数理解析研究所講究録
1047巻 1998年 47-55

ANALYTICITY AND SMOOTHING EFFECT FOR THE KORTEWEG DE VRIES EQUATION

加藤圭一(東京理科大・理) (KEIICHI KATO)
小川卓克(名古屋大・多元数理) (TAKAYOSHI OGAWA)

1. Introduction and Theorem

We study the smoothing effect for the following initial-value problem of the Korteweg-de Vries equation (KdV equation):

$$(1.1)\qquad \begin{cases} \partial_t v + \partial_x^3 v + \partial_x(v^2) = 0, & t, x \in \mathbb{R}, \\ v(0,x) = \phi(x). \end{cases}$$

Here the solution $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denotes the surface displacement of the water wave.

There are a lot of works for the study of KdV equation ([4], [5], [6], [8], [10], [14], [16], [21], [22], [24]). Among others, Kato [14] firstly extract the smoothing effect for the evolution operator of the linear part of the KdV equation $e^{t\partial_x^3}$. The Uniqueness result is obtained by Kruzhkov-Faminskii [21], Ginibre-Y. Tsutsumi [8] in the subspace of H^1. Later on, Kenig-Ponce-Vega [16] extended the Kato type smoothing effect and they showed that the KdV equation is well-posed in the Sobolev space $H^{3/4}$.

Along the elegant method in the series of papers, Bourgain [2] obtained L^2 well-posedness of the KdV equation in the periodic boundary condition. His argument also works for the Cauchy problem (1.1) and the global well-posedness is established. Furthermore, by refining the method of Bourgain, Kenig-Ponce-Vega [17] [18] proved some bilinear estimates involving the negative exponent Sobolev space and established the local well-posedness for the Cauchy problem in the negative Sobolev space $H^s(\mathbb{R})$ where $(-3/4 < s)$.

AMS Subject Classifications: 35Q53

On the other hand, a highly regular solution has also been studied by several authors. T.Kato-Masuda [20] obtained a global smooth solution and the analyticity for any point $(t,x) \in \mathbb{R} \times \mathbb{R}$. Hayashi-K.Kato [9] obtained the analytic smoothing effect for the nonlinear Schrödinger equation (see also K.Kato-Taniguchi [13]) and de Bouard-Hayashi-Kato [7] established the analyticity for KdV equations from the Gevrey initial data. Those results are basically obtained by using the commutation and almost commutation operators with the linear KdV equation.

Thanks to the paper [18], we have a time local solution of (1.1) with Dirac's delta as the initial data. Our problem in this note is to study the regularity of the solution with Dirac's delata as the initial data. In the following, we show that if the initial data is in some class which contains Dirac's delta, the solution is real analytic for $t \neq 0$.

More precisely, our result is the following:

Theorem 1.1. *Let $-3/4 < s$, $b \in (1/2, 7/12)$. Suppose that the initial data $\phi \in H^s(\mathbb{R})$ and for some $A_0 > 0$, it satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty.$$

Then there exists a unique solution $v \in C((-T,T),H^s) \cap X_b^s$ of the KdV equation (1.1) in a certain time interval $(-T,T)$ and the solution v is time locally well-posed, i.e. the solution continuously depends on the initial data. Moreover the solution v is analytic in time and space variables at any point $(t,x) \in (-T,0) \cup (0,T) \times \mathbb{R}$, where we define

$$\|f\|_{X_b^s} = \left(\iint \langle \tau - \xi^3 \rangle^{2b} \langle \xi \rangle^{2s} |\hat{f}(\tau,\xi)|^2 d\tau d\xi \right)^{1/2} = \|V(-\cdot)f(\cdot)\|_{H_t^b(\mathbb{R};H_x^s(\mathbb{R}))}$$

and $V(t) = e^{-t\partial_x^3}$ is the unitary group of the free KdV evolution.

Remark 1. A typical example of the initial data satisfying the assumption of the above theorem is the Dirac delta measure, since $(x\partial_x)^k \delta(x) = (-1)^k \delta(x)$. The other example of the data is $p.v.\frac{1}{x}$, where $p.v.$ denotes Cauchy's principal value. Any possible linear combination of those distributions with an analytic H^s data satisfying the assumption can be also the initial data. In this sense, Dirac's delta measure adding the soliton initial data can be taken as the initial data.

