m\,(r)$ is locally bounded near 0 and $q\,(r)$ satisfies

$$|q\,(r)| \leq \frac{\sqrt{3}}{2\,r}$$

near 0, then every l_k is of limit point type at 0. If $m\,(r) = b/r$ (b is a real constant) and

$$|q\,(r)|^2 \leq \left[\frac{3}{4} + b^2\right] \frac{1}{r^2}$$

near 0, then every l_k is of limit point type at 0 (see, e.g., Arai [1] and Yamada [10], where are more general results).

The Proof of Theorem C′. We shall make use of the Gilbert–Pearson theory (Gilbert–pearson [4], Behncke [2]), showing that the differential equation

$$l_k\, v = \left(-i\,\sigma_2 \frac{d}{dr} + q\,(r)\,\sigma_2 + q\,(r)\, I_2 + \frac{k}{r}\,\sigma_1\right) v = \lambda\, v \quad (\lambda < 0) \tag{15}$$

does not possess a subordinate solution at infinity, that is, any non-trivial soutions v and w of (15) for $\lambda < 0$ satisfy

$$\liminf_{r \to \infty} \frac{\displaystyle\int_{R_0}^{\infty} |v\,(s)|^2\, ds}{\displaystyle\int_{R_0}^{\infty} |w\,(s)|^2 ds} > 0 \tag{16}$$

for some $R_0 > 0$. To this end, for a solution $v = {}^t(v_1, v_2)$ of (15), we set

$$\tilde{v} = \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} \sqrt[4]{(2\,q - \lambda)/(-\lambda)}\, v_1 \\ \sqrt[4]{(-\lambda)/(2\,q - \lambda)}\, v_2 \end{pmatrix}.$$

Then, $\tilde{v}$ satisfies

$$\left(-i\,\sigma_2 \frac{d}{dr} + M\,\sigma_1 + Q\, I_2\right) \tilde{v} = 0\,, \tag{17}$$

where

$$M\,(r) = \frac{-q'}{2\,(2\,q - \lambda)}\,, \qquad Q\,(r) = \sqrt{(-\lambda)\,(2\,q - \lambda)}\,, \qquad P \equiv 0\,.$$

Under the assumptions (C.1) and (C.2) we have that the above M and Q satisfy the conditions (11), (12) and (14) in Lemma 2 and, therefore, any solution $\tilde{v}$ of (17) is bounded at infinity, which implies

$$C^{-1} \leq |\tilde{v}(r)|^2 \leq C \quad (r \geq R_0)$$

for some positive constants R_0 and $C > 1$. Thus, we have

$$0 < \liminf_{r\to\infty} \frac{|v(r)|^2}{\sqrt{2q(r)-\lambda}} \leq \limsup_{r\to\infty} \frac{|v(r)|^2}{\sqrt{2q(r)-\lambda}} < \infty .$$

The same estimate holds for w, which yields (16).

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