Remark 2. For a non-smooth initial data, it is known that the global in time solution has been obtained (see [5], [10]) by the inverse scattering method. Recently the analyticity for the inverse scattering solution with a weighted initial data was obtained by Tarama [23]. Since our method is based on the fact that the solution is in H^s, we don't know if our result is true globally in time.

By a almost similar argument of Theorem 1.1, one can also show the following corollary.

Corollary 1.2. *Let* $-3/4 < s$, $b \in (1/2, 7/12)$. *Suppose that for some* $A_0 > 0$, *the initial data* $\phi \in H^s(\mathbb{R})$ *and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{(k!)^3} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

then there exists a unique solution $v \in C((-T,T), H^s) \cap X_b^s$ *of the KdV equation* (1.1) *for a certain time interval* $(-T,T)$ *and for any* $t \in (-T,0) \cup (0,T)$ $v(t,\cdot)$ *is analytic function in space variable and for* $x \in \mathbb{R}$, $v(\cdot, x)$ *is of Gevrey* 3 *with respect to time variable.*

Remark 3. Both in Theorem and Corollary, the assumption on the initial data implies the analyticity and Gevrey 3 regularity except the origin respectively. In this sense, those results states that the singularity at the origin immediately disappears after $t > 0$ or $t < 0$ up to analyticity.

Remark 4. Some related results are obtained for the linear and nonlinear Schrödinger equations. For linear variable coefficient case, see Kajitani-Wakabayashi [11] and for nonlinear case, Chihara [3]. They give a global weighted uniform estimates of the solution with arbitrary order of derivatives in space variable. Even in our case, we expect that the similar uniform bounds are available.

2. Method of the proof

Our method is based on the following observation. Firstly, we introduce the generator of the dilation $P = 3t\partial_t + x\partial_x$ for the linear part of the KdV equation. Noting the commutation relation with the linear KdV operator $L = \partial_t + \partial_x^3$:

$$[L, P] = 3L,$$

it follows

$$LP^k = (P+3)^k L, \tag{2.1}$$

$$(P+3)^k \partial_x = \partial_x (P+2)^k \tag{2.2}$$

for any $k = 1, 2, \cdots$. Applying $P = 3t\partial_x + x\partial_x$ to the KdV equation, we have

$$\begin{aligned} \partial_t(P^k v) + \partial_x^3(P^k v) &= (P+3)^k Lv = (P+3)^k(-\partial_x(v^2)) \\ &= -\partial_x(P+2)^k v^2. \end{aligned} \tag{2.3}$$

We set $v_k = P^k v$ and $B_k(v,v) = \partial_x(P+2)^k v^2$. Noting that

$$\begin{aligned} (P+2)^l v &= (P+2)^{l-1} Pv + 2(P+2)^{l-1} v = \cdots \\ &= \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} 2^{l-j} P^j v, \end{aligned} \tag{2.4}$$

we see

$$\begin{aligned} B_k(v,v) =& \partial_x(P+2)^k(v^2) = \partial_x \sum_{l=0}^{k} \binom{k}{l} (P+2)^l v P^{k-l} v \\ =& \partial_x \sum_{l=0}^{k} \sum_{m=0}^{l} \binom{k}{l} \binom{l}{m} 2^{l-m} P^m v P^{k-l} v \\ =& \sum_{k=k_1+k_2+k_3} \frac{k!}{k_1!k_2!k_3!} 2^{k_1} \partial_x(v_{k_2} v_{k_3}). \end{aligned}$$

The above nonlinear term maintains the bilinear structure like the original KdV equation, since the Leibniz law can be applicable for the operator P. Now each v_k satisfies the following system of equations;

$$\begin{cases} \partial_t v_k + \partial_x^3 v_k + B_k(v,v) = 0, & t, x \in \mathbb{R}, \\ v_k(0,x) = (x\partial_x)^k \phi(x), \end{cases} \quad k = 0, 1, 2, \cdots. \tag{2.5}$$

Therefore we firstly establish the local well-posedness of the solution to the following infinitely coupled system of KdV equation in a suitable weak space:

$$\begin{cases} \partial_t v_k + \partial_x^3 v_k + B_k(v,v) = 0, & t, x \in \mathbb{R}, \\ v_k(0,x) = \phi_k(x), \end{cases} \quad k = 0, 1, 2, \cdots. \tag{2.6}$$

By taking $\phi_k = (x\partial_x)^k \phi(x)$, the uniqueness and local well-posedness yields that $v_k = P^k v$ for all $k = 0, 1, \cdots$.

According to Bourgain [2], we introduce the Fourier restriction space;

$$X_b^s = \{f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{X_b^s} < \infty\},$$

where

$$\|f\|_{X_b^s}^2 = c\iint \langle\tau-\xi^3\rangle^{2b}\langle\xi\rangle^{2s}|\hat{f}(\tau,\xi)|^2 d\tau d\xi = \|V(-t)f\|_{H_t^b(\mathbb{R};H_x^s)}^2.$$

It has been proven that the KdV equation is well-posed in the above space X_b^s when $s > -3/4$ with $b > 1/2$. The space where we solve the system (2.6) is infinite sum of copies of this space. Let $f = (f_0, f_1, \cdots, f_k, \cdots)$ denotes the infinite series of distributions and define

$$\mathcal{A}_{A_0}(X_b^s) = \{f = (f_0, f_1, \cdots, f_k, \cdots); f_i \in X_b^s \quad (i = 0,1,2,\cdots) \text{ such that} \|f\|_{\mathcal{A}_{A_0}} < \infty\},$$

where

$$\|f\|_{\mathcal{A}_{A_0}} \equiv \sum_{k=0}^{\infty} \frac{A_0^k}{k!}\|f_k\|_{X_b^s}.$$

The system (2.6) will be shown to be well-posed in the above space if $s > -3/4$ and $b > 1/2$ under the assumption for the initial data

$$\|\phi_k\|_{H^s} \le CA_1^k k! \quad k = 0,1,\cdots.$$

The well-posedness is derived by utilizing the contraction principle argument to the corresponding system of integral equations:

$$\psi(t)v_k(t) = \psi(t)V(t)\phi_k - \psi(t)\int_0^t V(t-t')\psi_T(t')B_k(v,v)(t')dt'. \tag{2.7}$$

The following estimates of linear and nonlinear part due to Bourgain [2] and refined by Kenig-Ponce-Vega [17] are our essential tools.

Lemma 2.1 ([2],[17],[18]). *Let $s \in \mathbb{R}$, $a, a' \in (0,1/2)$, $b \in (1/2,1)$ and $\delta < 1$. Then for any $k = 0,1,2,\cdots$, we have*

$$\|\psi_\delta\phi_k\|_{X_{-a}^s} \le C\delta^{(a-a')/4(1-a')}\|\phi_k\|_{X_{-a'}^s}, \tag{2.8}$$

$$\|\psi_\delta V(t)\phi_k\|_{X_b^s} \le C\delta^{1/2-b}\|\phi_k\|_{H^s}, \tag{2.9}$$

$$\|\psi_\delta\int_0^t V(t-t')F_k(t')dt'\|_{X_b^s} \le C\delta^{1/2-b}\|F_k\|_{X_{b-1}^s}. \tag{2.10}$$

Lemma 2.2 ([2],[17],[18]). *Let $s > -3/4$, $b, b' \in (1/2, 7/12)$ with $b < b'$. Then for any $k, l = 0,1,2,\cdots$, we have*

$$\|\partial_x(u_k v_l)\|_{X_{b'-1}^s} \le C\|v_k\|_{X_b^s}\|v_l\|_{X_b^s}. \tag{2.11}$$

Proof of Lemma 2.1 and 2.2. See [17] and [18]. □

From Lemma 2.2, it is immediately obtained the bilinear estimate for the nonlinearity for the system.

Corollary 2.3. *Let $s > -3/4$, $b, b' \in (1/2, 7/12)$ with $b < b'$. Then we have*

$$\|B_k(v,v)\|_{X^s_{b'-1}} \le C \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{k!}{k_1!k_2!k_3!} \|v_{k_2}\|_{X^s_b} \|v_{k_3}\|_{X^s_b}. \tag{2.12}$$

We construct a contraction map via the integral equations. Set a map $\Phi : \{v_k\}_{k=0}^{\infty} \to \{v_k(t)\}_{k=0}^{\infty}$ such that $\Phi = (\Phi_0, \Phi_1, \cdots)$ and

$$\Phi_k(\phi_k) \equiv \psi V(t)\phi_k - \psi \int_0^t V(t-t')\psi_T(t')B_k(v,v)(t')dt'.$$

We show that $\Phi_k : \mathcal{A}_{A_0}(H^s) \to \mathcal{A}_{A_1}(X^s_b)$ is contraction.

In fact, by using Lemma 2.1 and Lemma 2.2, we easily see that for any $k \ge 0$,

$$\begin{aligned} \|\Phi_k(v_k)\|_{X^s_b} \le & C_0\|\phi_k\|_{H^s} + C_1T^{\mu}\|B_k(v,v)\|_{X^s_{b'-1}} \\ \le & C_0\|\phi_k\|_{H^s} + C_1T^{\mu} \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{k!}{k_1!k_2!k_3!} \|v_{k_2}\|_{X^s_b} \|v_{k_3}\|_{X^s_b}. \end{aligned} \tag{2.13}$$

By taking a sum in k,

$$\begin{aligned} \|\!|\Phi|\!\|_{\mathcal{A}_{A_1}(X^s_b)} =& \sum_{k=0}^{\infty} \frac{A_1^k}{k!} \|v_k\|_{X^s_b} \\ \le & C_0 \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|\phi_k\|_{H^s} + C_1T^{\mu} \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{k!}{k_1!k_2!k_3!} \|v_{k_2}\|_{X^s_b} \|v_{k_3}\|_{X^s_b} \\ \le & C_0\|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1T^{\mu} \sum_{k=0}^{\infty} \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{A_0^{k_1}}{k_1!} \frac{A_0^{k_2}}{k_2!} \|v_{k_2}\|_{X^s_b} \frac{A_0^{k_3}}{k_3!} \|v_{k_3}\|_{X^s_b} \\ \le & C_0\|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1T^{\mu} \sum_{k_1=0}^{\infty} 2^{k_1} \frac{A_0^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} \frac{A_0^{k_2}}{k_2!} \|v_{k_2}\|_{X^s_b} \sum_{k_3=0}^{\infty} \frac{A_0^{k_3}}{k_3!} \|v_{k_3}\|_{X^s_b}. \end{aligned}$$

Hence we have

$$\|\!|\Phi(v)|\!\|_{\mathcal{A}_{A_1}(X^s_b)} \le C_0\|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1 e^{2A_0} T^{\mu} \|\!|v|\!\|^2_{\mathcal{A}_{A_1}(X^s_b)}$$

and also we have the estimate for the difference

$$\|\!|\Phi(v) - \Phi(\tilde{v})|\!\|_{\mathcal{A}_{A_1}(X^s_b)} \le C_1 e^{2A_0} T^{\mu} (\|\!|v|\!\|_{\mathcal{A}_{A_1}(X^s_b)} + \|\!|\tilde{v}|\!\|_{\mathcal{A}_{A_1}(X^s_b)}) \|\!|v - \tilde{v}|\!\|_{\mathcal{A}_{A_1}(X^s_b)}.$$

Choosing T small enough, the map Φ is contraction from

$$X_T = \{f = (f_0, f_1, \cdots); f_i \in X^s_b, \sum_0^{\infty} \frac{A_0^k}{k!} \|f_k\|_{X^s_b} \le 2C_0M_0\}$$

to itself, where $M_0 = \|\phi\|_{\mathcal{A}_{A_0}(H^s)}$. A similar argument in [1] gives us the uniqueness of the system of the solution. This shows the proof of well-posedness.

Hence under the assumption for the initial function

$$\|(x\partial_x)^k\phi\|_{H^s} \le CA_1^k k! \quad k = 0, 1, \cdots,$$

the corresponding solution to (KdV) satisfies the estimate

$$\|P^k v\|_{X_b^s} \le CA_0^k k! \quad k = 0, 1, \cdots. \tag{2.14}$$

Multiplying t to the both sides of the first equation of (2.5), we have

$$t\partial_x^3 v_k = -\frac{1}{3}Pv_k + \frac{1}{3}x\partial_x v_k + tB_k(v, v), \tag{2.15}$$

from which we gain the regularity of v with (2.14).

For a fixed point $(t_0, x_0) \in (0, T) \times \mathbb{R}$, we show that v is analytic near (t_0, x_0). Let $a(t, x) \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function near (t_0, x_0) such that supp$a \subset [t_0 - \varepsilon, t_0 + \varepsilon] \times [x_0 - \varepsilon^{1/3}, x_0 - \varepsilon^{1/3}]$. First we show that

$$\|aP^k v\|_{L^2_{t,x}(\mathbb{R}^2)} \le C_2 A_2^k k! \quad k = 0, 1, 2, \cdots, \tag{2.16}$$

for some positive constants C_2 and A_2. This is shown by using the following lemma:

Lemma 2.4. *Let $P = 3t\partial_t + x\partial_x$ be the generator of the dilation and $D_{t,x}$ be an operator defined by $\mathcal{F}_{t,x}^{-1}\langle|\tau| + |\xi|\rangle\mathcal{F}_{t,x}$.*
(1) For a freezing point (t_0, x_0), we suppose that $g \in X_{b-1}^r$ with supp $g \subset B_{2\varepsilon}(t_0, x_0)$ and $t\partial_x^3 g$, $P^3 g \in X_{b-1}^r$. Then for $b \in (0, 1]$ and $r \in (-\infty, 0]$, we have

$$\|\langle D_{t,x}\rangle^{3b} g\|_{L^2(\mathbb{R};H^r(\mathbb{R}))} \le C\{\|g\|_{X_{b-1}^r} + \|t\partial_x^3 g\|_{X_{b-1}^r} + \|P^3 g\|_{X_{b-1}^r}\}, \tag{2.17}$$

where the constant C depends on (t_0, x_0) and ε.
(2) If $g \in H^{\mu-3}(\mathbb{R}^2)$ with supp $g \subset B_{2\varepsilon}(t_0, x_0)$ and $t\partial_x^3 g$, $P^3 g \in H^{\mu-3}(\mathbb{R}^2)$. Then we have

$$\|\langle D_{t,x}\rangle^{\mu} g\|_{L^2(\mathbb{R}^2)} \le C\{\|g\|_{H^{\mu-3}(\mathbb{R}^2)} + \|t\partial_x^3 g\|_{H^{\mu-3}(\mathbb{R}^2)} + \|P^3 g\|_{H^{\mu-3}(\mathbb{R}^2)}\}, \tag{2.18}$$

where the constant C depends on (t_0, x_0) and ε.

From (2.16) and (2.15) we can show that

$$\|aP^k v\|_{H^{7/2}(\mathbb{R}^2)} \le C_3 A_3^k k! \quad k = 0, 1, 2, \cdots, \tag{2.19}$$

for some positive constants C_3 and A_3. (2.19) gives us immediately that

$$\sup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \|P^k v\|_{H^1(x_0 - \varepsilon^{1/3}, x_0 + \varepsilon^{1/3})} \le C_3 A_3^k k!, \tag{2.20}$$

for $k = 0,1,2,\cdots$. From this estimate (2.20) and (2.15) we have with some positive constants C_4 and A_4,

$$\sup_{t\in[t_0-\varepsilon,t_0+\varepsilon]} \|(t^{1/3}\partial_x)^l P^k v\|_{H^1(x_0-\varepsilon^{1/3},x_0+\varepsilon^{1/3})} \le C_4 A_4^{l+k}(l+k)!, \tag{2.21}$$

for $m,k = 0,1,2,\cdots$. This is shown by induction with respect to l. From (2.21) and (2.15) we have with some positive constants C_5 and A_5,

$$\sup_{t\in[t_0-\varepsilon,t_0+\varepsilon]} \|\partial_t^m \partial_x^l v\|_{H^1(x_0-\varepsilon^{1/3},x_0+\varepsilon^{1/3})} \le C_5 A_5^{m+l}(m+l)!, \tag{2.22}$$

for $m,l = 0,1,2,\cdots$, which shows that v is real analytic in (t,x) near (t_0,x_0).

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